APPENDIX to Circular-Letter No. 4 - H of 1954

THE HARMONIC DEVELOPMENT OF THE TIDE-GENERATING POTENTIAL

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§ 1. *Introduction.*

The harmonic development of the tide-generating potential is the basis of most work on tidal observations, and since 1883 the development given by Sir G.H. Darwin has been universally used and has been of remarkable value. But discrepancies between prediction and observation are serious and have been attributed to faulty « harmonic constants »; it has been assumed that if these were improved better predictions would be obained, and it has also been tacitly assumed that it is only necessary to consider the harmonic constituents as given by Darwin. Recent work, however, especially at the Tidal Institute, has shown that when the « Darwinian constituents » are removed from the tidal height there is a residue composed of constituents which are not included in his schedules. These are such that any slight improvements possible in the « constants » usually obtained are comparatively negligible.

The obvious course, therefore, was to make a more thorough development of the potential, and in view of the unknown nature of the residues great accuracy was obviously desirable, especially as the possibility of resonance has always to be considered. The development given in this paper, even if it proves to be needlessly thorough for practical tidal work, will cover the needs of research work, since it includes all terms whose coefficients (relatively to the greatest coefficient) are greater than 0.00010.

Darwin used the old lunar theory and referred everything to the orbit rather than to the ecliptic; his results are all given in the algebraic form, arithmetic being used only to decide what terms to omit. His development is a quasi-harmonic development because he retains factors in the coefficients and terms in the arguments which are considered as constant over fairly long intervals of time, such as a year, but which are really slowly variable. The present method [p. 306]
of development uses the results of the modern lunar theory and is essentially a numerical method throughout. The theoretical expansions for the longitude and latitude of the moon referred to the ecliptic, as given by Brown*, have been used, and the development is truly harmonic. Ferrel's development, published in 1874, was also truly harmonic, but it included only the most important terms.

The new schedules of constituents, as compared with the old schedules, contain many terms which, for modern purposes, are too large to be ignored; this matter is dealt with in § 9, and Table VI gives a comparison of the main terms as given in the old and in the new schedules. It is of interest to note that J.C. Adams verified Darwin's work and carried out the development so as to include more terms, but this work does not seem to have been published.

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* Monthly Notices Roy. Astron. Soc., vol. 65, p. 285 (1905).

One great aim of the author has been to reduce the subject to its very simplest form, and what credit is due for this must be equally shared with Prof. J. Proudman, for whose criticism and advice the author is deeply indebted.

A great deal of attention has been paid to the matters of notation and presentation of results; a prominent feature in both is the adoption of a special notation for the arguments, which is such that any argument is represented by a number and, what is very striking, if the terms in the expansions are arranged according to the argument-number, they are thereby automatically arranged according to « speed », which is very convenient.

The application of the new development to the analysis of observations and to predictions is not fully dealt with in this paper. Certain suggestions are made, however, concerning future practice.

§ 2. *Development of the Lunar Tide-generating Potential.*

Let
E = mass of the earth
M = mass of the moon
r = distance between centres of earth and moon
$1/c$ = mean value of $1/r$
ρ = radius of earth at given place P
λ = latitude of P
L = longitude of P, west of Greenwich
a = mean radius of earth
g = mean value of gravitational acceleration
and V = tide-generating potential due to the moon. (1)

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Then

$$V = \frac{\mu M \rho^2}{r^3}\left(P_2 + \frac{\rho}{r} P_3 + \frac{\rho^2}{r^2} P_4 + \dots\right) = V_2 + V_3 + V_4 + \dots, \quad (2)$$

where
$P_2 = 1/2\,(3\cos^2 \mathfrak{z} - 1)$,
$P_3 = 1/2\,(5\cos^3 \mathfrak{z} - 3\cos \mathfrak{z})$,
$P_4 = 1/8\,(35\cos^4 \mathfrak{z} - 30\cos^2 \mathfrak{z} + 3)$,
$\mathfrak{z}$ = geocentric zenith distance of the moon from P,
and μ = attraction between unit masses at unit distance apart
$= ga^2/E$.

The ultimate result of the development of V is a series of terms harmonic in time, and as only the relative values of these are usually of importance, it is convenient to have the greatest numerical coefficient approximately unity; hence we write

$$G = \frac{3}{4}\frac{M}{E} \cdot \frac{ga^2\rho}{c^3}, \tag{3}$$

and therefore

$$\left.\begin{aligned} V_2 &= 2/3\ (3\cos^2\vartheta - 1)\ .\ G\ (c/r)^3 \\ V_3 &= 2/3\ (5\cos^3\vartheta - 3\cos\vartheta)\ .\ G\ (c/r)^4\ (\rho/c) \\ V_4 &= 1/6\ (35\cos^4\vartheta - 30\cos^2\vartheta + 3)\ .\ G\ (c/r)^5\ (\rho/c)^2 \end{aligned}\right\} \tag{4}$$

The factor ρ/c is small and can be taken as equal to the value of the sine of the mean equatorial horizontal parallax, whose numerical value is $3422''{\cdot}70 \div 206265'' = 0{\cdot}0165937$.

The first stage in the further development of these functions is the separation of the long-period, diurnal, semi-diurnal, ter-diurnal, and quarter-diurnal species of constituents. Referring to fig. 1, let γ be the first point of Aries, M the place

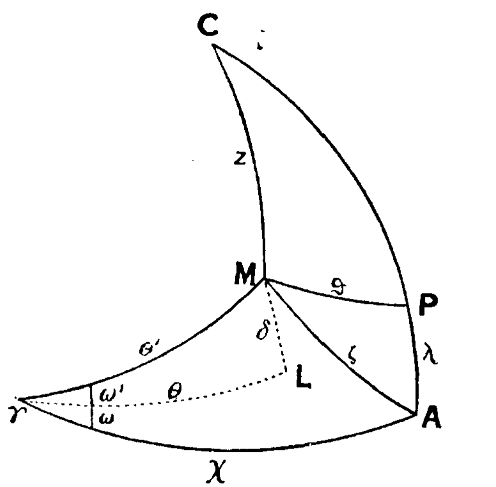

Fig. 1.

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of the moon, C the north Pole, P the given place, and A the intersection of the meridian of P with the equator, $\chi = \gamma$ A. Also let θ', z, ϑ ζ, and χ be respectively the geocentric zenith distances γM, MC, MP, MA and γA.

Then, from the spherical triangle MCP we have

$$\cos\vartheta = \sin\lambda\cos z + \cos\lambda\sin z\cos C,$$

and since the angle C increases at the rate of approximately 360° per mean lunar day, the expansion of V_2, V_3, and V_4 in terms of cos C, cos 2C, cos 3C, and cos 4C will separate the species of constituents; these expansions are expressed as series of terms involving functions of λ and G, multiplied by functions of z and C. The former will be called « geodetic coefficients », and it is desirable that these should all be expressed with the same maximum value, G, so that the numerical coefficients of the harmonic constituents ultimately obtained will give the chief index of their relative importance.

It is easy to verify, either directly, or by using the theory of spherical harmonics, that

$$\left.\begin{aligned}
V_2 &= (c/r)^3\,(G_0H_0 + G_1H_1 + G_2H_2)\\
V_3 &= (c/r)^4(0{\cdot}004947G_0'H_0' + 0{\cdot}011425G_1'H_1'\\
&\qquad + 0{\cdot}031935G_2'H_2' + 0{\cdot}013828G_3'H_3')\\
V_4 &= (c/r)^5(0{\cdot}000046G_0''H_0'' + 0{\cdot}000121G_1''H_1'' +\\
&\qquad 0{\cdot}000148G_2''H_2'' + 0{\cdot}000522G_3''H_3'' + 0{\cdot}000201G_4''H_4'')
\end{aligned}\right\} \quad (5)$$

where

$$\left.\begin{aligned}
G_0 &= 1/2\ G\ (1 - 3\sin^2\lambda)\\
G_1 &= G \sin 2\lambda\\
G_2 &= G \cos^2\lambda\\
&\ldots\ldots\ldots\ldots\\
G_0' &= 1{\cdot}11803G \sin\lambda\ (3 - 5\sin^2\lambda)\\
G_1' &= 0{\cdot}72618G \cos\lambda\ (1 - 5\sin^2\lambda)\\
G_2' &= 2{\cdot}59808G \sin\lambda \cos^2\lambda\\
G_3' &= G \cos^3\lambda\\
&\ldots\ldots\ldots\ldots\\
G_0'' &= 0{\cdot}12500G\ (3 - 30\sin^2\lambda + 35\sin^4\lambda)\\
G_1'' &= 0{\cdot}47346G \sin 2\lambda\ (3 - 7\sin^2\lambda)\\
G_2'' &= 0{\cdot}77778G \cos^2\lambda\ (1 - 7\sin^2\lambda)\\
G_3'' &= 3{\cdot}07920G \sin\lambda \cos^3\lambda\\
G_4'' &= G \cos^4\lambda\\
&\ldots\ldots\ldots\ldots\\
H_0 &= 2/3 - 2\cos^2 z\\
H_1 &= \sin 2z \cos C = 2\cos z \cos\zeta\\
H_2 &= \sin^2 z \cos 2C = 2\cos^2\zeta - \sin^2 z\\
&\ldots\ldots\ldots\ldots\\
H_0' &= \cos z\ (3 - 5\cos^2 z)\\
H_1' &= \sin z \cos C\ (1 - 5\cos^2 z) = \cos\zeta\ (1 - 5\cos^2 z)\\
H_2' &= \sin^2 z \cos z \cos 2C = \cos z\ (2\cos^2\zeta - \sin^2 z)\\
H_3' &= \sin^3 z \cos 3C = \cos\zeta\ (4\cos^2\zeta - 3\sin^2 z)\\
&\ldots\ldots\ldots\ldots\\
H_0'' &= 3 - 30\cos^2 z + 35\cos^4 z\\
H_1'' &= \sin 2z \cos C.\ (3 - 7\cos^2 z)\\
H_2'' &= \sin^2 z \cos 2C.\ (1 - 7\cos^2 z)\\
H_3'' &= \sin^3 z \cos z \cos 3C\\
H_4'' &= \sin^4 z \cos 4C
\end{aligned}\right\} \quad (6)$$

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The numerical factors in the geodetic coefficients are necessary for each coefficient to have the same maximum value.

The seconds forms of the expressions for H_1, H_2, H_1', H_2' and H_3' are obtained by using

$$\sin z \cos C = \cos \zeta,$$

derived from the spherical triangle MCA. The corresponding expressions for H_0''... H_4'' are not given, simply because they are not used; obviously there are simple relations between these and H_0, H_1 and H_2, and those are used, as will be shown later.

The development in terms of cos z and cos ζ is required in order to use the known harmonic expansions for the longitude and latitude of the moon referred to the ecliptic. Referring to fig. 1, let γ L be the ecliptic and let

$$\left.\begin{aligned} \theta &= \gamma L = \text{longitude of moon} \\ \delta &= LM = \text{latitude of moon} \\ \omega &= \text{angle } A\gamma L = \text{inclination of ecliptic to equator} \\ \text{and} \quad \omega' &= \text{angle } L\gamma M \end{aligned}\right\} \quad (7)$$

Then we have

$$\cos z = \sin \theta' \sin (\omega + \omega'),$$

$$\cos \zeta = \cos \theta' \cos \chi + \sin \theta' \sin \chi \cos (\omega + \omega'),$$

and

$$\cos \theta' = \cos \theta \cos \delta,$$

$$\sin \theta' \sin \omega' = \sin \delta,$$

$$\sin \theta' \cos \omega' = \sin \theta \cos \delta;$$

whence

$$\left.\begin{aligned} \cos z &= \sin \omega \cos \delta \sin \theta + \cos \omega \sin \delta \\ \cos \zeta &= \cos \delta \cos \theta \cos \chi + (\cos \omega \cos \delta \sin \theta - \sin \omega \sin \delta) \sin \chi \end{aligned}\right\} \quad (8)$$

[p. 310]

It is, of course, possible to expand V_2, V_3, and V_4 in terms of θ and δ direct, but a good deal of analysis would be necessary ; in addition to losing the present simplicity of the analysis, no real saving of arithmetical work would be achieved. The arithmetical expansions of cos z and cos ζ are first obtained from the expressions of the longitude and latitude of the moon, and the rest of the work simply consists of carrying out systematically the operations involved in (5) and (6). The value of ω used in (8) is the value on January 1, 1900, viz., $\omega = 23°27'8''\cdot26$.

§ 3. *Choice of Variables for Arguments.*

The expansions for V_2 and V_3 necessarily involve six independent variables in the arguments, and considerable attention has been paid to the choice of these. For reasons which will be appreciated later, the independent variables adopted are defined as follows —

$$\left.\begin{aligned}
\tau &= \text{local mean lunar time reduced to angle}\\
s &= \text{moon's mean longitude}\\
h &= \text{sun's mean longitude}\\
p &= \text{longitude of moon's perigee}\\
N' &= -N\text{, where N is the longitude of the moon's ascending node}\\
p_1 &= \text{longitude of sun's perigee}
\end{aligned}\right\} \quad (9)$$

These are taken in preference to the variables ordinarily used in lunar theory.

Mean solar time will be taken as commencing at midnight, and, analogously, local mean lunar time will be measured from the lower transit of the « mean moon ». Then if we write

$$t = \text{Greenwich mean solar time}$$

we have $$\chi = 15^\circ t + h - 180^\circ - L \quad (10)$$

and $$\tau = \chi - s + 180^\circ = 15^\circ t + h - s - L$$

At first sight the choice of t rather than τ as an independent variable seems simpler, but there are many conveniences attached to the choice of the argument of the principal lunar constituent as one of the independent variables, both in the presentation of the schedules and in actual application.

The « speeds » of the variables are all positive, and, as they are written, are in descending order of magnitude. The chief variables are τ, s, and h, and it is a curious fact that if we classify in terms of τ, with a sub-classification with regard to s, and a further sub-classification with regard to h, the constituents are completely separated into groups with no over-lapping of speeds. It is still more [p. 311]
curious that to the order required the same process can be continued for all the variables. Owing to this, a rather elegant and very useful form of presentation of the result is possible.

§ 4. *Numerical Data for Arguments.*

The numerical data for the arguments is given by Brown, or may be obtained from the « Nautical Almanac », 1917 and 1923. The origin of time is taken as midnight at Greenwich on January 0-1, 1900:—

$$\begin{aligned}
\tau &= 15^\circ t + h - s - L,\\
s &= 277^\circ{\cdot}0248 + 481267^\circ{\cdot}8906T + 0^\circ{\cdot}0020T^2 + \ldots,\\
h &= 280^\circ{\cdot}1895 + 36000^\circ{\cdot}7689T + 0^\circ{\cdot}0003T^2 + \ldots,\\
p &= 334^\circ{\cdot}3853 + 4069^\circ{\cdot}0340T - 0^\circ{\cdot}0103T^2 + \ldots,\\
N' &= 100^\circ{\cdot}8432 + 1934^\circ{\cdot}1420T - 0^\circ{\cdot}0021T^2 + \ldots,\\
p_1 &= 281^\circ{\cdot}2209 + 1^\circ{\cdot}7192T + 0^\circ{\cdot}0005T^2 + \ldots,
\end{aligned}$$

where T is a Julian century of 36,525 mean solar days.

The speeds per mean solar day are as follows:—

$\dot{\tau} = 360^\circ - 12^\circ{\cdot}19074939,$ $\qquad \dot{p} = 0^\circ{\cdot}11140408,$

$\dot{s} = 13^\circ{\cdot}17639673,$ $\qquad \dot{N}' = 0^\circ{\cdot}05295392,$

$\dot{h} = 0^\circ{\cdot}98564734,$ $\qquad \dot{p}_1 = 0^\circ{\cdot}00004707.$

The speeds per mean solar hour are not very important, and are omitted.

No provision is made in this paper for the discussion of observations other than those referred to Greenwich mean solar time.

§ 5. *The Argument-Number.*

The actual calculations have been facilitated very considerably by the use of a special notation for the arguments, and this notation has been retained in the schedules. All the arguments are linear functions of the standard variables, with integral coefficients, and it is very desirable to have a short method of writing such expressions as

$$2\tau - 3s + 4h + p - 2N' + 2p_1.$$

Now the various coefficients involved in the expressions for the arguments are only occasionally outside the range —4 to 4, and this suggests the use of a datum of five for each so as to avoid writing negative values as much as possible. In the case of τ, however, the coefficients are always taken as positive, and with this exception, if we add five to each of the coefficients in the above expression we shall get the *argument-number*

229·637.

This number will serve to denote the argument and may also be used to denote the term as a whole. It is divided into two parts for reasons explained later. [p. 312]

In rare instances the coefficients are outside the range —4 to 4 and in these cases we replace —6 by 1, —5 by 0, 5 by X and 6 by E. The addition (or subtraction) of arguments is quite simple; allowance has to be made for the datum 055·555, which should be subtracted (or added) either before or after the operation — the former method is most convenient when dealing with the addition of one argument to each of a series of arguments.

§ 6. *Methods of Calculation.*

The original data for the longitude, latitude and sine parallax of the moon were obtained from the Tables by Brown, and are given in Tables I to III in the notation of § 3; the coefficients in the expansions of the longitude and latitude were reduced to radians, and the coefficients in the sine parallax expansion divided by the absolute term 3422″·70 in order to get c/r. In these expansions the coefficients are given to six decimal places.

The procedure is substantially that indicated by equations (8), (5) and (6). It may be noted that H_0 and H_2 can be calculated together, since

$$2 \cos^2 \zeta = H_2 + 1/2\ H_0 + 2/3,$$

In the case of the terms arising from V_3, the expansions of $(c/r)^4 \cos z$ and $(c/r)^4 \cos \zeta$ were determined and used as follows:—

$$(c/r)^4 H_0' = (4/3 + 5/2 H_0) \ .\ (c/r)^4 \cos z,$$

$$(c/r)^4 H_1' = (-2/3 + 5/2 H_0) \ .\ (c/r)^4 \cos \zeta,$$

$$(c/r)^4 H_2' = H_2 \ .\ (c/r)^4 \cos z,$$

and

$$(c/r)^4 H_3' = \text{ter-diurnal part of } 2H_2 \ .\ (c/r)^4 \cos \zeta.$$

The terms arising from V_4 were found in a similar manner; $4 \cos^4 z$ was obtained from $(2/3 - H_0)^2$ and hence H_0'' was readily calculated; also we have

$$H_1'' = (2/3 + 7/2 H_0) \ .\ H_1,$$

$$H_2'' = (-4/3 + 7/2 H_0) \ .\ H_2,$$

$$H_3'' = \text{ter-diurnal part of } H_1 H_2,$$

$$H_4'' = \text{quarter-diurnal part of } 1/2\ H_2^2.$$

The terms resulting from V_4, however, were, except for one term, just too small to be incorporated in the schedules.

The order of variables adopted in § 3 was not altogether the best for actual calculations and certain modifications were made. Brown's arguments for $(\theta - s)$, δ and c/r are given in the form

$$a\,(s - h) + b\,(s - p) + c\,(s - N) + d\,(h - p_1),$$ [p. 313]

and if the variables be changed to s, h, p, N and p_1, this expression takes the form

$$As + Bh + Cp + DN + Ep_1$$

with the relation $A + B + C + D + E = 0$. If the datum 5 be used then the sum of the figures in each argument-number is 25 (*).

Knowing this, it was possible to omit systematically one figure of the argument-number, and so to save a considerable amount of writing. In the case of $\cos z$ and $\cos \zeta$, however, this relation was not the same for all terms, but they separated themselves into sets of which the characteristic was that the sum of the figures of an argument-number was constant within the set.

* In the Tables, however, it should be noted that the variable there used is — N, and not N, so that the relation just mentioned does not hold.

By rearranging the order of the variables the terms of a set were separated into large groups in which the only effective variables were s and h, if p be ignored as mentioned above. The advantage gained by grouping was enormous because of the amount of writing thereby eliminated, and in fact the calculations were greatly facilitated by these methods of grouping.

The actual multiplication of series was quite an easy matter, and very efficient current checks were available; the greatest trouble was in connection with the collection of coefficients contributing to a term in the expansion, and this part of the work was always done twice. The author acknowledges with thanks the great assistance he has received from the staff of the Tidal Institute in this laborious arithmetical work.

Certain methods of checking were used which may be illustrated from Table III. Consider the calculation of $(c/r)^3$ from (c/r); if we suppose that all the variables are made zero except s then each of these expansions reduces to seven group-terms: all the coefficients of terms whose argument-numbers start with six would be added together, and so on. Taking the abbreviated expansion of (c/r) and cubing it should give an expansion for the abbreviated value of $(c/r)^3$ whose terms should be equal to the group-terms obtained from the full expansion. This method of checking, with appropriate modifications, has been used with the groupings explained above, and it has been very efficient indeed. The difference between any two such group-terms obtained by the two methods has always been less than 0·000050; usually it has been much less than this. In the one case where the difference reached 0·000050 no error could be found, but as coefficients less than 0·000005 were ignored in this case the probability of serious error in any one term is not great. The coefficients in the final schedules are reduced to five decimals and terms with coefficients less than 0.00010 have been ignored; the figures given may be taken as accurate to within two in the last place.

The expansions of $(c/r)^3$, $(c/r)^4$, cos z and cos ζ are contained in the following Tables :—

Table I.—Expansion for the Longitude of the Moon: $(\theta - s)$.
Coefficients of sines to six decimals.

[p. 314]

Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.
55·654	5	555	−605	74·356	47	85·255	175
753	1	654	−138	455	−41	475	−219
775	6	65·356	−2	554	−119	86·254	−3
56·356	−12	455	0·109760	75·355	3728	9Ī·755	3
455	90	554	87	454	6	90·556	9
554	−3243	653	−12	575	−1996	91·555	67
576	7	675	−192	76·354	−37	92·356	6
57·355	−1026	66·355	9	574	2	554	−1
553	−37	454	−532	77·155	−5	576	−2
575	−267	575	3	375	3	93·355	70
58·354	−42	67·255	−64	8Ī·657	2	575	−28
574	−11	453	−6	855	1	94·354	−1
59·353	−1	68·254	−2	80·656	21	95·155	9
60·658	1	70·558	2	81·457	4	375	−19
61·635	1	756	13	655	186	595	2
657	36	71·557	40	82·456	71	XĪ·655	2
855	6	656	−1	555	2	X0·456	1
62·436	−2	755	149	654	−3	X1·455	10
535	−1	72·556	802	676	−2	675	−1
656	1000	655	−16	83·455	931	X3·255	5
755	−6	754	−2	675	−46	475	−5
63·435	−31	73·335	−2	84·256	3	X5·275	−2
457	13	357	1	355	−3	E1·355	1
556	−3	555	0·011490	454	−14		
655	0·022236	654	1	476	−1		
64·456	717	775	−3	575	1		

Table II.—Expansion for the Latitude of the Moon: (δ).
Coefficients of sines to six decimals.

Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.
55·566	−4	544	−59	466	33	84·366	4
665	−4847	566	24	565	−26	465	−3
56·444	−25	65·345	154	664	−6	564	−6
466	4	565	0·089504	75·245	8	85·365	300
565	23	66·344	−2	465	4897	585	−31
664	−27	465	2	564	4	86·364	−3
57·245	−1	564	−31	685	−14	90·666	2
465	−808	67·365	−75	76·464	−26	91·445	2
685	1	585	−11	77·265	−7	665	15
58·464	−36	68·364	−3	485	−2	92·466	6
59·463	−1	70·646	3	80·546	2	93·465	73
61·547	5	71·645	32	766	1	685	−1
745	3	667	2	81·545	18	94·464	−1
62·546	144	72·446	9	567	2	95·265	19
645	−2	545	−2	765	12	485	−5
63·545	3024	666	43	82·566	39	X1·565	6
765	−8	73·445	162	665	−1	X3·365	7
64·346	1	665	967	83·345	10	X5·165	1
445	−3	74·444	−4	565	568	E1·464	1

Table III.—Expansions for (c/r), $(c/r)^3$ and $(c/r)^4$. (Lunar.)

Coefficients of cosines to six decimals, except for $(c/r)^4$, where the coefficients are given to four decimals. [p. 315]

Argument-number.	c/r coefficient.	$(c/r)^3$ coefficient.	$(c/r)^4$ coefficient.	Argument-number.	c/r coefficient.	$(c/r)^3$ coefficient.	$(c/r)^4$ coefficient.
55·555	1·000000	1·004736	1·0095	74·356	37	167	3
654		7		455	−32	−143	−2
775		−34	−1	554	−88	−287	−4
56·455	4	−42	−1	75·355	2970	13442	210
554	−117	−318	−4	454	5	22	
57·355	−89	1463	32	575	−4	−48	−1
553	−3	−7		76·354	−30	−135	−2
575	−31	−105	−2	77·155	−4	−11	
58·354	−6	47	1	375	−3	−13	
574	−2	−6		$8\bar{1}$·855	1	6	
61·657	14	46	1	80·656	20	91	1
855	2	26	1	81·457	3	13	
62·656	422	1348	18	655	176	814	13
755	−3	−11		82·456	67	308	5
63·435	−14	−52	−1	555		−10	
457	6	19		654	−3	−14	
655	0·010025	31475	430	83·455	902	4189	66
64·456	337	1014	14	675	−3	−15	
555	−285	−866	−12	84·256	4	21	
654	−66	−208	−3	355	−3	−16	
65·455	0·054501	0·164395	2201	454	−14	−65	−1
554	44	133	2	85·255	182	1076	19
653	−6	−18		475		−4	
675	−209	−629	−8	86·254	−3	−16	
66·355	5	13		$9\bar{1}$·755	3	17	
454	−278	−846	−11	90·556	10	50	1
575	2	6		91·555	76	396	6
67·255	−35	−2	1	92·356	6	36	1
453	−3	−9		554	−1	−5	
475	−24	−77	−1	93·355	83	496	9
68·254	−1	−1		94·354	−1	−8	
70·756	9	43	1	95·155	12	85	2
71·557	27	83	1	$X\bar{1}$·655	3	15	
755	109	513	8	X0·456	2	11	
72·556	561	1771	24	X1·455	13	79	1
655	−11	−41	−1	X2·256		3	
73·335	−1	−6		X3·255	7	52	1
555	8249	26580	367	X5·055		4	
775	−4	−18		E1·355	1	9	

[p. 316]

Table IV.—Expansion for cos z. (Lunar.)

Coefficients of sines to six decimals.

Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.
55·556	−17	566	22	675	−10	575	−5
566	−4	655	−10	74·444	−4	775	−2
653	−2	754	2	456	175	84·356	19
655	−21825	65·345	141	466	30	366	4
665	−4442	355	143	555	−119	455	−15
675	−39	456	−2	565	−24	465	−3
56·444	−23	555	0·395818	654	−33	554	−31
454	−70	565	82032	664	−6	564	−6
456	21	575	−795	75·245	7	85·355	1329
466	4	775	6	255	6	365	275
555	120	66·344	−2	455	21684	375	−3
565	21	354	−2	465	4488	454	2
654	−108	356	−3	475	−44	575	−5
664	−25	455	26	554	17	86·354	−16
57·455	−4518	465	2	564	4	364	−3
465	−741	554	−647	653	−2	87·155	−4
653	−2	564	−28	675	−82	90·656	8
675	12	576	2	685	−17	666	2
58·454	−198	67·355	−451	76·355	4	91·445	2
464	−33	365	−69	454	−142	655	64
59·453	−7	553	−7	464	−24	665	14
463	−5	575	−78	77·255	−39	92·456	25
61·557	7	585	−13	265	−6	466	6
745	3	68·354	−17	463	−2	93·455	326
755	5	364	−3	475	−13	465	67
62·536	2	574	−3	485	−3	675	−2
546	132	70·646	3	78·254	−2	94·256	2
556	142	71·645	29	80·546	2	355	−2
645	−2	655	12	756	5	454	−6
655	−2	657	7	81·545	17	95·255	86
63·535	32	667	2	555	4	265	17
545	2769	72·446	8	557	9	96·254	−2
555	2020	456	4	567	2	X0·556	3
557	8	545	−2	755	52	X1·555	26
755	−40	656	197	765	11	565	6
765	−7	666	39	82·556	176	X2·356	3
64·356	2	73·435	−5	566	36	X3·355	33
445	−3	445	149	655	−5	365	6
455	−2	455	57	83·345	9	X5·155	6
544	−54	457	3	355	3	E1·455	4
554	−17	655	4282	555	2528	E3·255	3
556	637	665	885	565	524		

[p. 317]

Table V.—Expansion for $\cos\zeta$. (Lunar.)
Coefficients of cosines to six decimals.

Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.	Argument-number.	Coefficient.
107·855	7	655	52249	755	3	564	−6
109·655	11	146·456	−81	163·435	28	177·355	−47
115·955	13	545	5	457	−6	365	−15
117·755	79	555	−288	645	161	575	−8
118·754	7	644	−7	655	−10886	585	−3
119·555	63	654	422	164·446	5	180·656	−3
11X·554	8	147·445	−192	456	−261	181·645	6
11E·355	3	455	10318	545	−5	655	−29
124·856	−5	554	−3	555	290	182·456	−11
125·845	−4	653	8	654	50	555	−3
855	208	655	−6	656	7	656	20
126·656	−13	665	−32	666	5	666	9
755	−5	675	−11	165·435	4	183·445	32
854	5	148·355	−3	445	963	455	−138
127·435	−4	444	−9	455	−52590	655	444
645	−15	454	476	554	−42	665	192
655	786	149·453	17	653	6	675	23
128·456	−3	465	−6	655	−2264	184·456	18
654	60	152·536	−8	665	−963	466	7
129·445	−3	756	−40	675	94	555	−12
455	155	153·525	3	166·444	−5	565	−5
653	3	535	−187	454	168	654	−3
12X·454	18	557	−17	456	2	185·255	−14
133·955	−9	745	15	555	12	455	2250
134·756	−38	755	−1089	565	5	465	973
135·535	−12	154·534	6	654	−11	475	106
656	5	546	6	664	−5	675	−9
745	−60	556	−1559	167·455	−469	685	−4
755	3202	655	62	465	−161	186·454	−15
136·556	−75	754	−8	168·454	−21	464	−5
655	−35	155·335	13	464	−7	187·255	−4
754	46	535	83	171·557	−17	191·545	4
137·335	−4	545	−17788	755	−12	555	−10
545	−114	555	0·953747	172·546	29	755	5
555	6091	654	−5	556	−343	765	2
138·455	−12	755	−15	655	4	192·556	18
544	−8	765	−31	173·535	3	566	8
554	424	775	3	545	600	193·355	−8
139·345	−2	156·356	6	555	−4868	555	262
355	127	455	−24	755	−4	565	114
553	21	544	−5	174·356	−4	575	12
565	−4	554	1536	455	5	194·554	−3
13X·354	11	566	12	544	−12	195·355	138
142·856	−4	157·355	−96	554	42	365	60
143·635	−32	553	19	556	66	375	7
657	−4	555	−210	566	5	1X1·655	7
855	−94	565	−600	576	−3	665	3
144·646	5	575	77	175·345	31	1X2·456	3
656	−341	158·554	−15	355	−345	1X3·455	34
755	10	564	−29	555	41067	465	15
145·425	4	574	4	565	17788	1X5·255	9
435	−197	161·635	3	575	1916	265	4
457	−6	657	−17	176·354	4	1E1·555	3
556	42	855	3	455	3	1E3·355	3
635	5	162·646	7	554	−67		
645	−973	656	−477				

§ 7. *Development of the Solar Tide-generating Potential*

The tide-generating potential due to the sun is developed by methods similar to those already used, but the whole problem is much simpler in this case. The expansions for the longitude and sine-parallax of the sun contain very few terms, and the sun's latitude may be ignored.

Using subscripts to denote quantities corresponding to those used for the lunar potential, we have

$$\left.\begin{aligned} \theta_1 &= h + 0{\cdot}033501 \sin(h - p_1) + 0{\cdot}000351 \sin 2(h - p_1) \\ &\qquad + 0{\cdot}000005 \sin 3(h - p_1) + \ldots \\ c_1/r_1 &= 1 + 0{\cdot}016750 \cos(h - p_1) + 0{\cdot}000281 \cos 2(h - p_1) \\ &\qquad + 0{\cdot}000005 \cos 3(h - p_1) + \ldots \end{aligned}\right\} \qquad (11)$$

which may be obtained from the formulae of elliptic motion, with eccentricity 0·0167504.

The geodetic coefficients are the same as for the lunar potential, except that G is replaced by

$$G_1 = \frac{3}{4}\,\frac{S}{E}\,\frac{ga^2\rho^2}{c_1^3} = \frac{S}{M}\,\frac{c^3}{c_1^3}\,G, \qquad (12)$$

where S, M, E are respectively the masses of the sun, moon and earth. It is desirable, however, to retain G as the common coefficient of both lunar and solar constituents, and to absorb the factor $(S/c_1^3) \div (M/c^3)$ in the numerical coefficients of the solar constituents. Now, $1/c$ and $1/c_1$ are proportional to the sines of the mean equatorial horizontal parallaxes of the moon and sun, respectively; the former is accurately known, and its value is 3422″·70; also, it is definitely known and well established* that S/E multiplied by the cube of the mean equatorial horizontal parallax of the sun is equal to $2''{\cdot}26428 \times 10^8$. The mass of the moon is not very accurately known, but the best value** is apparently given by $E/M = 81{\cdot}53 \pm 0{\cdot}047$. We therefore obtain $G_1 = 0{\cdot}46040\ G$, and this numerical coefficient has been used.

The terms arising from the solar potential are given in the same schedules as those arising from the lunar potential, and are distinguished by inserting the appropriate geodetic coefficient for the solar terms, and leaving it as understood for the lunar terms.

§ 8. *Explanation of the Schedules.*

The harmonic terms in the development of the potential are contained in four schedules, numbered 0 to 3. The number of the schedule denotes the species; for example, Schedules 1 and 2 contain respectively all the diurnal and semi-diurnal terms, whatever be their source — lunar or solar, V_2 or V_3. Each term has a [p. 319] numerical coefficient and a geodetic coefficient, and the source of the term is indicated by the latter; terms arising from the lunar V_3 have geodetic coefficients G_0', G_1', G_2' or G_3', the suffix indicating the species. Terms arising from the

* Ball, « Spherical Astronomy », pp. 309 and 310.
** « Monthly Notices, Roy. Astron. Soc. », February, 1911.

solar V_2 have geodetic coefficients G_0, G_1 or G_2, and so have the terms from the lunar V_2: to distinguish between the two sources, the geodetic coefficient is not written in the schedules for the lunar V_2 terms, and at the top of each schedule is a note stating the geodetic coefficient that is supposed to be understood. The values of the geodetic coefficients, considered as functions of the latitude, are placed at the top of the schedule.

With each geodetic coefficient is associated either a sine or a cosine of the given argument; the necessary information is given at the top of the schedule; as examples, from Schedule 1 we have the terms

$$0{\cdot}37689\ G_1 \sin \{145{\cdot}555\}, \quad -0{\cdot}16817\ G_1 \sin \{165{\cdot}555\}$$

and

$$-0{\cdot}00108\ G_1' \cos \{145{\cdot}655\},$$

and from Schedule 2 we have terms

$$0{\cdot}90812\ G_2 \cos \{255{\cdot}555\}, \quad 0{\cdot}42286\ G_2 \cos \{273{\cdot}555\}$$

and

$$0{\cdot}000525\ G_2' \sin \{265{\cdot}555\},$$

curl brackets being used to denote the argument corresponding to the argument-number.

The numerical coefficients are given to five decimal places, and all terms with numerical coefficients less than 0·00010 are ignored. The latter rule just cuts out all the V_4 terms except one, a quarter-diurnal term, which has a coefficient of 0·00016, and this also is ignored.

As was mentioned in § 3, the terms, when arranged according to the argument-number, are thereby automatically arranged according to speed, as may be easily verified from the speeds given in § 4. The full argument-number will be used to denote the associated term.

Considered from the point of view of analysis of tidal observations, and assuming that the terms in the tide correspond to the terms in the potential, it is possible, using one year's observations only, to distinguish between terms whose arguments differ by multiples of h, but not between terms whose arguments differ by multiples of p, N' or p_1. If, therefore, we have several terms whose arguments are the same so far as τ, s and h are concerned, these must necessarily be regarded as one *constituent*.* The word « constituent » will be applied and restricted to a set of terms wholly inseparable within a year; we can therefore appropriately speak [p. 320] of the first three figures of the argument-number as the « constituent-number », and it is convenient to apply the term « group-number » to the first two figures of the argument-number; as an example, take the following:—

argument-number: 265·555,
constituent-number: 265,
group-number: 26,
species-number: 2, i.e., semi-diurnal.

---

* It is possible to infer the separate terms of a constituent if it be assumed that the harmonic terms in the tide have to one another the relations of corresponding terms in the potential, but this is a matter which hardly concerns us at the moment: the point is, that in actual analysis of records not exceeding one year, terms such as the above must be regarded as one.

It is impracticable, even were it desirable, to invent for each constituent a symbol corresponding to the symbols already in use; the constituent-number is much more serviceable than an initial can be, and it certainly conveys a great deal more. Certain symbols such as M_2, S_2, K_1 and O_1 are, however, well established, and may still be used. A list of Darwin's symbols, together with the corresponding constituent-number, is given later (Table VI). It should be remarked, however, that each tidal-constituent given by Darwin only includes a particular set of terms from the whole of the contributory terms to our constituents. Since, however, it is the same constituent that is dealt with, though his expression for it is not complete, there is no objection to regarding his symbols and our numbers as equivalent.

Two diagrams are given to illustrate the schedules. The first, Diagram A,

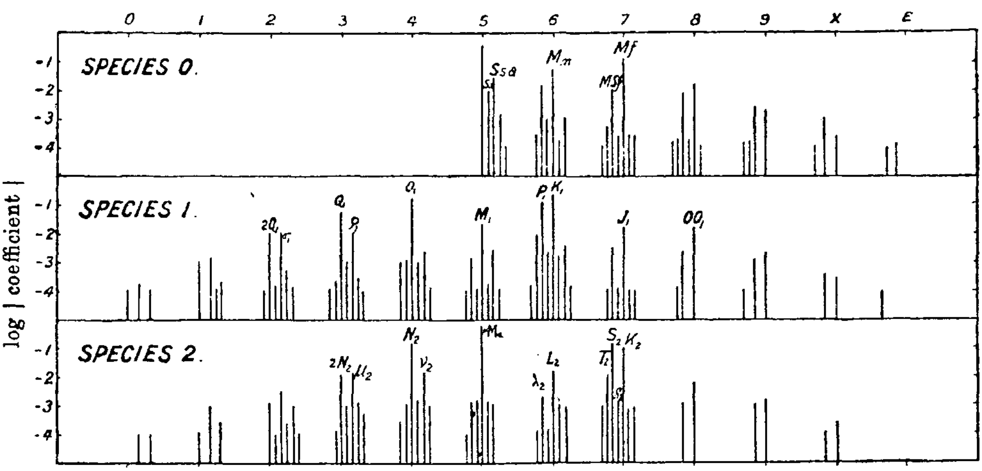

DIAGRAM A : constituents, separable in one year.

gives for the first three species a representation of the constituents, with speed in abscissa and the logarithm of the absolute value of the coefficient as ordinate, assuming that the contributory terms are additive regardless of sign. The logarithmic scale gives, to some extent, a false idea of the magnitude of the constituent, but the scale is convenient for representation; further, the difficulties of analysis are not caused by the large constituents, but by the small ones, and, from this point of view, the logarithmic scale does correspond roughly to the difficulties experienced [p. 321] by the harmonic analyses.

The speed scale is indicated by the figures at the top of the diagram; these, with the species-number, give the group-numbers, and the places of the constituents in the diagram can then be readily found. An increment of 1 in the group-number coresponds to an increase in speed of about 13° per mean solar day; the increase in speed for an increase of 1 in the constituent-number is about 1° per mean solar day.

Diagram B gives, on a more open scale than Diagram A, a representation of constituents separable in a period of about nineteen years. Terms whose

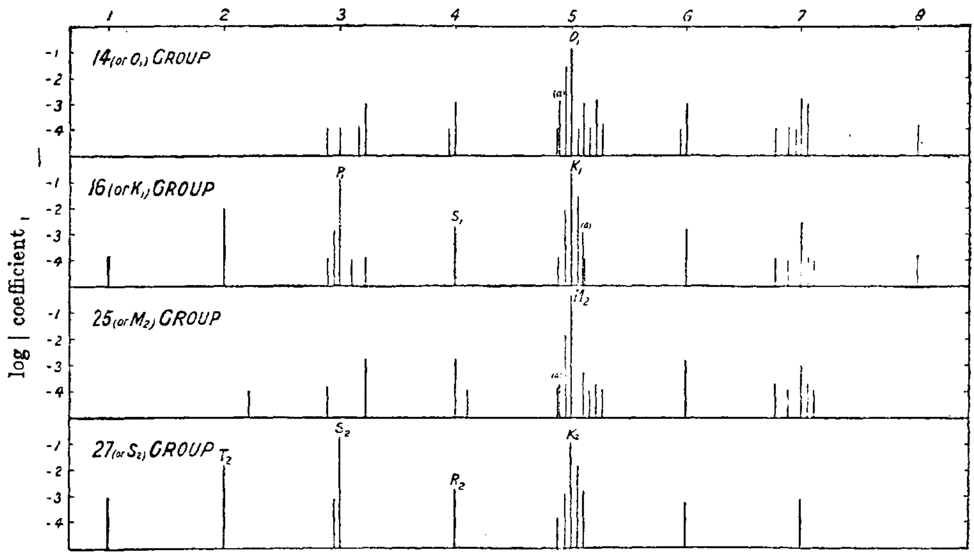

DIAGRAM B: terms separable in about nineteen years. (a): two distinct terms not separable from one another (see § 7).

arguments differ only in p_1 are regarded as one; in a few cases two terms may have nearly the same speed, though their arguments differ in p and N', as in the case of the two terms 145·455 and 145·535. The speed of the latter term is greater than that of the former by $\dot{p} - 2\dot{N}'$, which is comparatively small; these terms are marked by (a) on the diagram. Only four groups are illustrated.

The speed scale is indicated by the figures at the top of the diagram; these, with the group-number, give the constituent-number, and an increment of unity in constituent-number corresponds to an increment in speed of nearly 1° per mean solar day.

SCHEDULE O.

$G_0 = \frac{1}{2}G\,(1-3\sin^2\lambda)$, *associated with coefficients of cosines to five decimals.*

$G_0' = 1{\cdot}11803\ G\sin\lambda\ (3-5\sin^2\lambda)$, *associated with coefficients of sines to five decimals.* [p. 322]

(*When no geodetic coefficient is entered* G_0 *is understood.*)

Argument-number.	Coefficient.	
05 (or Ssa) group.		
055 ·555	50458	
555	23411	G_0
565	−6552	
575	64	
655	26	G_0'
056 ·554	−16	
554	1176	G_0
556	−61	G_0
057 ·355	73	
553	30	G_0
555	12	
555	7287	G_0
565	−181	
575	−40	
058 ·554	427	G_0
059 ·553	17	G_0
06 (or Mm) group.		
062 ·656	68	
063 ·445	−16	
645	−113	
655	1578	
665	−103	
064 ·456	51	
555	−44	
654	−10	
065 ·445	−542	
455	8254	
465	−535	
545	−24	G_0'
555	466	G_0'
565	73	G_0'
655	−442	
665	−179	
675	−47	
066 ·454	−43	
067 ·455	−116	
465	−58	

Argument-number.	Coefficient.	
07 (or Mf) group.		
071 ·755	26	
072 ·556	91	
073 ·545	98	
555	1370	
565	−88	
655	15	G_0'
074 ·554	−17	
556	48	
566	12	
075 ·345	−36	
355	677	
365	−44	
455	76	G_0'
465	12	G_0'
555	15642	
565	6481	
575	607	
585	−13	
076 ·554	−54	
564	−14	
077 ·355	−47	
365	−19	
08 group.		
081 ·655	42	
082 ·456	16	
656	26	
666	11	
083 ·445	22	
455	217	
465	−14	
555	13	G_0'
655	569	
665	236	
675	21	
084 ·456	28	
466	10	
555	−16	

Argument-number.	Coefficient.	
08 group—*contd.*		
085 ·255	54	
455	2995	
465	1241	
475	117	
555	38	G_0'
565	24	G_0'
675	−12	
086 ·454	−26	
09 group.		
091 ·555	20	
755	14	
092 ·556	32	
566	13	
093 ·355	25	
555	478	
565	200	
575	19	
095 ·355	396	
365	165	
375	16	
455	11	G_0'
0X group.		
0X1 ·655	23	
0X3 ·455	116	
465	48	
0X5 ·255	45	
265	19	
0E group.		
0E1 ·555	12	
0E3 ·355	19	

SCHEDULE 1.

$G_1 = G \sin 2\lambda$, *associated with coefficients of sines to five decimals.*

$G_1' = 0{\cdot}72618 G \cos\lambda\,(1-5\sin^2\lambda)$, *associated with coefficients of cosines to five decimals.*

(*When no geodetic coefficient is entered* G_1 *is understood.*) [p. 323]

Argument-number.	Coefficient.	
10 group.		
105·955	11	
107·755	46	
109·555	28	
11 group.		
115·755	−10	G_1'
845	21	
855	108	
117·555	−10	G_1'
645	53	
655	278	
118·654	21	
119·445	10	
455	54	
12 group.		
124·756	−13	
125·645	−23	G_1'
655	−58	G_1'
745	180	
755	955	
126·556	−16	
655	−11	
754	15	
127·455	−11	G_1'
545	218	
555	1153	
128·544	14	
554	79	
129·355	35	
13 (or Q_1) group.		
133·855	−23	
134·656	−61	

Argument-number.	Coefficient.	
13 (or Q_1) group—*contd.*		
135·435	−28	
545	−84	G_1'
555	−211	G_1'
635	−42	
645	1360	
655	7216	
755	−13	G_1'
855	−19	
136·456	−13	
555	−39	
644	11	
654	68	
137·445	258	
455	1371	
555	−18	G_1'
655	−78	
665	24	
138·444	11	
454	64	
139·455	−14	
14 (or O_1) group.		
143·535	−17	
·745	−20	
755	−113	
144·546	−15	
556	−130	
145·455	12	G_1'
535	−218	
545	7105	
555	37689	
645	16	G_1'
655	−108	G_1'
665	14	G_1'
755	−243	
765	−40	
146·544	12	
554	115	
147·355	−21	
455	−21	G_1'
545	14	
555	−491	
565	107	
148·554	−33	

Argument-number.	Coefficient.	
15 (or M_1) group.		
152·656	−14	
153·645	−63	
655	−278	
154·656	15	
155·435	17	
445	−197	
455	−1065	
545	98	G_1'
555	−661	G_1'
565	86	G_1'
645	85	
655	−2964	
665	−594	
675	17	
156·555	16	
654	−18	
157·445	16	
455	−566	
465	−124	
158·454	−24	
16 (or K_1) group.		
161·557	42	G_1
162·556	1029	G_1
163·535	14	
545	−199	
555	30	
555	17554	G_1
557	−11	G_1
755	−26	
164·554	−147	G_1
556	−423	G_1
165·455	−36	G_1'
545	1050	
555	− ·16817	G_1
555	− ·36233	
565	−7182	
575	154	
655	−13	G_1'
166·554	−423	G_1
167·355	−26	
553	−11	G_1
555	−756	G_1
565	29	
575	14	
168·554	−44	G_1

[p. 324]

SCHEDULE 1—*continued.*

Argument-number.	Coefficient.	
17 (or J_1) group.		
172·656	−24	
173·445	−17	
645	18	
655	−566	
665	−112	
765	−89	G_1'
174·456	−18	
555	16	
175·445	·87	
455	−2964	
465	−587	
475	13	
555	−241	G_1'
655	46	
665	29	
675	17	
176·454	15	
177·455	12	
18 (or OO_1) group.		
182·556	−32	
183·545	−16	
555	−492	
565	−96	
185·355	−240	
365	−48	
455	−40	G_1'
465	−16	G_1'
555	−1623	
565	−1039	
575	−218	
585	−14	
19 group.		
191·655	−15	
193·455	−78	
465	−15	
655	−59	
665	−38	
19 group—*contd.*		
195·255	−19	
455	−311	
465	−199	
475	−42	
1X group.		
1X3·555	−50	
565	−32	
1X5·355	−41	
365	−27	
1E group.		
1E3·455	−12	

SCHEDULE 2.

$G_2 = G\cos^2\lambda$, *associated with coefficients of cosines to five decimals.*
$G_2' = 2{\cdot}59808\, G \sin\lambda \cos^2\lambda$, *associated with coefficients of sines to five decimals.*
(*When no geodetic coefficient is entered* G_2 *is understood.*)

Argument-number.	Coefficient.	
20 group.		
207·855	15	
209·655	18	
21 group.		
215·955	27	
217·755	111	
219·555	69	
22 group.		
225·755	−27	G_2'
855	259	
226·656	−12	
227·555	−27	G_2'
645	−25	
655	671	
22 group—*contd.*		
228·654	54	
229·455	130	
22X·454	15	
23 (or $2N_2$) group.		
234·756	−31	
235·535	−14	
645	−27	G_2'
655	−156	G_2'
745	−86	
755	2301	
236·556	−40	
655	−25	
754	36	
237·455	−29	G_2'
545	−104	
555	2777	
238·554	189	
239·355	85	
24 (or N_2) group.		
243·635	−15	
855	−56	
244·656	−147	
245·435	−63	
545	−97	G_2'
555	−569	G_2'
556	14	
645	−648	
655	17387	
755	11	G_2'
246·456	−33	
555	−94	
654	163	
247·445	−123	
455	3303	
555	15	G_2'
655	17	
665	−12	
248·454	153	

[p. 325]

SCHEDULE 2—*continued.*

Argument-number.	Coefficient.	
25 (or M_2) group.		
252·756	−11	
253·535	−40	
755	−273	
254·556	−314	
655	14	
255·455	32	G_2'
535	47	
545	−3386	
555	·90812	
655	86	G_2'
665	16	G_2'
755	53	
765	19	
256·554	276	
257·355	−52	
455	17	G_2'
555	107	
565	−51	
575	18	
26 (or L_2) group.		
262·656	−33	
263·645	24	
655	−670	
264·456	−10	
555	17	

Argument-number.	Coefficient.	
26 (or L_2) group—*contd.*		
265·445	95	
455	−2567	
545	−31	G_2'
555	525	G_2'
565	99	G_2'
645	−12	
655	643	
665	283	
675	40	
267·455	123	
465	59	
27 (or S_2) group.		
271·557	101	G_2
272·556	2479	G_2
273·545	94	
555	·42286	G_2
555	72	
274·554	−354	G_2
556	92	G_2
275·455	29	G_2'
545	−147	
555	7858	
555	3648	G_2
565	3423	
575	372	
276·554	92	G_2
277·555	78	G_2

Argument-number.	Coefficient.	
28 group.		
283·655	123	
665	54	
285·445	−12	
455	643	
465	280	
475	30	
555	48	G_2'
565	31	G_2'
29 group.		
293·555	107	
565	46	
295·355	53	
365	23	
555	168	
565	146	
575	47	
2X group.		
2X3·455	17	
2X5·455	32	
465	28	

SCHEDULE 3.

$G_3' = G\cos^3\lambda$, *associated with coefficients of cosines to five decimals.*

Argument-number.	Coefficient.	
32 group.		
327·655	−17	G_3'
33 group.		
335·755	−56	G_3'
337·555	−57	G_3'

Argument-number.	Coefficient.	
34 group.		
345·645	18	G_3'
655	−326	G_3'
347·455	−61	G_3'
35 group.		
355·545	66	G_3'
555	−1188	G_3'

Argument-number.	Coefficient.	
36 group.		
363·655	17	G_3'
365·455	67	G_3'
655	−25	G_3'
665	−11	G_3'
37 group.		
375·555	−155	G_3'
565	−68	G_3'

§ 9. *Comparison with Darwin's Results.*

Darwin's schedules are not directly comparable with those now given, as his expansion is not purely harmonic. The constituents he gives are of the general form J cos $(\sigma t + u)$, where σ is the appropriate speed, J is a function of the inclination of the moon's orbit to the equator, and u depends upon the position of the intersection of the equator and orbit. [p. 326]

Darwin's practice is to replace J and u by their mean values within the interval of time considered, and each set of observations is treated with different values of J and u. His theoretical « mean coefficient » is the mean value of J cos u over a period of about nineteen years, the period of revolution of the node. He shows that J cos $(\sigma t + u)$ can be expanded in the form

$$\Sigma J_r \cos(\sigma t + r\Omega),$$

where Ω is the longitude of the node; J_r is not quite constant, but partly depends upon the longitude of the node and upon the inclination of the orbit to the ecliptic. Darwin's mean coefficient is taken as equal to Jo in the above expansion. (This is not the mean value of J, however, which is somewhat larger than Jo: his theory and practice are not quite in conformity in this respect.)

Further expansion by the above method would be very difficult, but it can be shown that one of Darwin's constituent would yield ultimately a set of terms whose arguments would be identical, but for the part dependent on N'. On looking through the new schedules, such sets of terms can be readily picked out; the greatest numerical coefficient in each set should be very nearly equal to Jo, or Darwin's mean coefficient. It will be noticed, however, that in some cases several such sets of terms may be contributory to a « constituent » as defined in § 8. In all cases only one coefficient, the greatest, is extracted to represent each set, and in Table VI those terms (or representative terms) with coefficients greater than 0.00400 are set forth for comparison with Darwin's results. The constituent-number only is given to represent the argument. In those cases where Darwin has compounded two terms to form one constituent the comparison is made separately; the compounded terms are bracketed. In the case of M_1 three terms are given, of which two are compounded by Darwin; the third term is the true M_1.

Generally speaking, there is fair agreement, except in the case of μ_2 and the true M_1; the cause of the latter discrepancy has been ascertained to be due to certain approximations made by Darwin in expanding V_3.

The constituents omitted by Darwin and indicated in Table VI are considered to be decidedly worthy of consideration; their combined effect is by no means negligible.

Table VI.—Comparison of New Expansion with Darwin's.

Name.	Number.	Coefficient.	Darwin's coefficient.	Per cent. difference.
	055	0·50458	0·50448	0·0 [p. 327]
Mf	075	0·15642	0·15654	0·1
Mf	075	677		
Mm	065	0·08254	0·08272	0·2
Mm	065	466		
Mm	065	−442		
Ssa	057	0·07287	0·07286	0·0
Ter-mensual	085	0·02995	0·03032	1·2
Evect. mthly.	063	0·01578	0·01510	4·3
Msf	073	0·01370	0·01242	9·3
Sa	056	0·01176		
	083	0·00569		
	093	0·00478		
	058	0·00427		
O_1	145	0·37689	0·37712	0·1
K_1	165	−0·36233	−0·36230 }	0·0
K_1	165	−0·16817	−0·16814 }	0·0
P_1	163	0·17554	0·17550	0·0
Q_1	135	0·07216	0·07302	1·2
M_1	155	−0·02964	−0·02970 }	0·2
M_1	155	−0·01065	−0·01044 }	2·0
M_1	155	−661	−0·00990	49·8
J_1	175	−0·02964	−0·02970	0·2
OO_1	185	−0·01623	−0·01624	0·0
ρ_1	137	0·01371	0·01416	3·3
σ_1	127	0·01153	0·00900	21·9
	162	0·01029		
$2Q_1$	125	0·00955	0·00974	2·0
	167	−0·00756		
	173	−0·00566		
	183	−0·00492		
	147	−0·00491		
S_1	164	−0·00423		
	166	−0·00423		
M_2	255	0·90812	0·90852	0·0
S_2	273	0·42286	0·42274	0·0
N_2	245	0·17387	0·17592	1·2
N_2	245	−569		
K_2	275	0·07858	0·07858 }	0·0
K_2	275	0·03648	0·03646 }	0·1
ν_2	247	0·03303	0·03412	0·3
μ_2	237	0·02777	0·02188	21·2
L_2	265	−0·02567	−0·02574 }	0·3
L_2	265	643	646 }	0·5
L_2	265	525		
T_2	272	0·02479	0·02486	0·3
2N	235	0·02301	0·02346	2·0
λ_2	263	−0·00670	−0·00660	1·5
	227	0·00671		
	285	0·00643		
M_3	355	−0·01188	−0·01198	0·8

The effect of taking mean values of J and u over a period of a year is readily investigated; the process is practically equivalent to taking a mean value of N′ (or N) in the set of terms obtained by expanding $J \cos(\sigma t + u)$.

Suppose that

$$J \cos (\sigma t+u) = J_0 \cos \sigma t + J_1 \cos (\sigma t + N) + \cdots;$$ [p. 328]

then, if bars denote mean values of functions of N, we have

$$J \cos (\sigma t+u) - \overline{J} \cos (\sigma t+\overline{u}) = J_1 \cos (\sigma t+N) - J_1 \cos (\sigma t+\overline{N})$$
$$= 2J_1 \sin 1/2 (\overline{N} - N) \sin (\sigma t + 1/2 N + 1/2 \overline{N}).$$

Therefore the effect is to leave a residual harmonic term with coefficient approximately equal to $J_1 \sin (\overline{N} - N)$; at the ends of the yearly period this has the approximate value of 1/6 J_1. Now the size of J_1 is not to be judged by the size of J_0, and large residues may be left by the smaller constituents. On looking through the schedules it will be found that there is a possibility of residuals of coefficients 0.011, 0.005, 0.005, ..., in the long-period constituents, 0.012, 0.012, ..., in diurnal constituents, and 0.006, ..., in semi-diurnal constituents. These residues are by no means negligible, especially when there are other constituents of this order which are not taken into account; the total effect of these may be important.

§ 10. *Considerations regarding Application to the Analysis and Prediction of Tides*

The application of the schedules to the analysis and prediction of tides requires mature consideration, though it has been borne in mind during their preparation. In Darwin's paper on the abacus, he gives a method of analysis of the solar constituents which may be applied more generally. Essentially he regards the constituents of the 27 (or S_2) group over a short interval of time as one constituent, and afterwards separates the various constituents T_2, S_2, R_2, K_2, ... by considering the variations in certain quantities derived by harmonic analysis. This method may be generalised with considerable advantage. Considering each constituent as effectively a function of τ, s and h only will simplify the application of such a method as this; it ought to simplify most methods.

If the variables p, N' and p_1 were absent, we should get constant coefficients for the constituents; actually their coefficients and arguments will vary very slowly, and it would probably be sufficient to tabulate for January 1 of each year the appropriate multiplying factor and change of phase; this would be a generalised form of $J \cos (\sigma t+u)$, as given by Darwin. But, for reasons already given, mean values over long periods are inadmissible; if, however, the multiplying factor and phase-shift be changed slowly but discontinuously at short intervals of time, the errors may be made negligible. Linear interpolation in J and u should suffice for this purpose. There seems to be no difficulty in doing this, either in analysis or in prediction. [p. 329]

Referring to the constituent 265, the chief term is 265-455, whose speed differs by p from the speed of 265.555; but there is no reason why the speed of the constituent 265 should be modified on this account, as any correction necessary would be automatically applied in using the variable coefficients and phases as indicated above.

To sum up, it is proposed—

(1) that the constituents be regarded as functions of τ, s, and h, with appropriate speeds;

(2) that analyses and predictions should be made with variable coefficients and phase corrections, automatically applied if possible, such coefficients and phase corrections being regarded as constants only over a sufficiently short interval of time.

The translation of these proposals into practical methods, however, is a matter for careful consideration.

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Corrections have been made as follows :
§ 7 : Coefficient of 2''.26428 is 10^8.
Schedule 1 : Coefficient of 165.455 is G'_1.
Schedule 2 : Coefficient of 277.555 is 78.
In addition, Table VI should include : 157 0.00566.
