PRECISE ASTRONOMIC POSITIONS FROM PROJECTED STAR POSITIONS ANALYTICALLY PROCESSED

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*Part II**

Position finding by the intersection of circles of equal altitudes of celestial bodies is a principle well-known to all marine and air navigators. As applied to navigation great accuracy is neither expected nor particularly desirable, the operating circumstances normally allowing tolerance of positional inaccuracies of the order of 1/4 to 1 or even miles.

So far as is known to this writer, no attempt has been made to apply the principle just mentioned to the determination of precise astronomic positions.

The resources of the equatorial stereographic projection and of the plane analytic geometry appear well suited to such application. Applying these resources we can in a *2-star* method stereographically project the equal altitude circles, write an equation for each of them, and then solve these equations simultaneously.

This solution yields the x, y coordinates of either one or two positions. If the circles are tangent (which will be the case if two stars were observed that differed 180° or 0° in azimuth) the solution yields the coordinates of only one point, — that of the point of mutual (external or internal) tangency. This will not normally be the case, so that, as in marine and air navigation, the equal-altitude circles will intersect in two positions, one of which is the projected position of the observer. There is thus some ambiguity regarding which of the two is the observer's position, but, again as in standard navigation practice, this ambiguity can usually be resolved easily.

Figure 1 shows two projected equal-altitude circles, and sets forth the projection formulas.

Ordinarily, the solution of two simultaneous equations of the second degree in x and y is an exacting task. This in particularly true when the equations are of the general from:

$$Ax^2 + Bxy \times Cy^2 + Dx + Ey + F = 0$$

which, of course, can represent a conic, a degenerate conic, a point, a line or lines, or no locus. In the case under discussion, however, it is known in advance that the equations are those of circles, and their simultaneous solution is accordingly greatly simplified.

If three stars instead of two stars are observed, new analytic processing possibilities become available. With this *3-star* method all ambiguity regarding the position of the observer, such as was present with the 2-star method, disappears, in that three circles can intersect or be tangent at only one point. Moreover, the observer's projected position is arrived at, not by solution of equations of the

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second degree in x and y, but of equations of the first degree. This represents an important gain in the computation. Equations of the first degree in x and y are those of straight lines, and in the case of intersecting or tangent equal-altitude circles, these lines are the *radical* axes of the circles taken in pairs. It will be recalled that the radical axis of two circles is the straight line passing through the two points (or point) in which the circles intersect (or are tangent). The equation of this line is arrived at quite simply by subtracting the equation of the one circle from that of the other.

Figure 2 shows a third circle added to the two circles of Figure 1. The three radical axes of these three circles are shown in dashed lines in Figure 2; the method of arriving at the equations of the axes is also illustrated.

The simultaneous solution of any two of the three radical axis equations such as might be gotten by the method of Figure 2 yields the x, y coordinates that should satisfy the third equation. In practice, however, this will probably not be the case, small personal errors being practically unavoidable in observation. The proper way to compensate for this is to consider the radical axis equations as a system of simultaneous linear equations, and to solve them by application of the principle of least squares.

A short dicussion of the method of least squares is given at the end of this article.

The radical axis and least squares solution principles can be applied to a *multi-star* method by simple extension of the 3-star method. The term multi-star predicates successive observations of a single star, or observation of a number of stars at about the same time. Considering the mathematical ridigity of the methods, the intervals of time elapsing between successive observations of a single star in the one case, or the differences in azimuths in the second case, are not particularly critical to the methods or to the final results. This is not true in marine and air navigation practice, where a good « cut » is very desirable by reason of the graphic plot involved.

Figure 3 is supplied as of interest and use in systematizing the computation work of the multi-star method. As noted before, the application of the principle of least squares to this method is included at the end of this article.

To examine the possibilities of arriving at a precise astronomic position by method of constant-azimuth star sights, let us begin by describing the observation procedure, and then recalling certain principles of the analytic geometry of space and of the gnomonic projection, rather than, as heretofore, of the stereographic projection.

The observation procedure can be described thus:

1. *Set up and accurately level any standard observing instrument.* Make no attempt to set the line of sight of the instrument in the meridian, — even if indeed this is possible. Clamp the instrument in azimuth, thus assuring a constant azimuth, thus assuring a constant azimuth during the observations.

2. Note the times when a number of identified stars transit the vertical cross-hair of the instrument. Do *not* measure the altitudes. Observe half of these stars with the telescope direct and the other half with the telescope reversed. The only restriction on the selection of stars is that at the instant of line-of-sight transit their altitudes (i.e., their declinations) must be different one from the others.

3. Unclamping the instrument in azimuth, swing it to some new azimuth. Make no attempt to measure the azimuth through which the instrument is *swung;* this azimuth does not enter into the computation.

4. Repeat the observation procedure of step 2.

The following evolve from the observation procedure:

The GHA's and declinations of the stars that transit the vertical cross-hair (at each setting of the instrument) are the spherical coordinates of points that lie in — and mathematically determine — the plane of the line of sight of the (azimuth-clamped) instrument.

The equation of each plane can be written, using the coordinates just mentioned, and the equations of the two planes taken together represent the equation of straight line in which the two planes intersect.

Each of the two planes passes through the center of the celestial sphere (each), as does the line in which the planes intersect. We can also note that inasmuch as the observer is in each of the two planes, he is on the line in which they intersect.

The direction cosines of the line of intersection can be gotten from the equations of the planes.These direction cosines lead directly to the latitude and longitude of the observer.

However, a detailed discussion of the equations of planes and of direction cosines properly belongs in the realm of the analytic geometry of space, whereas this article undertakes to discuss or consider the relation of the plane geometry to precise astronomic positions.

(Nevertheless, to diverge a bit, the determination of the equation of each of the two planes of the constant-azimuth technique is entirely similar to the determination of the equation of plane of the constant — altitude technique —, the latter being exemplified by the astrolabe method. Moreover, the observer's positions in both « constant » methods proceeds from the direction cosines of a line passing through the center of the celestial sphere. In the constant-azimuth method this line is the *normal* to the plane in which the stars lie (and which determine its equation). This fact prompted the writer to study the matter at some length, with the result that a computation method — not at all complicated — can be standardized and can be used with either the constant-azimuth or the constant-altitude procedures. The inference is that precise astronomic positions can be arrived at without measuring any altitudes or azimuths, and without concern with a pre-computation (assumed) position from which to work to the final position, and employing a more-or-less single computational procedure. Opinions as to the desirability of this possibility will no doubt vary; in any event, one *negative* advantage can be cited, namely, that if no altitudes or azimuths are measured the possibility of reading and/or recording such angles erroneously is completely eliminated! A description of the simplified and standardized computation method for the constant-azimuth and the constant-altitude techniques is planned for Part III of this series).

Reverting to the central theme of this article —, precise astronomic positions from projected star positions — let us in connection with the constant-azimuth technique consider a plane tangent at the north celestial pole, and then gnomonically project the various star positions gotten with the technique onto that plane. These projected star positions will, according to the principle of the gnomonic projection, lie along (and, in fact, determine) two intersecting straight lines, with the observer's projected position at the point where the two straight lines intersect. The equations of these lines can be written and the simultaneous solution of the two equations leads to the x, y coordinates of the observer's projected position. These coordinates lead by reverse projection directly to the observer's terrestrial latitude and longitude.

Figure 4 illustrates the projection, and sets forth the projection formulas. The application of the principle of least squares to arrive at the equation of each of the two lines is described at the end of this article.

Although the constant-azimuth technique can be used in any latitude, this writer has considered the method as somewhat more useful or applicable in high northern or sub-arctic latitudes, for the following reasons:

(a) The projection formulas are very simple. (This is, to be sure, also true in the case of the equatorial gnomonic projection, but it is definitely not true at a latitude intermediate between 0° and 90°).)

(b) The need for precise astronomic positions may be somewhat more apparent in high northern latitudes than in other latitudes, considering the extensive surveying and development likely in such latitudes in the future. Moreover, in areas where no maps or charts are available the problem, present in some standard methods of position determination, of selecting a position to use in the computation process is entirely avoided with the constant-azimuth method. It is this writer's belief that extensive areas in high northern latitudes have not been mapped or charted.

c) The rate of change in azimuth in high northern latitudes is much more sensitive for observation purposes than the rate of change of altitude. As an extreme case, the rate of change of the altitude of stars to an observer at the north pole is practically zero during a normal observational epoch.

Thus, Figure 4 and the accompanying projection formulas apply to the north polar gnomonic projection.

It remains, now to discuss and illustrate the application of the principle of least squares to the computation problems posed in this article. The problems can be phrased thus:

1. Given the equations of a number of straight lines intersecting in a point, to determine the x, y coordinates of that point. (This is the 3-star and multi-star situation).

2. Given the coordinates of a number of points along a straight line, to determine the equation of the line. (This is the constant-azimuth situation).

In Problem 1 we have a system of linear equations of the form:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$\cdots\cdots\cdots\cdots$$
$$a_nx + b_ny + c_n = 0$$

The «a » in each of these *observations* equations represents a (D-D) subtraction as in Figure 2; the « b » represents a (E-E) subtraction; the « c » represents a (F-F) subtraction.

We first add all the observation equations to get the first *normal* equation, shown as (1) below.

We then multiply each observation by the coefficient of « x » in it, add the equations together, getting the second normal equation, shown as (2) below.

We then multiply each observation equation by the coefficient of « y » in it, and then add the equations to get the third normal equation, shown as (3) below.

(1) $\Sigma ax + \Sigma by + n.c = 0$ (n=No. of equations in the system)

(2) $\Sigma a^2x + \Sigma aby + \Sigma a.c = 0$

(3) $\Sigma abx + \Sigma b^2y + \Sigma b.c = 0$

The solution of these simultaneous equations leads to the x, y, coordinates of the observer's projected position.

In Problem 2 we have a system of linear equations of the form:

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$
$$\dots\dots\dots\dots$$
$$ax_n + by_n + c = 0$$

The « x » in each of these observation equations is a (cot. cos) product; the « y » in each equation is a (cot. sin) product, as gotten from Figure 4.

We first add all the observation equations to get the first normal equation, shown as (4) below.

We then multiply each observation equation by the coefficient of « x » in it, then add the equations, and get the second normal equation, shown as (5) below.

We then multiply each observation equation by the coefficient of « y » in it, add the equations as before, and get the third normal equation, shown as (6) below.

(4) $a\Sigma x + b\Sigma y + n.c = 0$

(5) $a\Sigma x^2 + b\Sigma y + \Sigma x.c = 0$

(6) $a\Sigma xy + b\Sigma y^2 + \Sigma y.c = 0$

The solution of these simultaneous equations leads to evaluation of the coefficients a, b, and c, and thus to the equation of the first of the two intersecting straight lines of the constant-azimuth method. The equation of the other line is gotten in an identical manner. As shown in Figure 4, stars S_1 S_2 S_3 S_4 determine the equation of $L_{1 \cdot 2 \cdot 3 \cdot 4}$, and stars S_5 S_6 S_7 S_8 determine the equation of $L_{5 \cdot 6 \cdot 7 \cdot 8}$.

Two Stars Method

$$C_1: x^2 + y^2 + D_1x + Ey + F_1 = 0$$
$$C_2: x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

where for star No. 1

$$D_1 = \frac{-2 \cos GHA_1 \cos Dec_1}{(\cos Z D_1 \pm \sin Dec_1)}$$

$$E_1 = \frac{-2 \sin GHA_1 \cos Dec_1}{(\cos Z D_1 \pm \sin Dec_1)}$$

$$F_1 = \frac{\cos^2 Dec_1 - \sin^2 Z D_1}{(\cos Z D_1 \pm \sin Dec_1)^2}$$

The sign in the denominator is governed as follows: an observer in the northern hemisphere would normally choose the south celestial pole as the projecting point. In such case, stars with north declination would call for use of the plus sign, and stars of south declination of the minus sign. An observer in the southern hemisphere would reverse this arrangement.

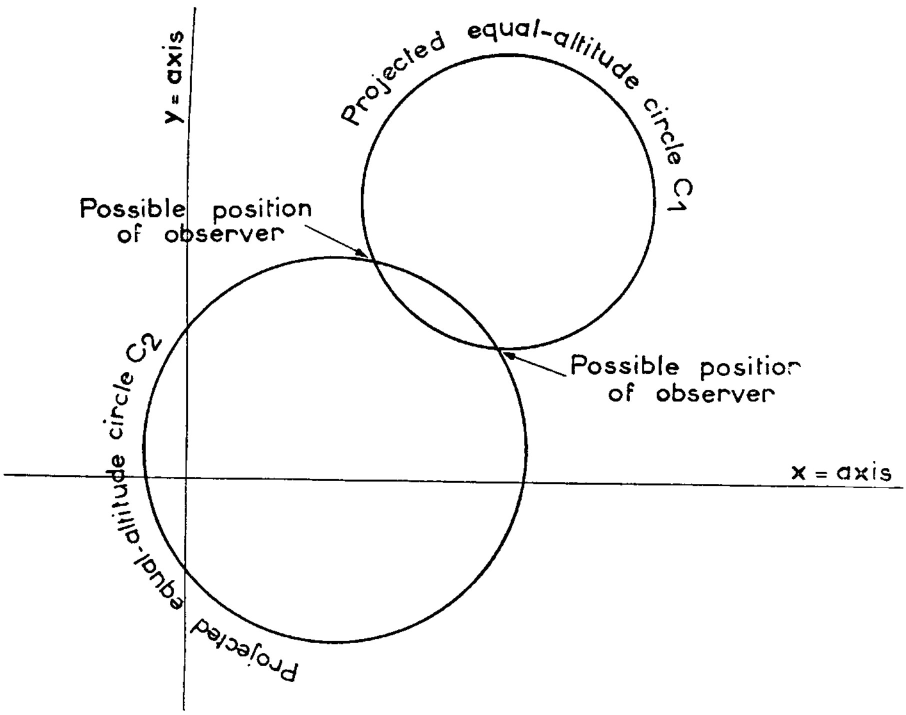

Fig. 1

The equation of each radical axis is arrived at by subtraction of the equations, taken in pairs, of the circles. For example:

$$C_1: x^2 + y^2 + D_1x + E_1y + F_1 = 0$$
$$(-)\quad C_2: x^2 + y^2 + D_2x + E_2y + F_2 = 0$$
$$L_{1\cdot 2}: (D_1-D_2)x+(E_1-E_2)y+F_1-F_2)=0$$

The point of intersection of $L_{1.2}$, $L_{1.3}$, and $L_{2.3}$ is the projected observer's position.

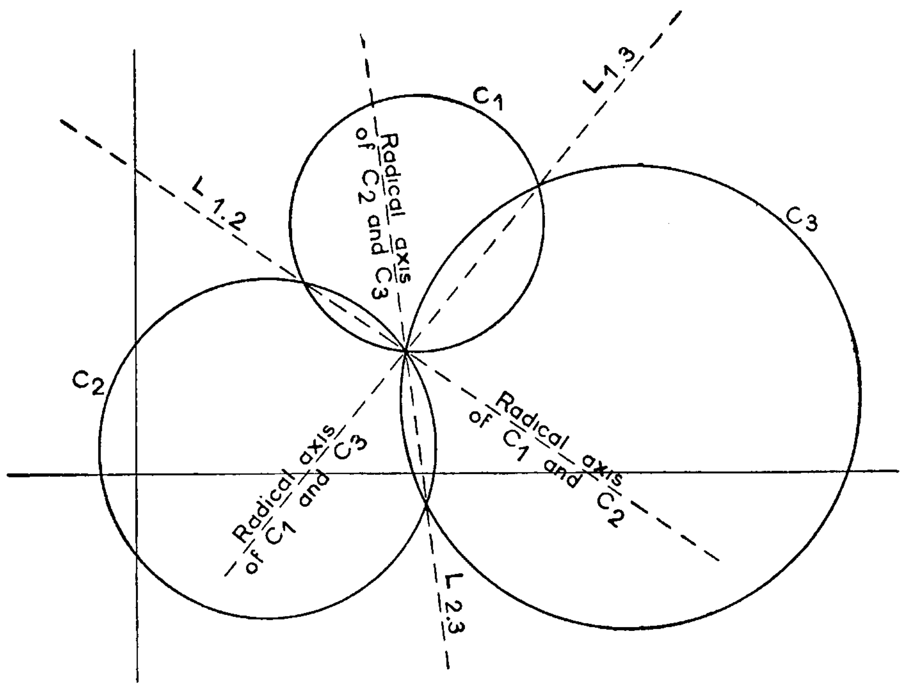

Fig. 2

The diagram shows all possible radical axes that can be gotten with 5 sights, each *of which* leads, of course, to the equation of a circle of equal altitudes. The nomenclature is similar to that used in Figure 2.

To generalize, with n sights the number fo radical axes possible is: 1 + 2 + 3... + (n—1). Thus, the 5 circles in the diagram lead to 10 axis equations, as shown.

These 10 radical axes will (in theory, at least) intersect in the projected position of the observer in a manner similar to those shown in Figure 2.

The diagram also suggests a method of systematizing or otherwise « keeping *track* » of the computatoin work.

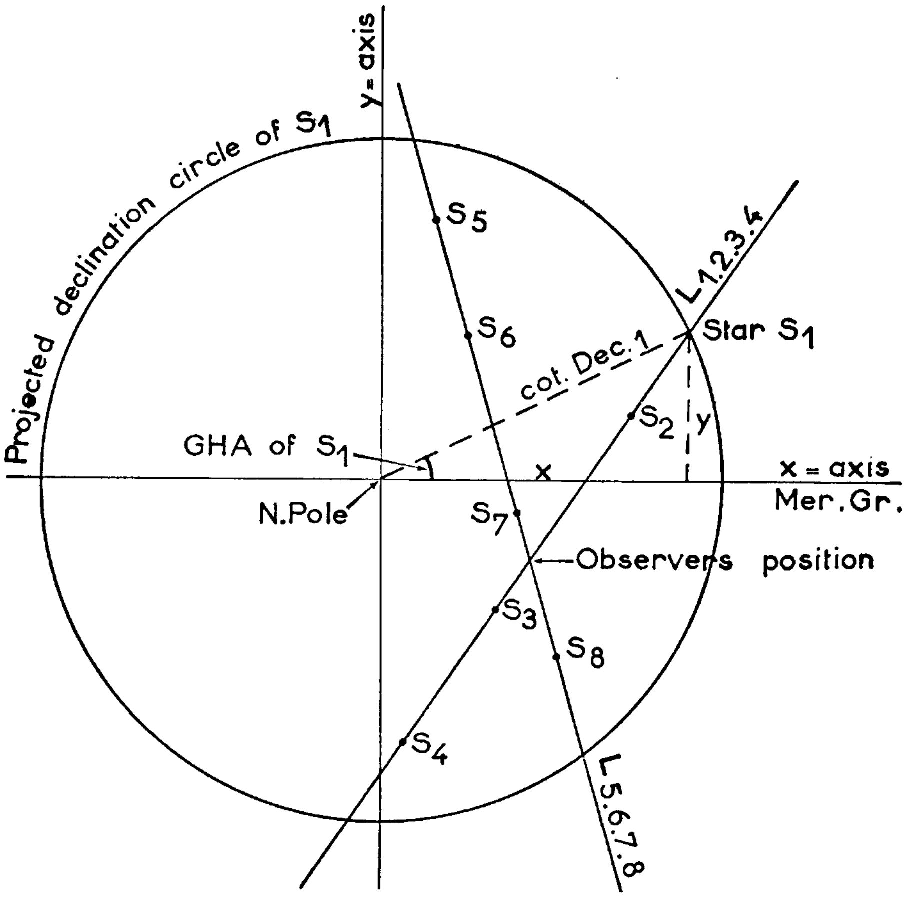

Fig. 3

S_1, S_2, S_3... S_8 represent gnomonically projected star positions. The projection formulas are:

$$x = \cot \text{Dec} \cos \text{GHA}$$
$$y = \cot \text{Dec} \sin \text{GHA}$$

Each of the two lines whose intersection marks the observer's projected position has an equation of the form:

$$ax + by + c = 0$$

which is determined by the star positions on the line.

	C_1	C_2	C_3	C_4	C_5
C_1		$L_{1.2}$	$L_{1.3}$	$L_{1.4}$	$L_{1.5}$
C_2			$L_{2.3}$	$L_{2.4}$	$L_{2.5}$
C_3				$L_{3.4}$	$L_{3.5}$
C_4					$L_{4.5}$
C_5					

Fig. 4