HARMONIC ANALYSIS OF A SHORT PERIOD OF TIDAL OBSERVATIONS

Separation of two constituents of approximately equivalent angular velocity

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The harmonic analysis of tidal observations over a short period makes it practically impossible to separate two constituents whose angular velocity is approximately equivalent.

Let us examine 30 days of observations, during which 721 hourly heights were obtained, and let us analyse these heights according to the least-squares method, by means of which the waves may most satisfactorily be separated.

Among other waves let us examine the average solar semi-diurnal constituent S_2. In both normal equations for this wave, the coefficient of the term in S_2 is about 360, while in the other « separable » semi-diurnal waves (μ_2, N_2, M_2, L_2), it is less than 22; in the diurnal waves (Q_1, O_1, K_1) it is under 3; and in the quarter-diurnal waves (M_4, MS_4) less than 1, which is the value attained for the term relating to mean level (1). As coefficient distribution is nearly similar in the other normal equations, the solving of the two systems formed by the various equations very adequately separates the various unknown quantities.

Let us now consider the declinational semi-diurnal wave K_2, whose angular speed of 30°.082 closely approximates that of S_2 (30°.000); the coefficients of the terms in S_2 and K_2 in the normal equations relating to S_2 are: in one of the systems (sine): 360 for S_2 and 346.14 for K_2; and in the other system (cosine): 361 for S_2 and 347.87 for K_2.

In other words, the terms in S_2 and K_2 in the equations particularly designed for determining S_2 may be written :

$$360\,(X_{S2} + 0.9615\,X_{K2}) \qquad 361\,(Y_{S2} + 0.9636\,Y_{K2})$$

This may not strictly apply in the case of the other normal equations, but as the term coefficients in S_2 and K_2 are small, it is legitimate to consider that the solution of the systems of normal equations, carried out by disregarding K_2, does not supply the constituents X_{S2} and Y_{S2} of S_2, but supplies the constituents $(X_{S2} + 0.96\,X_{K2})$ and $(Y_{S2} + 0.96\,Y_{K2})$ of the component R of S_2 and 0.96 K_2 at the mean instant of the observations (2).

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(1) See *Bulletin d'Information du C.O.E.C.*, VI, 9 (nov. 1954); tables of coefficients of normal equations appear on pages 403 and 404.

(2) If a similar study were made of group (K_1 P_1), it would be discovered that analysis supplies the constituents of the component of K_1 and 0.95 P_1.

If a method other than the least-squares method is used in forming the normal equations, the confusion of S_2 and K_2 is even more complete, i.e. the coefficient of K_2 in the component instead of amounting to 0.96, approaches unity even more closely.

In the case of 30 days of observations, a similar line of reasoning may be followed for the groups (Q_1 P_1), (K_1 P_1), (μ_2 $2N_2$), (N_2 ν_2), L_2 λ_2), S_2 T_2), (S_2 R_2). It will be noted in particular that in the case of S_2, analysis supplies the (S_2 K_2 T_2 R_2) group only, whose separation requires a more extensive series of observations; but as we theoretically have the relationships $K_2/S_2=0.27$, $T_2/S_2=0.06$, $R_2/S_2=0.01$, in first approximation we can neglect T_2 and S_2 and consider only group (S_2 K_2).

When only 15 days of observations are analysed, a group (M_2 N_2) is added.

Separation of the two waves in a group whose component has been supplied by harmonic analysis is usually accomplished by computation, assuming that the amplitude relationship of the two waves is equivalent to their theoretical relationship, and that the difference in phase lag is in ratio to the difference in their angular velocities. This assumption is borne out to a certain extent by the fact that the hydrodynamic influences characterizing the waves are essentially dependent on the cycle of such waves, and consequently almost identically affect waves with a nearly equivalent cycle. But if the harmonic analysis of two series of observations has been made, say of one month, taken at an interval of a few months, no additional assumption need be resorted to, and elements of both waves may be derived from both values of the component by process of computation.

We shall proceed to show that in this case, the problem may also be solved by a simple graphic method which has the added advantage of applicability to analyses exceeding two in number, providing, of course, the reference periods of such analyses are sufficiently close together for the nodal factors f and arguments u to be considered constant, i.e. providing the first and last observations are not more than a year apart.

PRINCIPLE OF GRAPHIC SOLUTION

a) *Case of two series of observations.*

Analysis supplies, for instants t_1 t_2, phases ω_1 ω_2 and amplitudes R_1 R_2 of the component of a two-wave group of angular velocity a and b. All these quantities constitute the data of the problem.

Let us take as unknown values amplitudes A and B of the two-constituents and their phases α_1 β_1 at instant t_1 with respect to the first analysis.

Figures 1 and 2 (plate I) show the constituents and their components at instants t_1 and t_2. Let the triangle of figure 2 be rotated around 0 to bring constituent A into the position occupied at instant t_1 (figure 3, plate I); the amplitude of the rotation is $\alpha_2 - \alpha_1 = - a(t_2 - t_1)$ (1); and the component, of course, retains its amplitude R_2, but its phase becomes $\omega'_2 = \omega_2 + a(t_2 - t_1)$; whereas the phase of constituent B is then written $\beta'_2 = \beta_2 + a(t_2 - t_1)$.

By superposing figures 1 and 3 (figure 4, plate I), the two constituents A are made to coincide, and the extremity N'_2 of constituent B is derived from N_1 by rota-

(1) As it is preceded by the minus sign in the expression of the argument, the phase decreases as the time increases.

tion about M_1 through an angle equal to $\beta'_2 - \beta_1 = \beta_2 - \beta_1 + a(t_2 - t_1) = (a - b)(t_2 - t_1)$.

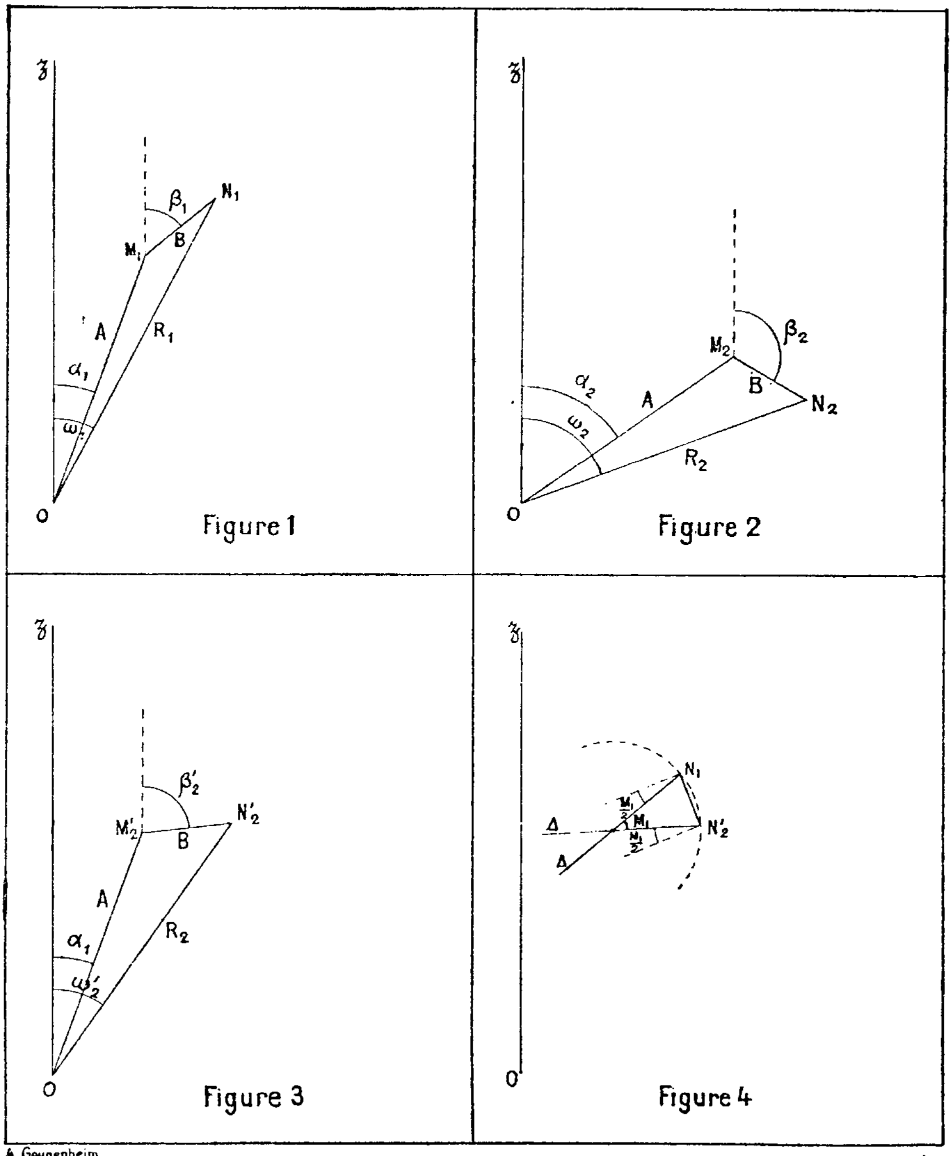


Planche I

Plate I.

N_1 and N'_2 being known from the results of analysis, and since ON_1 and ON'_2 are vectors whose respective amplitudes are R_1 and R_2 and whose phases are ω_1 and $\omega_2 + a(t_2 - t_1)$, M_1 may easily be derived. Two loci of this point are obtained, one being the bisector of $N_1 N'_2$ and the other the segment containing $(a - b)(t_2 - t_1)$ constructed on $N_1 N'_2$. Actually it is more convenient, although slightly less rational, to take as loci of M_1 the two straight lines which, at N_1 and N'_2, and in the appropriate direction, enclose angle $1/2\,(a - b)(t_2 - t_1)$ in conjunction with the direction perpendicular to $N_1 N'_2$. We shall designate these straight lines by Δ.

Numéro de l'analyse	Date de référence	Δt en jours	Résultat de l'analyse pour le groupe (K, P) ω	Résultat de l'analyse pour le groupe (K, P) R	aΔt	ω' = ω - aΔt	Valeurs relevées sur le graphique B (vraie grand.)	Valeurs relevées sur le graphique β'	(a-b) Δt	$\beta'_1 = \beta' - (a-b)\Delta t$
			°	cm.	°	°	mm.	°	°	°
1	16·7·50	0	14,9	41,7	0	14,9	97,2	43,8	0	44,8
2	16·9·50	62	73,1	24,9	61,1	12,0	91,4	172,9	122,2	50,7
3	16·11·50	123	113,5	37,8	121,2	352,3	100,0	293,4	242,5	50,9
4	16·2·51	215	236,7	32,9	211,9	24,8	104,0	110,2	63,8	46,4
5	16·6·51	335	335,2	43,2	330,2	5,0	96,4	357,8	300,4	57,4
						Moyennes adoptées	mm 97,8			° 50,0 = β_1

Couple	Δt du couple	(a-b)Δt	Valeurs relevées sur le graphique A (vraie grand.)	Valeurs relevées sur le graphique α_1
		°		
1·4	215	63,8		
4·2	153	-58,4		
2·3	61	120,2		
3·5	212	57,9		
Moyennes adoptées (Point M_1 placé à vue)			mm 336,7	° 7,0

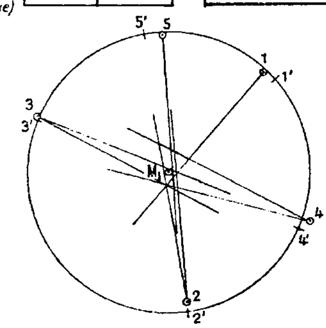
RÉSULTATS

A l'instant moyen (31-12-1950) $f(K_1) = 1{,}111$ $H(K_1) = \frac{33{,}67}{1{,}111} = 30^{cm},3$

$f(P_1) = 1$ $H(P_1) = \frac{9{,}78}{0{,}95} = 10^{cm},3$

Rapport $\frac{P_1}{K_1}$ = 0,340 (Valeur théorique 0,331)

	pour K_1	pour P_1
Phase	7°,0	50°,0
$(V_0 + u)^{oh}_{Gr.}$	203,0	156,7
Long. Est	141,4	141,4
Situation k	351°,4	348°,1



Echelle $\frac{1}{4}$

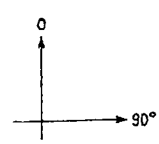


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Planche II

Plate II.

Numéro de l'analyse	Date de référence	Δt en jours	Résultat de l'analyse pour le groupe ($S_2 K_2$): ω	Résultat de l'analyse pour le groupe ($S_2 K_2$): R en cm.	aΔt	ω' = ω − aΔt	Valeurs relevées sur le graphique: B	Valeurs relevées sur le graphique: β'	(a−b)Δt	β'_1 = β' − (a−b)Δt
			°	cm.	°	°	mm.	°	°	°
1	16·7·50	0	123,7	11,0	0	123,7	57,8	236,2	− 0	236,2
2	16·9·50	62	101,4	18,2	0	101,4	40,0	101,5	− 122,2	223,1
3	16·11·50	123	76,0	13,7	0	76,0	62,6	353,7	− 242,5	236,2
4	16·2·51	215	119,3	18,1	0	119,3	62,9	162,8	− 63,8	226,6
5	16·6·51	335	99,1	8,5	0	99,1	57,7	285,5	− 300,4	225,9
						Moyennes adoptées..	mm 56,2			229,9 = β_1

Couple	Δt du couple	(a−b)Δt	Valeurs relevées sur le graphique: A (vraie grand^r)	Valeurs relevées sur le graphique: α_1
		°	mm	°
1·4	215	− 63,8	149,3	98,3
4·2	153	53,4	129,6	109,7
2·3	61	−120,2	136,3	96,2
3·5	212	− 57,9	154,2	102,8
Moyennes adoptées...			mm 142,3	° 101,7

R É S U L T A T S

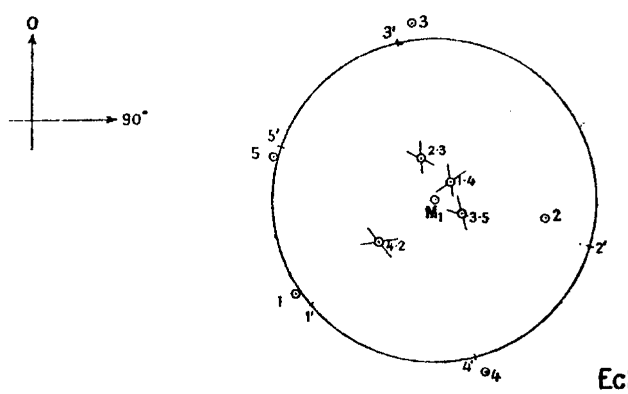
A l'instant moyen (31-12-1950)

$f(S_2) = 1$ $H(S_2) = 14^{cm},2$

$f(K_2) = 1{,}310$ $H(K_2) = \frac{5{,}62}{0{,}96 \times 1{,}310} = 4^{cm},5$

Rapport $\frac{K_2}{S_2} = 0{,}314$ (Valeur théorique 0,272)

	pour S_2	pour K_2
Phase	101°,7	229°,9
$(V_0+u)^{0h}_{Gr.}$	0,0	225,2
2 x Long. Est	282,8	282,8
Situation k	24°,5	17°,9



Echelle $\frac{1}{2}$

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Planche III

Plate III.

Knowledge of M_1 immediately supplies A and α_1, elements of vector OM_1; moreover, amplitude B is the common value of $M_1 N_1$ and $M_1 N'_2$. β_1 is obtained by plotting bearings β'_1 and β'_2 of $M_1 N_1$ and $M_1 N'_2$, and by taking the mean of β'_1 and $\beta'_2 - (a - b)(t_2 - t_1)$ to get $\beta_1 = 1/2\,(\beta'_1 + \beta'_2) - 1/2\,(a - b)(t_2 - t_1)$.

b) *Case of a large number of series.*

By proceeding as in the previous case, a point N' may be made to correspond to each series, and all such points must (1) describe a circle, whose centre M_1 and radius B are unknown, and (2) be located on radii produced from M_1 and enclosing angles proportional to the time intervals between analyses.

An overall solution may be found through trial and error plotting (by means of a series of concentric circles drawn on tracing paper) of the circle best fitting points N'_1, but this does not allow for the angular condition with respect to relative directions of radii $M_1 N'$.

It therefore appears more advisable to combine the N' points into sets of two, thus reducing the problem to the previous case. Each set supplies two straight lines Δ, the loci M_1, but all the straight lines are not strictly convergent, owing to observational errors, approximations due to methods of analysis, and the neglect of such small waves as T_2 and R_2 during separation of the ($S_2 K_2$) group. The most likely point of convergence is selected by inspection, the point M_1 thus obtained and the origin O supply a vector OM_1, whose length A and bearing α_1, i.e. elements of constituent A at time t_1, are plotted (example 1, plate II).

When the point of convergence of the straight lines can only be selected by inspection with difficulty, because the straight lines are too scattered, the points M_1 relating to each pair of N' points are plotted on the graph, the corresponding elements A and α_1 are plotted and the respective mean values for the various A and α_1 values are adopted. The mean point M_1 thus obtained is then plotted on the graph (example 2, plate III).

The amplitude of constituent B is obtained by averaging the lengths of the various vectors $M_1 N'$ measured on the graph; its phase β_1 is then determined by plotting with a protractor bearings $\beta'_1\ \beta'_2\ \beta'_3 \ldots$ of directions $M_1 N_1$, $M_1 N'_2$, $M_1 N'_3 \ldots$ and averaging quantities $\beta'_1\ \beta'_2 - (a - b)(t_2 - t_1)$, $\beta'_3 - (a - b)(t_3 - t_1) \ldots$ or equivalently, by taking

$$\beta_1 = \frac{\beta'_1 + \beta'_2 \ldots \beta'_n}{n} - (a - b)(t_0 - t_1),$$

where the quantity t_0 designates the mean nistant of analysis.

EXAMPLES

Both examples supplied below relate to a series of 5 analyses (1) of 30 days of observations (721 hourly heights) carried out at 2 to 4-month intervals over an entire year. As the series were begun on the first day of the month at 0 hours (U.T.) the mean times $t_1\ t_2 \ldots$ are the sixteenth of the month at 0 hours.

Example 1 (plate II) deals with the separation of group ($K_1 P_1$); the graph is to one-quarter scale. As straight lines Δ are sufficiently convergent, mean point M_1 was selected by inspection. In example 2 (plate III), however, relating to the separation of group (S_2 K_2), and in which the graph is to one-half scale, straight lines Δ are fairly scattered, probably owing to the neglect of waves T_2 and R_2. Here the various points M_1 have been constructed so that the adopted point M_1 might be derived by computing averages.

The plates show in detail the computations required for the graphic separation of the constituents. Such computations are not as tedious as would appear from the above description. It may be noted that in the two examples under consideration, which are most frequently used in practice, the difference a — b has the same value, except for the sign.

The points N' (the circled points identified by the number of the analysis) are plotted by bearing and distance from the origin by means of the data in columns R and ω'; the Δ lines are produced from these points by using the angles appearing in column (a — b) Δ t, divided by 2.

On both graphs, points N'' have been obtained by describing the circle of centre M_1 and radius B, and by plotting on the circle the ends of the radii of bearing $\beta_1 + (a - b)\ \Delta\ t$. Points N' should be located at these points to ensure rigorous determination of M_1 : the discrepancies N' and N'' therefore indicate the amount of uncertainty affecting the final result. Points N'' are shown by the points identified by the number of the analysis and the symbol '. The actual dimensions of the N' N'' differences are of the order of 1 centimetre, with one exception (Point 2—2' in plate III), reaching 2 centimetres. It will thus be realized, particularly as regards the separation of K_1 and P_1, that there is no need for recourse to 5 monthly analyses as in the previous examples, but that three are largely sufficient.

(1) By the least-squares method.