CORRECTION OF VARIANCE FOR EXTRANEOUS CONTRIBUTION OF GRADIENT

by Carsten M. WINGER, Oceanographic Publications Branch, Division of Oceanography, U. S. Navy Hydrographic Office.

ABSTRACT

The drawing of isolines of standard deviation for a function (temperature, for example) necessitates having a sufficient number of stationary points, sufficiently close together, for which the frequency distributions of the function are known. Over ocean areas, however, this necessity is not met, and will not be met, presumably, for a very long time. The best that can be done at present is to deal with the observations for a finite area over a finite period of time, combining them as elements of a single frequency distribution for some point near the center of the area in question. This procedure introduces extraneous spatial deviations, giving the variance of our function too large a value. The magnitude of this error increases with the size of the area and with the numerical value of the gradient, and also varies to some extent with the latitude and the orientation of the isolines of the function. This report proposes a formula for the correction of this error.

Correction of Variance (σ^2) for Extraneous Contribution of Gradient

Consider a scalar function F (y, t), continuous in the plane. Let us choose the area of consideration (A) small enough so that the isolines of F are reasonably straight, and such that the gradient can be considered to change at a constant rate with distance.

We are concerned with the construction of a set of isolines of standard deviation of F over a region of which our area A is an integral part. Interpolation in the completed pattern, then, will enable us to determine the standard deviation of F for every point of A. However, if we inject into our problem the not uncommon situation, especially over ocean areas, of lacking sufficient stationary points of observation, sufficiently close together, to permit the drawing of the pattern in question, a rather considerable difficulty presents itself. In order to obtain adequate data for any one frequency distribution, we are forced to combine the observations for a finite area, rather than for a point; and this procedure tacitly involves extraneous spatial deviations, giving the variance of F too large a value.

Though the following method for evolving a formula for correcting this value of variance is applicable to any rectangular area, as well as to other geometrical configurations, since it is felt that this correction would normally involve a function related to the earth's grid, we choose our area A as an approximate rectangle bounded by meridians and parallels of latitude, p degrees of latitude long and p degrees of longitude wide. For convenience, we choose one degree of latitude as our unit of length. The magnitude of p, of course, should be as small as possible to enhance the credibility of our assumptions; and yet large enough to give sufficient observations to permit the consideration of frequency distributions. At the present time, over most ocean areas, we must have $p \geqq 1$.

To facilitate the mathematics, we must assume that our gradient does not change with time, but only with distance, which limits the length of our time interval; and in order that our theory may be applied to any particular area, the observations should be reasonably uniformly distributed in the area.

In accordance with the foregoing assumptions, we have :

$$\frac{\partial^2 F}{\partial^2 y} = a, \quad \frac{\partial^2 F}{\partial y \, \partial t} = o,$$

where a is an arbitrary constant. Then :

$$F(y,t) = \frac{a y^2}{2} + \left(\frac{\partial F}{\partial y}\right)_c \cdot y + Fc\,(t),$$

where $\left(\frac{\partial F}{\partial y}\right)_c$ is the gradient of F at point C, the center of our « square », and does not vary with time; and $F_c(t)$ is the value of F at point C at any time t.

Letting $\bar{F}$ be the mean value of F at an arbitrary point and σ, its true standard deviation ; and assuming a normal distribution with $v = \frac{F - \bar{F}}{\sigma}$, the weighted squared deviation Δ^2 from $\bar{F}_m$ for our arbitrary point is given by

$$\Delta^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} l^{-\frac{v^2}{2}} \left(\bar{F} - \bar{F}_m\right)^2 dv =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} l^{-\frac{v^2}{2}} \left[\sigma v + \left(\bar{F} - \bar{F}_m\right)\right]^2 dv = \sigma^2 + \left[\bar{F} - \bar{F}_m\right]^2$$

where $\bar{F}_m$ is the average value of F for the entire area A and the point m is taken as the midpoint of the isotherm having the value $\bar{F}_m$ on the mean chart.

We are interested in the surface integral of this expression over A, so we wish to express $(\bar{F} - \bar{F}_m)$ as a function of y for given values of ϕ and α, the former denoting the latitude of the center of A and the latter denoting the angle of orientation of the isolines of F relative to the parallels of latitude (Fig. 1), taking counter-clockwise measurement as positive. For convenience of expression, we adopt the following notation in connection with Fig. 1 :

$$l_1 = + \tfrac{p}{2} (\cos \alpha - \sin \alpha \cos \phi)$$

$$l_2 = + \tfrac{p}{2} (\sin \alpha \cos \phi + \cos \alpha)$$

$$l_3 = y \tan \alpha$$

$$l_4 = y \tan \alpha + p \sec \alpha \cos \phi$$

$$l_5 = \left[y + \tfrac{p}{2} (\cos \alpha - \sin \alpha \cos \phi)\right]\left[-\cot \alpha\right] - \tfrac{p}{2} \sin \alpha \left[1 - \cos \phi \tan \alpha\right]$$

$$l_6 = \left[y - \tfrac{p}{2} (\cos \alpha - \sin \alpha \cos \phi)\right]\left[-\cot \alpha\right] + \left[\tfrac{p}{2} \sin \alpha + p \cos \phi \sec \alpha \left(1 - \frac{\sin^2 \alpha}{2}\right)\right]$$

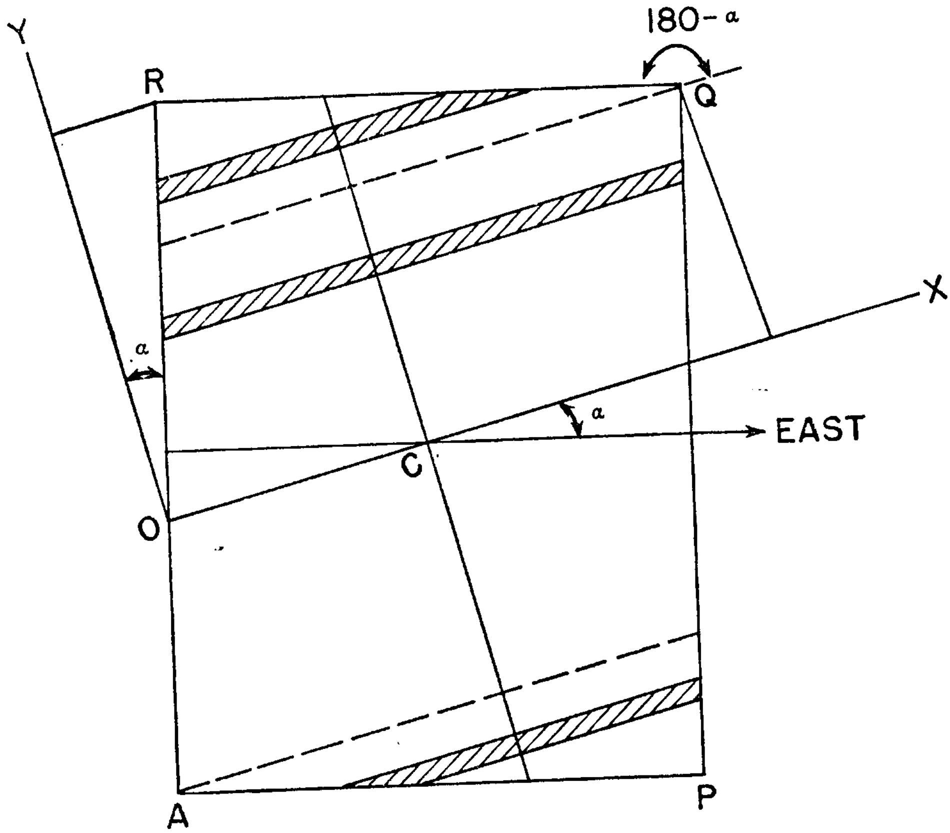

Fig. 1. — Basic diagram for deriving correction formula.

A P Q R is a p-degree « square » with C as its center.
ϕ = Latitude of C.
Equation of line RA : $y = x \cot \alpha$.
Equation of line QP : $y = \cot \alpha \, [x - p \sec \alpha \cos \phi]$.
Equation of line RQ : $y = -\tan \alpha \, [x - (p/2) \sin \alpha - p \cos \phi \sec \alpha \, (1 - \overline{1/2} \sin^2 \alpha)] + (p/2) \, [\cos \alpha - \sin \alpha \cos \phi]$.
Equation of line AP : $y = -\tan \alpha \, [x + (p/2) (\sin \alpha - \cos \phi \sec \alpha \sin^2 \alpha)] - (p/2) \, [\cos \alpha - \sin \alpha \cos \phi]$.

Points :
Q $[(p/2) \sin \alpha + p \cos \phi \sec \alpha \, (1 - \overline{1/2} \sin^2 \alpha), \; (p/2) \, . \, (\cos \alpha - \sin \alpha \cos \phi)]$.
R $[(p/2) \sin \alpha \, (\tan \alpha \cos \phi + 1), \; (p/2) \, . \, (\sin \alpha \cos \phi + \cos \alpha)]$.

Then :

$$p^2 \cos \phi \, \overline{F}_m = \int_{-l_2}^{-l_1} \int_{+l_5}^{+l_4} \overline{F} \, dx \, dy + \int_{-l_1}^{+l_1} \int_{+l_3}^{+l_4} \overline{F} \, dx \, dy + \int_{+l_1}^{+l_2} \int_{+l_3}^{+l_6} \overline{F} \, dx \, dy = \frac{a p^4 \cos^2 \phi}{24} \left[1 - \sin^2 \alpha \sin^2 \phi \right] + p^2 \cos \phi \, \overline{F}_c$$

Hence :

$$\Delta^2 = \sigma^2 + \left[\frac{a}{2} y^2 + \left(\frac{\partial \bar{F}}{\partial y} \right)_c . y - \frac{ap^2}{24} (1 - \sin^2 \alpha \sin^2 \phi) \right]^2$$

By the definition of Δ^2, our deviations are being taken with respect to $\bar{F}_m$. If σ_u^2 denote the uncorrected variance for point M, for the interval $0 \leqq \alpha \leqq \cot^{-1} \cos \phi$ we can write :

$$p^2 \cos \phi \, \sigma_u^2 = \int_{-l_2}^{-l_1} \int_{+l_5}^{+l_4} \Delta^2 \, dx \, dy + \int_{-l_1}^{+l_1} \int_{+l_3}^{+l_4} \Delta^2 \, dx \, dy + \int_{+l_1}^{+l_2} \int_{+l_3}^{+l_6} \Delta^2 \, dx \, dy$$

or

$$\sigma_u^2 = \sigma^2 + \frac{p^2}{12} \left(\frac{\partial \bar{F}}{\partial y} \right)_c^2 \left[1 - \sin^2 \alpha \sin^2 \phi \right] + \frac{a^2 p^4}{5.32.24} \left[(1 - \sin^2 \alpha \sin^2 \phi)^2 + 3 \sin^2 \alpha \cos^2 \alpha \cos^2 \phi \right]$$

From considerations of symmetry, our formula also holds for the interval $-\cot^{-1} \cos \phi \leqq \alpha \leqq 0$; and, in similar fashion to the above, it can be shown to hold for the interval $\cot^{-1} \cos \phi \leqq \alpha \leqq \pi . -\cot^{-1} \cos \phi$. Therefore, it can be said to hold for all values of α.

Substituting the expression found previously for $\bar{F}_m$ in our general expression for $\bar{F}$ gives the ordinate y_m of point M as :

$$y_m = -\frac{1}{a} \left(\frac{\partial \bar{F}}{\partial y} \right)_c \left[1 - \sqrt{1 + \frac{a^2 p^2 (1 - \sin^2 \alpha \sin^2 \phi)}{12 \left(\frac{\partial \bar{F}}{\partial y} \right)_c^2}} \right]$$

For a uniform gradient, a = 0; and our correction formula simplifies to

$$\sigma_u^2 = \sigma^2 + \frac{p^2}{12} \cdot \left(\frac{\partial F}{\partial y} \right)_c^2 \left[1 - \sin^2 \alpha \sin^2 \phi \right]$$

This immediately raises the question as to the magnitude of the ratio of the first term (C_1, say) of our correction to the second term (C_2) in our general formula.

Let $\bar{F}_1$ be the value of F at the northwest corner of our area on the mean chart, and let $\bar{F}_2$ be the value at the southeast corner. We then have :

$$F_1 = \frac{a}{2}\left[\frac{p}{2}(\sin\alpha\cos\phi + \cos\alpha)\right]^2 + \left(\frac{\partial F}{\partial y}\right)_c\left[\frac{p}{2}(\sin\alpha\cos\phi + \cos\alpha)\right] + \bar{F}_c,$$

$$F_2 = \frac{a}{2}\left[-\frac{p}{2}(\sin\alpha\cos\phi + \cos\alpha)\right]^2 + \left(\frac{\partial F}{\partial y}\right)_c\left[-\frac{p}{2}(\sin\alpha\cos\phi + \cos\alpha)\right] + \bar{F}_c.$$

Hence :

$$a = \frac{\bar{F}_1 + \bar{F}_2 - 2\bar{F}_c}{\left[\frac{p}{2}(\sin\alpha\cos\phi + \cos\alpha)\right]^2} \text{ and } \left(\frac{\partial F}{\partial y}\right)_c = \frac{\bar{F}_1 - \bar{F}_2}{p(\sin\alpha\cos\phi - \cos\alpha)}$$

Substitution in our general correction formula gives

$$\sigma_u^2 = \sigma^2 + (\bar{F}_1 - \bar{F}_2)^2\left[\frac{1 - \sin^2\alpha\sin^2\phi}{12(\sin\alpha\cos\phi + \cos\alpha)^2}\right]$$

$$+ (\bar{F}_1 + \bar{F}_2 - 2\bar{F}_c)^2\left[\frac{(1 - \sin^2\alpha\sin^2\phi)^2 + 3\sin^2\alpha\cos^2\alpha\cos^2\phi}{45(\sin\alpha\cos\phi + \cos\alpha)^4}\right]$$

$$= \sigma^2 + (\bar{F}_1 - \bar{F}_2)^2 x_1 + (\bar{F}_1 + \bar{F}_2 - 2\bar{F}_c)^2 x_2, \text{ say.}$$

Letting $r = \dfrac{\bar{F}_c - \bar{F}_1}{\bar{F}_2 - \bar{F}_c}$, we have $\dfrac{C_1}{C_2} = \left[\dfrac{r+1}{r-1}\right]^2 \cdot \dfrac{x_1}{x_2}$.

For $0 \leqq \alpha \leqq \pi/2$, $x_1/x_2 \geqq 3.75$; and since $\left[(r+1)/(r-1)\right]^2$ decreases as r increases, we can arrive at a lower limit for C_1/C_2 if we can determine, for the particular function F(y, t) in hand, a maximum value for r.

Two functions for which this correction procedure is of prime importance are sea temperature and air temperature over the oceans. For these functions, with $0 \leqq \alpha \leqq \pi/2$, we will rarely have a value of r greater than 2, so that the ratio C_1/C_2 will generally be greater than 30. Even for r = 3, this ratio will be equal to, or greater than, 15.

From our general formula it follows that these results are therefore also true for $-\frac{\pi}{2} \leqq \alpha \leqq 0$, sa we can conclude that, for temperatures over the oceans, we can use the simpler formula

$$\sigma^2 = \sigma_u^2 - \frac{p^2}{12}\left(\frac{\partial T}{\partial y}\right)_c^2\left[1 - \sin^2\alpha\sin^2\phi\right].$$

Figures 2 and 3 are nomograms for the evaluation of our correction for various combinations of our parameters. Figure 3 also enables one to determine values of C_1/σ_u^2. The intersections of our σ_u — lines with our 100 % — line give the lower limit for σ_u for any given combination of parameters.

Obviously, our correction will always be positive. For a given latitude, it will increase as the size of our area increases, and as our gradient increases numerically, and, for a given gradient, will decrease as the isotherms take on more of a north-south orientation.

As a concrete example, consider the following excerpt from the mean chart for February :

Two-degree square (p = 2) bounded by parallels of latitude 38° N and 40° N, and meridians 148° E and 150° E. Hence $\phi = 39$. Also, $\bar{T}_c = 45°$ F., $\alpha = 15°$, $\left(\frac{\partial T}{\partial y}\right)_c = 5$ F°, and $\sigma_u = 4.4$ F°.

We are interested in evaluating graphically our correction

$$C_1 = p^2 \frac{1}{12} \left(\frac{\partial T}{\partial y}\right)_c^2 \left[1 - \sin^2 \alpha \sin^2 \phi\right]$$

Entering Fig. 2 with the given values of ϕ and α, we drop vertically to the line k = 5 and read the corresponding value along the right-hand margin. This would be our correction for p = 1. However, for p = 2, the case in hand, we proceed horizontally in Fig. 3 (which is but a continuation of Fig. 2) to the line p = 2 and read the corresponding value along the base of our nomogram. This gives us the value of our desired correction, namely $C_1 = 8$. Hence $\sigma^2 = 19.36\text{-}8 = 11.36$, or $\sigma = 3.4$. Proceeding vertically from $C_1 = 8$ to the interpolated dashed line $\sigma_u = 4.4$, the corresponding point along our right-hand margin gives 41 % as the percent correction to σ_u^2. This corresponds to a correction to σ_u of $(1 - \sqrt{1 - .41})\,100 = 23$ %.

In Figure 4, curve A represents the distribution for point C with $\sigma = 3.4$; and curve B represents the distribution with $\sigma_u = 4.4$. The corresponding intervals within which 90 % of the observations lie are also shown.

If we next consider this same point with $\sigma = 3.4$, $\left(\frac{\partial T}{\partial y}\right)_c = 5$ and p = 5, we find $\sigma_u^2 = 62.56$. Curve D of Figure 4 represents this distribution, with $\sigma_u = 7.9$, for which our 90 % interval is 32.0 — 58.0, a range of 26 as compared to only 11.2 for curve A. This is an extreme case, with $\left(\frac{\partial T}{\partial y}\right)_c = 5$, but does illustrate the effect of p = 5 as compared to p = 2.

Curve C represents a more normal case, with p = 5, $\sigma = 3.4$ and $\left(\frac{\partial T}{\partial y}\right)_c = 3$, giving $\sigma_u = 5.5$. Our 90 % range is 18 as compared to 11.2 for curve A, an appreciable difference. At present, it appears that we must operate with p = 5 in many areas (if at all); and the foregoing figures show the necessity of considering the correction in question.

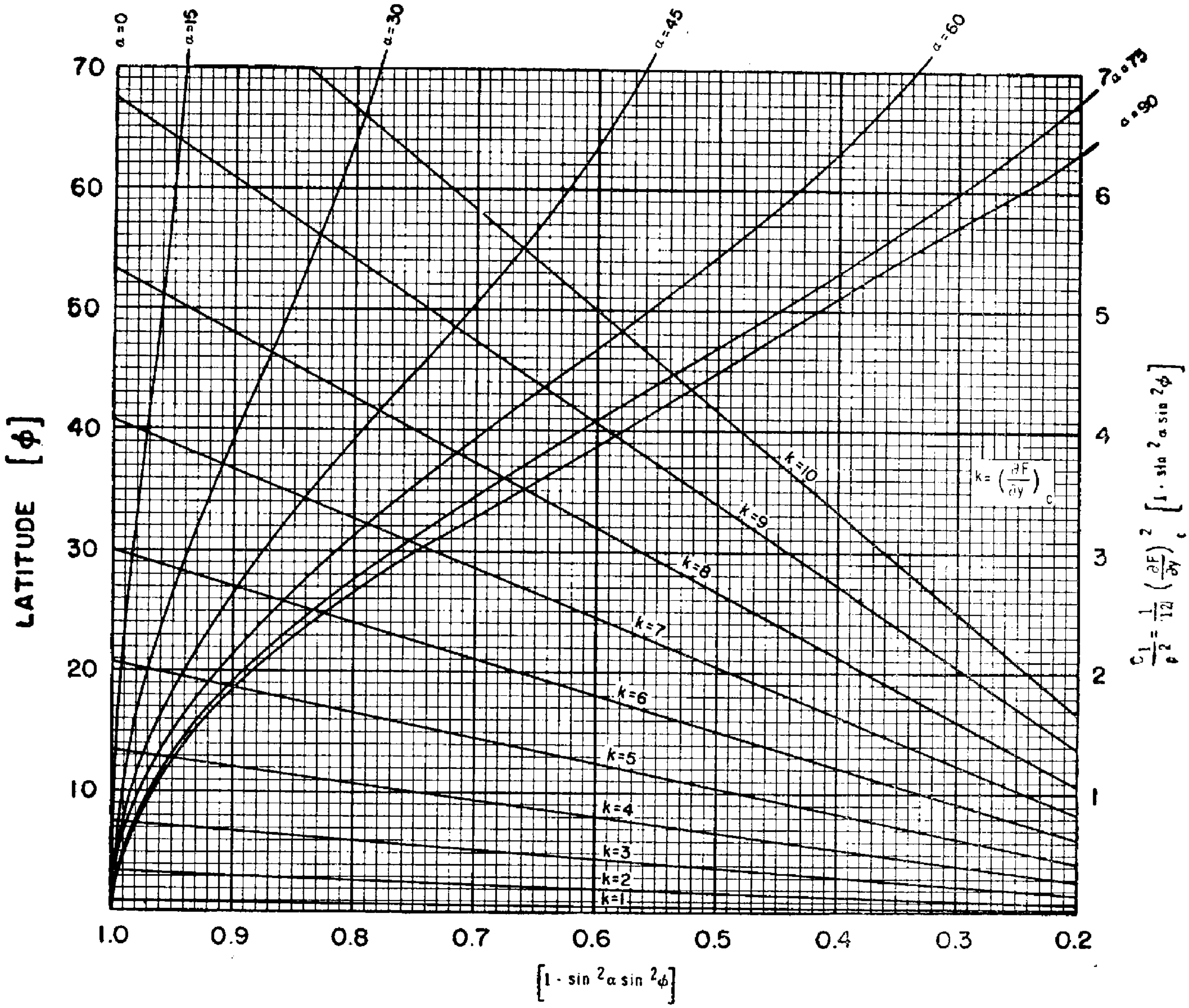

Fig. 2. — Part one of nomogram for calculating corrections.

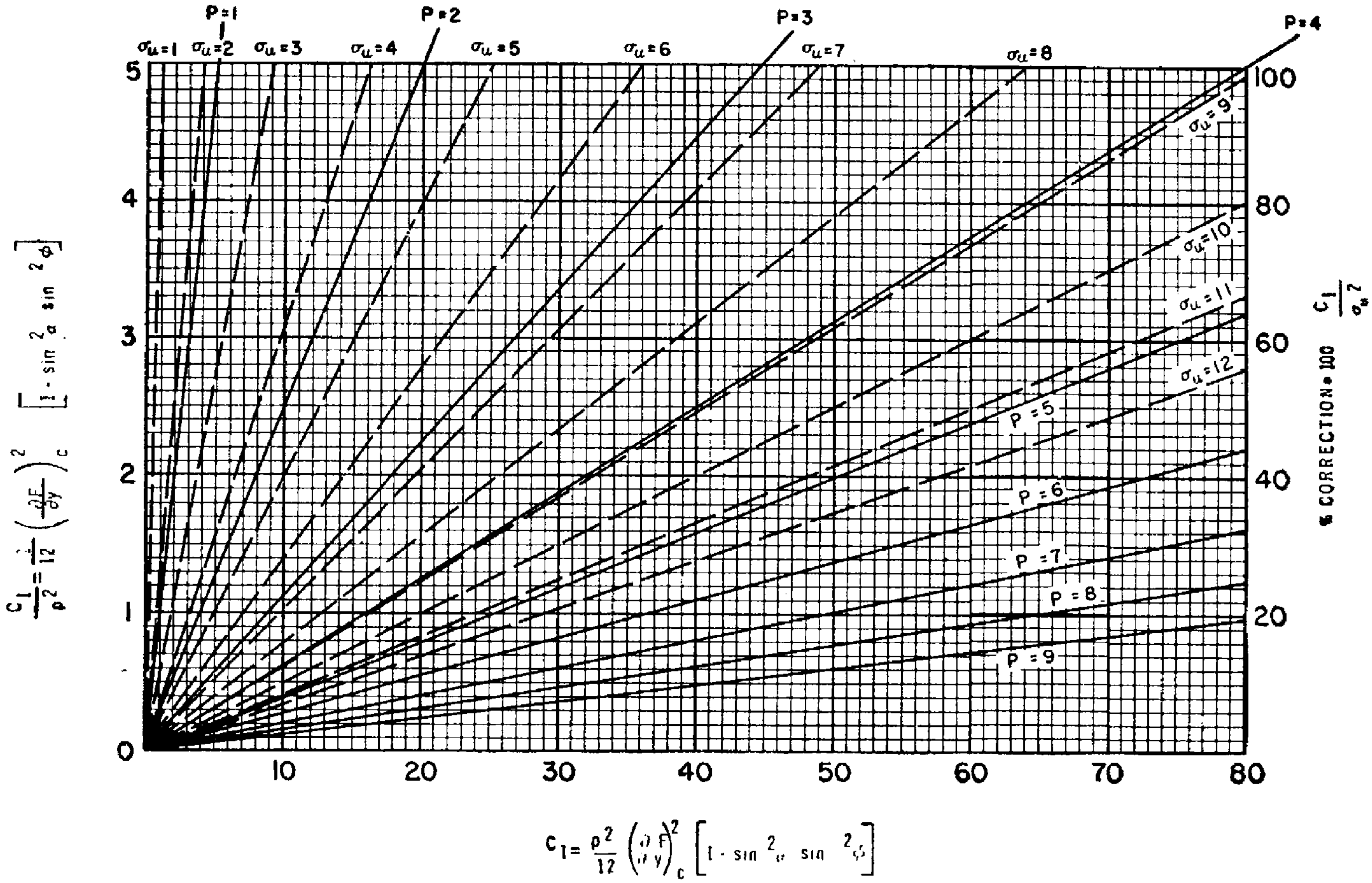

Fig. 3. — **Part two of nomogram for calculating corrections.**

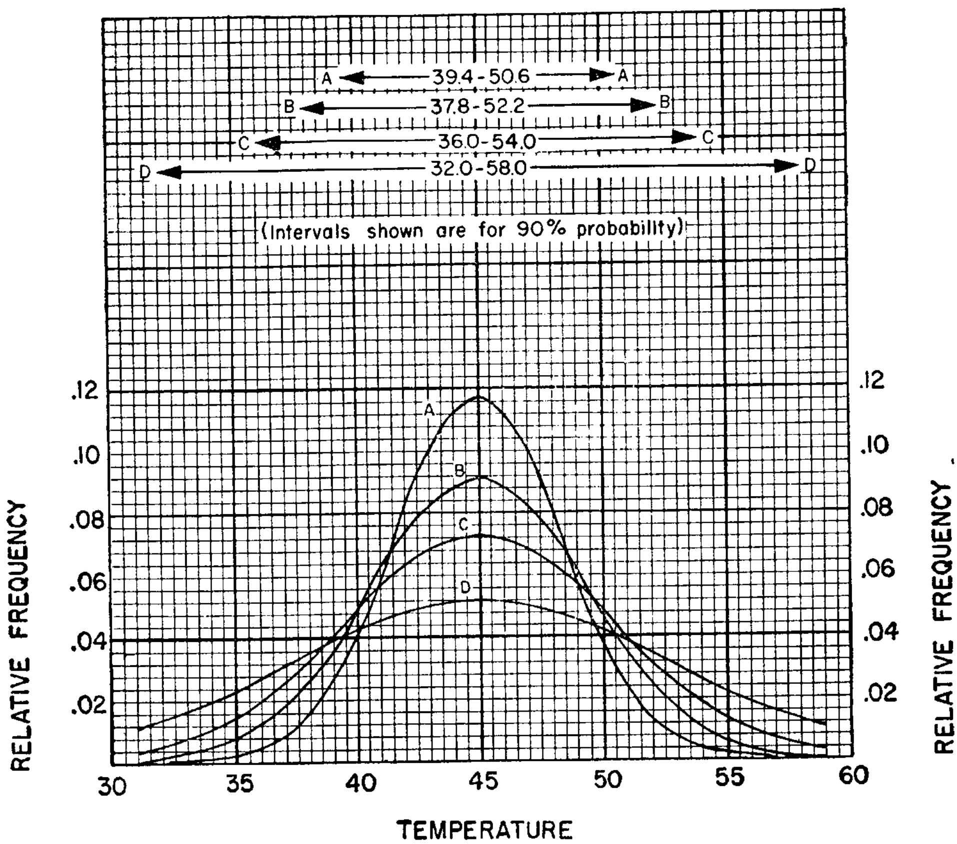

Fig. 4. — Curves illustrating correction-effect.

In conclusion, though admittedly the situations met in practice will never conform exactly to those proposed under our assumptions, nevertheless, the judicious use of the formula developed in this paper permits a reckoning of the order of magnitude of our correction, and hence gives an approximation to the magnitude itself.