NEW METHODS OF SHIP POSITION FINDING
FROM CELESTIAL OBSERVATIONS

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In this article I have tried to show in a few lines my short tabular methods for computation of altitude and azimuth (Tables K₁), as well as for direct computation of the observed position (Tables K₁₁); the latter is independent of the conventional Marcq de Saint-Hilaire method, and Longitude and Latitude methods.

In order to enable the reader to get a clear idea of the computation of a position line and the observed position respectively, I have shown extracts of these tables and examples of work too.

TABLES K₁

The main principle of computing the altitude and azimuth by these tables consists of the following:

Instead of solving the astronomical triangle directly or dividing it into two right-angled spherical triangles by dropping a perpendicular from the zenith or the celestial body, three right-angled spherical triangles formed by the intersection of the celestial equator, the celestial horizon, the lower branch of the celestial meridian, the vertical circle and the hour circle of the celestial body are used. These three triangles have one common pole at the point where the hour circle of a celestial body intersects the celestial horizon. This point is marked by letter g in Figure 1.

From the first triangle Wfg (see Figure 1) with known elements 90°—s (complement of the local hour angle) and 90°—φ (complement of the latitude) the angle C and angular distance M are to be computed. Then from the second triangle Jhg with known C and M+δ (algebraic sum of M and declination) the computation of the altitude (V) and angular distance F is to be made. The angular distance ω+F (azimuth angle plus F) is to be found from the third triangle PngN with known φ and 180°—s. The value F subtracted from ω+F gives the azimuth angle (ω), and in this way the whole problem is completely solved.
FORMULÆ FOR COMPUTATION OF THE ALTITUDE

Applying the Napier rules to the following triangles it is found:

From the triangle Wfg (Figure 1):
\[ \cos C = \sin (90^\circ - \varphi) \cdot \sin [90^\circ - (90^\circ - s)] \]
\[ \cos C = \cos \varphi \cdot \sin s \]
\[ \cos [90^\circ - (90^\circ - s)] = \cot (90^\circ - M) \cdot \cot (90^\circ - \varphi) \]
\[ \cos s = \tan M \cdot \tan \varphi \]
\[ \tan M = \cot \varphi \cdot \cos s \]

or
\[ \cos (90^\circ - \varphi) = \sin C \cdot \sin (90^\circ - M) \]
\[ \sin \varphi = \sin C \cdot \cos M \]
\[ \cos M = \frac{\sin \varphi}{\sin C} \]

From the triangle Jgh:
\[ \cos (90^\circ - V) = \sin C \cdot \sin (M + \delta) \]
\[ \sin V = \sin C \cdot \sin (M + \delta) \]
\[ \cos (M + \delta) = \cot V \cdot \cot C \]
\[ \cot V = \cos (M + \delta) \cdot \tan C \]

or
\[ \cos C = \sin V \cdot \sin (90^\circ - V) = \sin V \cdot \cos V \]
\[ \sin V = \frac{\cos C}{\cos V} \]
Figure 2 shows the arguments M, C and altitude (V) for two celestial bodies (J and J₂) whose declinations (δ and δ₂) are of the same name as latitude (φ) but local hour angle of the first body (s) is less and local hour angle of the second body (s₂) is greater than 90°.

From this figure it is seen that:

a) M is of the same name as latitude (above the horizon) if the local hour angle is less than 90°; but M is of the contrary name (below the horizon) if the local hour angle is greater than 90°.

b) C never exceeds 90°.

FORMULAE FOR COMPUTATION OF THE AZIMUTH

Applying the Napier rules to the triangles mentioned below it is found:

From the triangle PegN (Figure 1):

\[
\begin{align*}
\cos (90° - \varphi) &= \cotg (180° - s) \cdot \cotg [90° - (\omega + F)] \\
\sin \varphi &= \cotg (180° - s) \cdot \tan (\omega + F) \\
\tan (\omega + F) &= \sin \varphi \cdot \tan (180° - s) \\
\tan (\omega + F) &= -\sin \varphi \cdot \tan s
\end{align*}
\]
From the triangle $Jgh$ (Figure 1):
\[
\cos C = \cot (90^\circ - F) \cdot \cot (M + \delta)
\]
\[
\cos C = \tan F \cdot \cot (M + \delta)
\]
\[
\tan F = \cos C \cdot \tan (M + \delta)
\]

From the triangle $PngN$ (Figure 1):
\[
\omega = (\omega + F) - F
\]

The azimuth angle is reckoned from the elevated pole of the observer to $180^\circ$, E or W depending on the local hour angle.

Formulae 1 to 7 developed from these three triangles render a very simple tabulation of the values necessary for the computation of the altitude and azimuth.

**INFLUENCE OF VARIOUS ARGUMENTS IN THE FINAL RESULT OF THE COMPUTED ALTITUDE**

The changes in altitude due to a change of various arguments are determined by formulae 8 to 14b developed from Figures 3, 4 and 1.

*The change in altitude ($dV$) due to a small change in hour angle ($ds$) is derived from Figure 3.*

\[
\sin \omega = \frac{dV}{ds \cdot \cos \varphi}
\]
\[
dV = ds \cdot \cos \varphi \cdot \sin \omega
\]

In this case $dV$ represents the correction to the altitude due to a small change (minutes of arc) in hour angle and is written as $ks$.
\[
ks = ds \cdot \cos \varphi \cdot \sin \omega
\]
\[
ds \cdot \sin \omega = U1
\]
\[
ks = U1 \cdot \cos \varphi
\]
In the similar way the other altitude corrections are determined. \( k_C \), i.e. the correction to the altitude due to a small change (minutes of arc) of \( C \) is developed from Figure 1.

\[
dV = dC \cdot \cos (90^\circ - F) \cdot \sin 90^\circ \\
dV = dC \cdot \cos (90^\circ - F) 
\]

\( \cos (90^\circ - F) = i_C \), i.e. index of altitude correction due to a change of 1 minute of arc of \( C \).

\[
dV = dC \cdot i_C 
\]

\( k_C = dC \cdot i_C \) ....................................................9b

\( k_{\varphi} \), i.e. the correction to the altitude due to a small change (minutes of arc) of latitude (\( d\varphi \)) is developed from Figure 3.

\[
dV = d\varphi \cdot \cos \omega 
\]

\( \omega = i_{\varphi} \), i.e. index of altitude correction due to a change of latitude (\( d\varphi \)).

\[
dV = d\varphi \cdot \cos i_{\varphi} 
\]

\( k_{\varphi} = d\varphi \cdot \cos i_{\varphi} \) ....................................................10b

\( k_M\delta \), i.e. the correction to the altitude due to a small change (minutes of arc) of \( M + \delta \) is developed from Figures 4 and 1.

\[
dV = \cos V = \frac{d\delta}{dV} \\
d\delta = d(M + \delta) \cdot \cos V 
\]

\( d(M + \delta) \) is a small change in the value \( (M + \delta) \). From Figure 1 it is seen that small change in altitude due to a small change in \( (M + \delta) \) is influenced by \( \cos V \) in the same way as already seen in the formula 13 for declination.

\[
dV = d(M + \delta) \cdot \cos V 
\]

\( \cos V = i_M\delta \), i.e. index of altitude correction due to a change of 1 minute of arc of \( (M + \delta) \).
Tables Ki are based on these formulae (1 to 14b) and give the computed values of altitude and azimuth for all latitudes and all celestial bodies. The interpolations, rules for signs and logarithmic operations are avoided when the sight is worked using these tables.

The use of Tables Ki consists of the following:

(A) SOLUTION FOR A POSITION LINE FROM THE ASSUMED POSITION

1. Enter Table I with the local hour angle (s) and assumed latitude (πi), rounded to whole or half degrees, and pick out the values: ω + F (azimuth angle plus F), M and C (see extracts of these Tables on Fig. 7).

2. Label M with the sign according to the short remarks printed in the head of Table I (M is of the same name as latitude when the local hour angle is less than 90°; if the local hour angle is greater than 90°, M is of the contrary name).

3. Add algebraically M and declination (δ).

4. With the value M + δ thus found and the value C found in step 1 enter Table II, using the next lower whole or half degrees of these values, and find: F, V (altitude), iMδ and iC.

The Tables are constructed for every whole degree and half degree only, and for remaining minutes of arc for the values M + δ and C, Table IV will be used as interpolation Table, from which kMδ and kC can be taken out as corrections respectively.

The values iMδ and iC respectively represent the indices of corrections serving as the entering argument in Table IV. Signs for kMδ and kC are given at the head of Table II above iMδ and iC.

5. The azimuth angle (ω) is always found by subtracting F from ω + F. The azimuth angle is measured through 180°; label it N or S to agree with the latitude, and E or W to agree with the local hour angle.

If an extreme accuracy of the azimuth is desired, F may be interpolated by inspection. That will only be necessary for very high altitudes.

6. Enter Table IV with iMδ at the side, using the nearest tabulated value, and the remainder of minutes of arc of M + δ at the top and take out kMδ (the correction to the altitude for minutes of arc of M + δ). The Table is so arranged that it may be entered at the top with both whole numbers and tenths.

7. kC, i.e. the correction to the altitude for the remainder of minutes of arc of C, is to be found in the same way as kMδ in step 6, but entering Table IV with iC at the side and with minutes of arc of C at the top.

8. The computed altitude (Vr) is found by applying the correction kMδ and kC to the tabulated altitude (V) found in step 4. The signs for these corrections are printed at the head of Table II above iMδ and iC (already mentioned in step 4).

9. Find the intercept (ΔV) by taking the difference between the true altitude (Vp) and the computed altitude (Vr). If the true altitude is the larger, the intercept is towards (+). If the true altitude is the smaller, the intercept is away (−).
EXAMPLE 1

Finding the Ship's Observed Position from Simultaneous Observations of Two Celestial Bodies.

Solution from the Assumed Position by Tables K1.

21st February 1950, the GMT 1726 dead reckoning position of a ship was $\varphi_z = 34° 51,5'N$, $\lambda_z = 38° 06,4'W$. About this time the navigator observed Sun's lower limb with a marine sextant, as follows: Greenwich Mean Time $T_s = 17^{h}26^{m}40,8^s$, true altitude $V_p = 30° 51,6'$. 21 s. later on, i.e. at $T_s = 17^{h}27^{m}01,8^s$ the true altitude of the Moon was taken $V_p = 65° 27,3'$. Find the observed position (Pp), i.e. the observed latitude ($\varphi_p$) and the observed longitude ($\lambda_p$).

NOTE: Extracts of Tables K1 which are necessary for solution of this example are shown in Figure 7.

Solution of the Sun Sight:

$$V_p = 30°51,6'$$
$$T_s = 17^{h}26^{m}40,8^s$$
$$S = 78°13,9'$$
$$\lambda_i = 38°13,9W$$
$$s = 40°W$$
$$\varphi_i = 35°N$$

$$\omega + F = 154,3°$$
$$M = + 47°34,3'$$
$$C = 58°13,7'$$

$$F = 21,4$$
$$\omega = N 132,9°W$$

$$V = 30°17,6'$$
$$78,9 \text{ kM} = + 23,4$$
$$36,3 \text{ kC} = + 5,0$$

$$- V_r = 30°46,0'$$
$$V_p = 30°51,6'$$

$$\Delta V = + 5,6'$$
Solution of the Moon Sight:

\[ V_p = 65^\circ 27,3' \quad T_s = 17^h 27^m 01,8^s \quad S = 29^\circ 08,3' + 360 \]
\[ \delta = + 11^\circ 19,6' \quad \lambda_{iz} = - 38^\circ 08,8 \ W \]
\[ + 19,1 \quad s = 351^\circ \ 00,0 \]
\[ \delta = + 11^\circ 38,7' \quad s = 9^\circ \ E \]
\[ \varphi = 35^\circ \ N \]

\[ \omega + F = 174,8^\circ \]
\[ M = + 54^\circ 39,9' \quad C = 82^\circ 38,3' \]
\[ M + \delta = 66^\circ 18,6' \]

\[ F = 16,3 \quad V = 64^\circ 55,3' \]
\[ \omega = N \ 158,5^\circ \ E \quad 95,1 \ kM = + 17,7 \]
\[ 27,3 \ kC = + 2,3 \]
\[ \varphi = 65^\circ 15,3' \quad \Delta V = + 12,0' \]

Fig. 6.
**NEW METHODS OF SHIP POSITION FINDING**

**EXTRACTS FROM TABLES K₁**

<table>
<thead>
<tr>
<th>Tabl. I</th>
<th>$\delta$</th>
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<td>$0'$</td>
</tr>
<tr>
<td>1°</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>90°</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

Fig. 7.
(b) SOLUTION FOR A POSITION LINE FROM THE DEAD RECKONING POSITION

The computed altitude and azimuth for the dead reckoning position (Pz) are found in the same way as for the assumed position (Pi) mentioned before under heading A, except for the following:

In step A1: The local hour angle and the latitude are not to be rounded to a whole or half degree, and Table I is entered with the next lower whole or half degrees of these entering arguments.

In addition to step A8: The application of corrections kM8 and kC to the tabulated altitude found in Table II is not enough to find the computed altitude in this case. Corrections to the altitude for the remainders of minutes of arc of the local hour angle and the dead reckoning latitude are still to be applied. Table IV gives correction to the altitude (kφ) for minutes of arc of latitude when entering with the azimuth angle (ω) at the side, using the nearest tabulated value, and the minutes of arc of latitude (δ) at the top. The sign for this correction is printed near the entering azimuth angle. The correction for the remainder of minutes of arc of the local hour angle is found in Table III in the following way: enter the Table with the azimuth angle (ω) at the left hand side (using the nearest tabulated value) and the minutes of arc (whole numbers and tenths) of the local hour angle (δ) at the top and pick out the value U1; then enter the same Table with U1 (whole numbers and tenths) at the foot and the latitude (φ) at the right hand (using the nearest tabulated value) and take out ks, i.e. the correction to the altitude for minutes of arc of the local hour angle. The sign for this correction is always minus and is printed at the foot of the Table.
EXAMPLE 2

Working the Sight from the Dead Reckoning Position
by Tables $K_1$

In order to enable easier comparison between procedures of working sights from the assumed position and the dead reckoning position, in this example are used the same data (date and GMT of observation of the Sun, dead reckoning position and true altitude) as in example 1.

NOTE: Extracts of Tables $K_1$ which are necessary for solution of this example are shown on Figure 9.

Solution of the Sun Sight:

\[
\begin{align*}
V_p &= 30^\circ 51,6' \\
\delta &= -10^\circ 36,0' \\
s &= 78^\circ 13,9' \\
\lambda_z &= 38^\circ 06,4' \\
\phi_z &= 34^\circ 07,5'W \\
\theta + F &= 154,6^\circ \\
\omega &= N 132,5^\circ W \\
M + \delta &= 37^\circ 31,4' \\
V &= 31^\circ 04,9' \\
\lambda &= 78,5 \text{ km$\delta$} = + 1,1 \\
K &= 37,5 \text{ kmC} = + 0,3 \\
K_p &= -14,6 \\
K_s &= -4,6 \\
V_T &= 30^\circ 47,1' \\
V_p &= 30,51,6 \\
\Delta V &= + 4,5' \\
\end{align*}
\]

From Figure 8 it is seen, that the same position line is obtained with the intercept + 5,6 found in example 1 and laid off from the assumed position (Pi), and the intercept + 4,5 found in this example and laid off from the dead reckoning position (Pz).

Fig. 8.
### Tabl. I - $S$

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<th>$V$</th>
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<th>$iC$</th>
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### Tabl. III

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### Tabl. IV

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</table>

**Fig. 9.**
CONCLUSION ABOUT TABLES K₁

We are thus briefly acquainted with Tables K₁. In order to determine which is the best to use in actual practice, a comparison was made between Tables K₁ and a series of more than twenty other Tables of Celestial Navigation. The use of Tables K₁ is found to be the least involved, except American Tables H.O. 214, in the amount of work in turning pages, entering Tables, additions and subtractions, rules for signs, interpolation of Tables and interpretation of diagrams etc. Tables K₁ have, however, a greater advantage in having only 200 pages in place of 2400 used in Tables H.O. 214.

DIRECT SOLUTION OF THE SHIP'S OBSERVED POSITION

The method mostly used in the navigational practice of today is the Marcq de St. Hilaire Method. This method is eighty years old and during this relatively long period of time nothing of it has been changed except the way of computing the altitude and azimuth. However, the method of Marcq de St. Hilaire is an indirect one because when the altitudes and azimuths of two celestial bodies have been computed, it is necessary to make one additional procedure, either graphical or arithmetical, in order to obtain the coordinates of the observed position; therefore a direct method of computing the observed position would be welcome.

From simultaneous observations of two celestial bodies we obtain their Greenwich hour angles, declinations and true altitudes. From the spherical triangle PnJ₁J₂, with the difference of Greenwich hour angles (ΔS) of the observed two bodies (see Figure 10) and their declinations (δ₁ and δ₂) the angular distance between these two bodies (90°—Vx) and the angle A can be computed in the same way as the zenith distance and azimuth angle are computed from the astronomical triangle. Then from the spherical triangle ZJ₁J₂, with true altitudes V₁ and V₂ and angular distance 90°—Vx the angle B can be computed. This value B subtracted from A gives the angle v (called the parallactical angle). Finally, from the spherical triangle PnZJ₂ with the declination δ₁, true altitude V₂ and the angle v, the local hour angle (σ₂) and the latitude (φ) of the observer can be computed; combining the local hour angle with the Greenwich hour angle the observed longitude is determined.

Accordingly, the true position of a ship is determined without using a dead reckoning or assumed position. Attention is drawn to the fact that the coordinates of the observed position can be computed also by means of the parallactical angle of the first (eastern) body J₁. For the simplicity of the procedure it is advisable to always name the western body as J₂ and regarding this one to compute angles A and B, and furthermore with its parallactical angle to find the final result of the latitude and local hour angle. Exception of this occurs only in the case when western body is in the vicinity of the meridian; in this case eastern body has to be named as J₁. Using this principle the formulae and the scheme of work remain always the same.

Sights of one celestial body taken at different times may be also used for direct computation of the observed position. In this case the main principle is the same as already mentioned before for the simultaneous observations of two celestial bodies. The only difference would be that the true altitude from the first observation has to be corrected for the ship's run between the first and the second observation, i.e. the correction of the true altitude from the first observation for the difference of latitude and the difference of longitude of the ship's run.
The procedure of logarithmic computation of five unknown elements \((Vx, A, B, \varphi, \text{ and } s_2)\) in two spherical triangles is long and unsuitable for use in navigation, because up-to-date practical work on board requires fast working and simple methods of finding ship's position at sea. With the aim of making direct computation of the ship's observed position shorter and simpler I devised Tables \(K_{11}\).

**TABLES \(K_{11}\)**

*(Tables of Computed Latitudes and Local Hour Angles)*

By this Table method the direct solution of the ship's observed position can be carried out without computation of the auxiliary values \(Vx, A, B\) and \(\varphi\). The observed latitude and local hour angle are obtained directly from these Tables (see extracts in Figures 11 and 12) using true altitudes of two stars as entering arguments. Comparing the local hour angle with the Greenwich hour angle the observed longitude is determined.

Tabulation of these results was possible because the constant values of declination and difference of Greenwich hour angles (or difference of sidereal hour angles) are used. For changes in minutes of altitudes, declinations and difference
of Greenwich hour angles the corrections are applied by Multiplication Table using indices of corrections (ik and iks) given in the main Tables. Signs for these indices are given in the main Tables for the positive changes of the following arguments: true altitudes, declinations and difference of Greenwich hour angles. When changes in these arguments are negative, signs are opposite. Multiplication of the mentioned indices and changes in minutes of the corresponding arguments (with their signs) gives the corrections to the latitude and local hour angle. This multiplication is made by Multiplication Table which is similar to Table IV in Tables K1.

Tables K11 can be applied in the first place to the selected pair of stars because their relative positions differ only slightly. It is also possible that the Tables be compiled for two observations of the Sun or Moon (see extract in Figure 13, but the second observation must be made after a certain interval of time which is equal to that tabulated, e.g. 45 degrees of Greenwich hour angle difference. For small differences of this interval the correction of latitude and local hour angle will be made by Multiplication Table in the same way as for the differences in other arguments.

It would be necessary to tabulate two results for latitude and local hour angle, because two circles of position intersect in two points, but as they are usually very far apart it would not be difficult to decide which is the one in question.

It is quite clear that the disadvantage of these Tables is the bulk of the volume, but by the following arrangements it is made possible for it to be divided into several volumes for certain zones of latitudes similar to Tables H.O. 214.

At the beginning of Tables K11 one short Table comprehending only two sheets is supplied giving for each zone of 5° of latitude and for each 15° of local hour angle of the First Point of Aries the names of selected pairs of stars which are the most convenient for observation and computation, as well as the number of the page where these pairs of stars are tabulated. In this way for a certain volume it is not necessary to tabulate two results for latitudes and local hour angles, and tabulation can be reduced to only those altitudes which come into consideration for the selected volume, i.e. the selected zone of latitudes. Selection of stars which have to be used for observation for the given latitude and local hour angle of the First Point of Aries is already applied in the American Tables H.O. 249, Vol. I.

Similar short Table will be also introduced at the beginning of main Tables for the Sun or Moon tabulated for a certain zone of latitudes. From that short Table the difference of Greenwich hour angle used for tabulation and number of page where data are tabulated are taken out.

**EXAMPLE 3**

*Direct Computation of the Ship's Observed Position from Observations of Two Stars; Solution by Tables K11.*

On 6th February 1939, the GMT 1648 dead reckoning position of a ship was \( \varepsilon_z = 42° 13,2'N, \lambda_z = 18° 19,0'E. \) About this time the navigator observed Rigel and Markab as follows:

- **Rigel**: Greenwich Mean Time \( T_s = 16^h 45^m 23^s, \) true altitude \( V_ip = 31° 32,8', \) true azimuth \( \omega_p = 142°. \) The ship was on course \( K_p = 310°, \) speed 7 knots.

- **Markab**: \( T_s = 16^h 48^m 11,8', \) true altitude \( V_2 p = 31° 28,3'. \)

Find the latitude and longitude of the observed position (\( \varepsilon_p \) and \( \lambda_p \)) at the moment of the second observation.
Note:

1. Extracts of Tables $K_{11}$ necessary for solution of this example are shown in Figures 11 and 12.

2. Explanations of additional abbreviations used in this example are listed below:

- $K_{11}$ = True course.
- $D_1$ = Rhumb line distance.
- $Nm$ = Nautical mile.
- $N$ = North.
- $E$ = East.
- $R$ = Departure.
- $\Delta \phi$ = Difference of latitude.
- $\Delta \lambda$ = Difference of longitude.
- $\Delta s$ = Difference of local hour angle.
- $\Delta S \gamma$ = Difference of Greenwich hour angles of the First Point of Aries.
- $\Delta T_s$ = Difference of Greenwich Mean Time.
- $\omega_{p}$ = True azimuth.
- $V_{1p}$ = True altitude of the first celestial body.
- $V_{2p}$ = True altitude of the second celestial body.
- $k_{\phi}$ = Correction to the altitude for minutes of arc of latitude,
  
  $k_{\phi} = \Delta \phi \cdot \cos \omega$.

- $k_{s}$ = Correction to the altitude for minutes of arc of hour angle,
  
  $k_{s} = \Delta s \cdot \cos \phi \cdot \sin \omega$.

- $V_{1k}$ = True altitude from the first observation corrected to the moment of the second observation.
- $V_1$ = Altitude of the first celestial body.
- $V_2$ = Altitude of the second celestial body.

- $360^\circ-\alpha$ = Sidereal hour angle.
- $\Delta(360^\circ-\alpha)$ = Difference of sidereal hour angles.

- $\varphi$ = North latitude.
- $S$ = Greenwich hour angle; South.
- $s$ = Local hour angle; Seconds (of time).
- $W$ = West.
Correction of the first observation altitude to the moment of the second observation

Rigel: \( \varphi = 142^\circ \)  
Kp = 310\(^\circ\)  
D\(_1\) = 0.3 Nm  
\( \Delta \varphi = 0.2' N \)  
\( k_\varphi = -0.2 \)  
\( R = 0.2' W \)  
\( \Delta \lambda = \Delta \delta = +0.3' \)  
T\(_{s1}\) = 16h45m23.3s  
T\(_{s2}\) = 16h48m11.8s  
\( \Delta T_s = 2m48.5s \)  
\( \Delta S_\varphi = -42.3 \)  
\( \Delta \delta = -42.0' \)  
ks = +19.1

Taking out data for both stars from the Nautical Almanac we obtain the following data necessary for the solution by Tables K\(_{11}\).

Rigel: \( \psi_k = 31^\circ 51.7' \)  
Markab: \( \psi_2 = 31^\circ 28.3' \)

Difference between given and tabulated coordinates \( \delta_1 \) and \( \delta_2 \) is found only for \( \delta_1 \) in the amount of \(-0.1'\). That means, from Tables K\(_{11}\) with whole or half degrees of known altitudes of Rigel and Markab (\( \psi_k \) and \( \psi_2 \)) as the entering arguments we shall pick out tabulated latitude (\( \varphi \)) and local hour angle (\( \lambda_s \)), and adding to these values the corrections for the remainders of minutes of arc of these two altitudes and for \(-0.1'\) difference of declination of Rigel we obtain the observed latitude (\( \varphi_p \)) and local hour angle or Markab (\( \lambda_2 \)). Comparing the local hour angle \( \lambda_2 \) with the Greenwich hour angle \( \lambda_2 \) the observed longitude (\( \lambda_p \)) is determined.

From Tables K\(_{11}\) (see extracts in figures 11 and 12) we obtain:

For \( \psi_k = 32^\circ \) and \( \psi_2 = 31^\circ 30' \):  
\( \varphi = 41^\circ 57.5'N \)  
\( \varphi_p = 42^\circ 07.9'N \)  
\( s_2 = 60^\circ 47.1' \)  
\( s_2 = 60^\circ 47.2'W \)  
\( S_2 = 42^\circ 32.4' \)

Solution of the same example by the Marcq de St- Hilaire method gives the following result of the observed position: \( \varphi_p = 42^\circ 07.8'N, \lambda_p = 18^\circ 14.6'E \).
**Extract from Tables K11**

**V₂ MARKAB**  \(\delta₂ = +14°52.7'; \Delta(360°-\alpha) = 92°28.7'\)

<table>
<thead>
<tr>
<th>(V₁) RIGEL</th>
<th>(31°)</th>
<th>(31°30')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta₁ = -8°16.4')</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>31 30</td>
<td>42 51.1</td>
<td>61 17.0</td>
</tr>
<tr>
<td>32</td>
<td>42 18.4</td>
<td>61 23.9</td>
</tr>
<tr>
<td></td>
<td>-112 - 90</td>
<td>+23 - 122</td>
</tr>
</tbody>
</table>

**Fig. 11.**

**NOTE.** — 1. This extract from Tables K11, is computed for every whole degree and half degree of altitudes \(V₁\) and \(V₂\) of the mentioned bodies for 1st January 1939, i.e. for the given values of declinations \(\delta₁\) and \(\delta₂\) and difference of sidereal hour angles \(\Delta(360°-\alpha)\). For difference in minutes of arc between given and tabulated altitudes and changes in minutes of arc of \(\delta₁\), \(\delta₂\), and \(\Delta(360°-\alpha)\) in later months and years, the corrections of the tabulated results \(\varphi\) and \(s₂\) (observed latitude and local hour angle) are obtained from the Multiplication Table using indices of corrections \(k\varphi\) and \(k₂\) (see Figure 12).

2. The corrections being taken out from Multiplication Table entering with the indices of corrections \(k\varphi\) and \(k₂\) at the left side and with minutes of arc of the difference between given and tabulated values of the arguments \(V₁\), \(V₂\), \(\delta₁\), \(\delta₂\) and \(\Delta(360°-\alpha)\) at the top. When these differences (given minus tabulated) are positive the signs of corrections are the same as those for indices of corrections given in the main table; when these differences are negative the signs are contrary.
### EXTRACT FROM TABLES $K_{11}$

#### Multiplication Table

<table>
<thead>
<tr>
<th>$i$</th>
<th>$k$</th>
<th>$s$</th>
<th>$V_1'$</th>
<th>$V_2'$</th>
<th>$d_1'$</th>
<th>$d_2'$</th>
<th>$\Delta (360^\circ - \alpha)'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
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<td>0,6'</td>
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<td>0,6'</td>
</tr>
</tbody>
</table>

#### Figure 12

### EXAMPLE 4

**Direct Computation of the Ship's Observed Position from Observations of Two Stars; Solution by Tables $K_{11}$**

In this example the same data are taken as in example 3 with the only difference that the date of observation is 15 years later, i.e. 6th February 1954. In connection with this the coordinates of stars are changed and now read as follows:

Rigel: $\delta_1 = + 8^\circ 15,2' \ (\text{diff.} + 1,3'), \ 360^\circ - \alpha_1 = 281^\circ 54,6'$

Markab: $\delta_2 = + 14^\circ 57,5' \ (\text{diff.} + 4,8'), \ 360^\circ - \alpha_2 = 142^\circ 23,0, S_2 = 42^\circ 43,0'$

$$\Delta = 267 \ 31,6$$
$$360$$

$$\Delta (360^\circ - \alpha) = 92^\circ 28,4' \ (\text{diff.} - 0,3')$$
Altitudes remain unchanged: $V_1 = 31° 51,7'$ and $V_2 = 31° 28,3'$.

From Tables $K_{11}$ (see extracts in Figures 11 and 12) we obtain:
For $V_1 = 32°$ and $V_2 = 31° 30'$:

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$\varphi$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8,3'</td>
<td>1,7'</td>
<td>41° 57,5'N</td>
<td>60° 47,1'</td>
</tr>
</tbody>
</table>

For differences:

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\Delta (360° - \varphi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+101</td>
<td>+46</td>
<td>-52</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varphi_p$</th>
<th>$s_2$</th>
<th>$\lambda_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>42° 11,7'N</td>
<td>60° 50,6'W</td>
<td>18° 07,6'E</td>
</tr>
</tbody>
</table>

Solution of the same example by the Marcq de St. Hilaire method gives the following result of the observed position: $\varphi_p = 42° 11,6'N$, $\lambda_p = 18° 07,6'E$.

**EXAMPLE 5**

Direct Computation of the Ship's Observed Position from Sun Sights Taken at Different Times; Solution by Tables $K_{11}$.

31st January 1954, the GMT 0631 dead reckoning position of a ship was $\varphi_z = 33° 16,6'N$, $\lambda_z = 27° 40,5'E$. About this time the navigator observed the Sun's lower limb, as follows: Greenwich Mean Time $T_s = 06h31m16s$, true altitude $V_{1p} = 15° 01,9'$. Declination $\delta_1 = -17° 30,8'$, Greenwich hour angle $S_1 = 274° 27,2'$, true azimuth $\omega_p = 123,3°$.

The ship made good a distance of 52,7 nautical miles on course $K_p = 141°$ and at $T_s = 10h02m04s$ (Greenwich Mean Time) the Sun's lower limb was observed again, as follows: true altitude $V_{2p} = 39° 44,1'$; declination $\delta_2 = -17° 28,5'$, Greenwich hour angle $S_2 = 327° 08,9'$.

Find the latitude and longitude of the observed position ($\varphi_p$ and $\lambda_p$) at the moment of the second observation.

Note: Extracts of Tables $K_{11}$ necessary for the solution of this example are shown in Figure 13.
EXTRACT FROM TABLES $\kappa_{11}$

(for Sun Sights taken at Different Times)

$$\delta = -17^{\circ}30'; \quad \Delta S = 52^{\circ}30'$$

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$\varphi$</th>
<th>$i_k$</th>
<th>$\varphi$</th>
<th>$s_1$</th>
<th>$i_k$</th>
<th>$s_1$</th>
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</thead>
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<tr>
<td></td>
<td>$V_1k$</td>
<td>$V_2$</td>
<td>$\delta_1$</td>
<td>$\delta_2$</td>
<td>$\Delta S$</td>
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<td>15 30</td>
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<td>109</td>
<td>85</td>
<td>85</td>
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<tr>
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<td>32 49,8</td>
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<td>108</td>
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</table>

Multiplication Table

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<th>$\varphi$</th>
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<td>22'</td>
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</tbody>
</table>

Fig. 13.
Correction of the first observation altitude to the moment of the second observation

\[
S_1 = 274° 27,2' \\
S_2 = 327° 08,9'
\]

\[
\Delta S = 52° 41,7' \\
K_p = 141°, D_l = 52,7 \text{ Nm}
\]

\[
\Delta \phi = 41,0'S \quad \text{......} \quad k_p = + 22,5 \\
R = 33,2'E \\
\Delta \lambda = \Delta s = -39,5' \quad \text{......} \quad k_s = + 27,6
\]

\[
V_{1k} = 15° 52,0'
\]

From Tables K_{11} (see extracts in Figure 13) we obtain:

For \( V_{1k} 15°30' \) and \( V_2 39°30' \):

\[
\phi = 32° 45,9'N \\
\lambda_p = 123,3°
\]

\[
V_{1p} = 15° 01,9' \\
V_{2p} = 15° 30,9'
\]

\[
\text{For differences:} \\
V_{1k} + 22' \times x + 13 = + 2,9 \\
V_2 + 14,1' \times x - 109 = - 15,4 \\
\delta_1 - 0,8' \times x - 10 = + 0,1 \\
\delta_2 + 1,5' \times x + 108 = + 1,6 \\
\Delta S + 11,7' \times x + 10 = + 1,2
\]

\[
\phi_p = 32° 36,3'N \\
\lambda_p = 28° 23,1'E
\]

In this case the hour angle of the first observation (\( s_1 \)) is tabulated in Tables K_{11} because the local hour angle of the second observation is small (in the vicinity of the observer's meridian).

Solution of the same example by the Marcq de St. Hilaire method gives the following result of the observed position: \( \phi_p = 32° 36,5'N, \lambda_p = 28° 23,1'E. \)

**COMPARISON BETWEEN THE METHODS OF MARCQ DE ST. HILAIRE AND DIRECT SOLUTION OF THE SHIP'S OBSERVED POSITION**

Taking into consideration that the position lines can be reckoned by the Short Method Tables and that there is a possibility of determining the observed position by plotting the position lines on the Mercator Plotting Sheet or directly on the chart, we may generally say that for the present, however, Marcq de St. Hilaire's method is shorter.

When Tables K_{11} for finding ship's observed position are used, the computation is so much reduced that finding results of the observed latitude and local hour angle becomes in some measure similar to the method of finding results of computed
altitudes and azimuths from the dead reckoning position by Tables H.O. 214. The reason for making a comparison between these Tables is that they represent the maximum of achievement in efforts to make the procedure of computing ship’s observed position by the direct method and Marcq de St. Hilaire shorter and simpler. It is also taken into consideration that whereas Tables K₁₁ are tabulated only for selected celestial bodies, Tables H.O. 214 enable the computation of altitude and azimuth of all bodies. Furthermore the work by Tables K₁₁ requires more corrections than by Tables H.O. 214, although Tables K₁₁ need no special graphical or numerical solution for obtaining the coordinates of the observed position from the computed position lines. Moreover, in case two stars are observed, in computing the observed position by Tables K₁₁, only Greenwich hour angle of one star need to be found.

The methods of computing the intercept and azimuth have been improved over a period of eighty years. Here also an attempt at the improvement of the computation of a position line is given with the aim to make it faster and simpler. Tables K₁, which might be classed among Tables of Computed Altitude and Azimuth, are outstanding in this respect. Whether it will be possible to carry out further improvement of the direct method of solution of the ship’s observed position, remains to be seen.

Note: More details about these Tables and also some modifications of Tables K₁ (e.g. Tables K₅ reducing 200 pages of Tables K₁ to 14 sheets only) may be found in my book published at the end of May 1955 by the Hydrographic Institute of the Yugoslav Navy under the same title, which in Yugoslav language reads: « Nove metode astronomskog odredjivanja pozicije broda ».

Tables K₁ will be published at the end of 1957 (200 pages, hard cover, size over all 9.4 by 11.8 inches) containing extensive explanation in English and so enabling it to be used by the mariners of other countries.