SHALLOW-WATER TIDES

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1. — INTRODUCTION

The purpose of the present work is to describe the theoretical basis and method of practical application of computations with regard to overall corrections of tidal predictions obtained by the British Admiralty method.

Such corrections, however, are determined in terms of harmonic constants arising in shallow water (shallow-water components), and these are derived from the theoretical investigation of the propagation of progressive waves in canals of simple geometrical aspect.

It was therefore apparent to the writer that a complete theoretical account might appropriately be given herein. This decision was reached in view of the difficulty which might confront the reader, and in fact experienced by the writer, in obtaining a comprehensive bibliography on the development of shallow-water components, which, in our opinion, are not altogether satisfactorily derived in the Admiralty Manual of Tides.

We may anticipate by saying that the theoretical semi-amplitudes of the waves derived herein have been checked against the values supplied by RAUSHELBACH and reproduced in Dr. DOODSON'S *article entitled The Analysis of Tidal Observations,* which appeared in *Philosophical Transactions,* Vol. 227, Series A, pp. 223-279. The agreement obtained is almost perfect.

In order that the present article may not be restricted to a limited number of experts, it has been given an elementary character.

Acknowledgment is hereby made to Dr. DOODSON for his kindness in granting us an interview in July 1950 at the Tidal Institute. A large part of this work is in fact based on data that was made available to us on that occasion.

2. — GENERAL ASPECT OF PROBLEM

The development in harmonic series of the elevation of the equilibrium tide supplies all the important waves that must be obtained at any place as the exclusive result of the forces of celestial attraction.

In shallow water, particularly in estuaries and inlets, such waves are subject to distortions of a hydrodynamical nature that may altogether conceal the regular pattern of the astronomical tide.

Careful study of the apparent irregularities that occur shows that these may be represented as resulting from the action of new harmonic constituents of hydrodynamical origin. It may therefore readily be seen that the study is a complex one. We may however reduce its difficulty if we assimilate the tide-wave taken as a whole to a progressive wave analysed as a special case of the flow of fluids, by using Lord Rayleigh's device and considering the wave surface as stationary, while the water flows in the opposite direction to the direction of propagation of the wave.

It should also be remarked that in estuaries and inlets, the oscillations most frequently occurring are of the *stationary* type. They may be considered as the resultants of the reflection of a progressive wave on the inner barrier delimiting the bay or hydrographic figure.

Mathematics in their present status are as yet powerless to solve satisfactorily the problem of stationary oscillations. But in shallow-water areas, friction decreases the reflection of progressive waves on the inner barrier to such a degree that the motion of such waves can be assumed to occur in a canal of unlimited extent, and this constitutes one of the basic assumptions of the theory described in this article.

It may be added that the present study definitely supplies the shallow-water components that should be considered during analysis. Experience has already shown that such components may be used in any and all cases.

3. — PROGRESSIVE WAVE THEORY

When we throw a stone into the unruffled waters of a lake, we observe the formation of ripples which originate at the point of impact and spread in the shape of circles. As a result of friction, the oscillations become weaker until they vanish altogether. This is the standard type of progressive wave.

If we apply to such oscillations a perfectly coordinated force, i.e. synchronized with the oscillations generated and strong enough to counteract friction, these will be maintained in amplitude and in their original form. In this case, we may consider the oscillations as resulting from the translation of a free progressive wave whose motion is unaffected by friction and which is the seat of a constant transformation of potential energy into kinetic energy, and vice versa.

Let us now examine the tidal phenomenon in estuaries and inlets. It will readily be understood that oscillations occurring therein are caused and maintained by the oceanic tide, while the local action of the forces of celestial attraction are practically negligible.

For the horizontal component of the luni-solar perturbing force responsible for the origin of the tides to produce any appreciable effect, it must act on seas of great extent and depth, in order that the lateral displacement of large masses of water may appreciably raise the level of the sea at those points towards which flow occurs, and lower it at other points. It is therefore legitimate, if we consider the tide at the mouth of a canal as a source of energy counteracting friction, to assume that the tide-wave entering the canal is a free and progressive wave.

Since we are not dealing with the problem of reflection of the progressive wave, we must consider the canal in which it is propagated as being of *indefinite* extent, an assumption which, as we have already noted, actually applies in a large number of practical cases.

The speed of propagation of a progressive wave in a canal may be more or less approximately determined by measuring the time-interval elapsing between High Water at the entrance (bar) and any point in the canal x kilometres beyond. As soon as we know the speed of propagation, we can compute the length of the wave by measuring the time-interval between two High Waters (crests of the wave) at an identical point in the canal. The length of the wave is equal to vT, which is much greater than the depth of the canal. A *long wave* is therefore involved.

A study of the free motion of a progressive wave in a canal with vertical sides, a rectilinear axis, a horizontal bottom, and constancy of width and depth enables determination of the speed of propagation v of the wave in terms of depth.

Figure 3A shows the profile of a progressive wave propagated in an ideal canal such as the one described above.

It is clear that in cross-section BE, where the figure represents Low Water, the water-level will increase until High Water is reached. This will of course only occur through the displacement of the mass of liquid in the shape of *currents*, which result from the oscillatory movement proper. In order to comprehend fully the formative process of such currents, let us assume that the depth h in the canal is of very great extent, and that owing to the extreme length of the wave as compared with the semi-amplitude of the vertical oscillation, the currents have practically no vertical components. Let us also assume that the profile of the wave, in the present instance, is rigid and incapable of deformation, and that an observer travelling with the wave is aware of events occurring beneath him in the various sections of the wave as it progresses, in the direction shown by the arrow, at speed v_0. Under such imaginary circumstances, it is readily apparent that the process taking place may be assimilated to the flow of water within a tube represented in longitudinal section in Figure 3A.

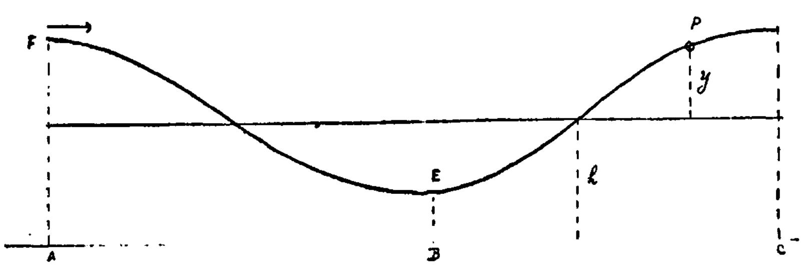


Fig. 3A

If the speed of propagation is constant, it is clear that the flow occurs at a constant rate, which means that the volume of water flowing during the unit of time through all cross-sections of the wave is constant. Since the areas of the cross-sections are different, the flow will only be constant if the rate varies from one cross-section to the next. The observer will therefore be able to check, by measuring the rate of flow at various points of the wave-profile, that in the cross-sections of maximum area the rate of flow will be minimum (crests of the wave), and maximum in the cross-sections of minimum area (troughs of the wave).

Therefore, if the profile of the wave moves at a rate equal to v , it is clear that the differences between this latter value and the values ascertained by the observer will be *currents*, which can be measured by an additional observer stationed on the bank of the canal.

By taking v' as the speed of the current, the rate of flow is $-(v_0 - v')$.

In view of the constant amount of flow, in the case of a canal of width b we can put:

$$-b(h + y)(v_0 - v') = \text{Constant} \qquad (3\text{ a})$$

in which $b(h + y)$ is the area of the canal cross-section corresponding to point P located at height y above mean level.

Owing to the condition expressed by (3 a), as regards the cross-sections corresponding to the points where $y = O$, we can put:

$$- bhv_0 = \text{Constant} \qquad (3\text{ b})$$

since where $y = O$ (absence of tide), the current is necessarily nil.

Let as now assume that the constant is identical in the expressions (3 a) and (3 b). By equating and extracting the value of v', we get:

$$v' = v_0 \frac{y}{h + y} = v_0 \frac{y}{h}(1 + \frac{y}{h})^{-1}$$

But as y is small in relation to h, we may apply the binomial theorem and disregard in the result all powers of $\frac{y}{h}$ above the first power. Whence:

$$v' = v_0 \frac{y}{h} \qquad (3\text{ c})$$

This expression may be regarded as correct for pure progressive waves, which can only exist at greater depths. From (3 c), the inference may likewise be drawn that the current is maximum and is propagated in the same direction as the wave at High Water (y max. positive), that it is nil when $y = O$, and maximum and opposite to the direction of propagation of the wave at Low Water (y max. negative).

We may now examine the case of any progressive wave propagated in a canal of which the depth h is no longer such that y may be considered small with reference to it.

We must then first consider the potential and kinetic energy of a particle located in the wave-profile, say at P (Fig. 3A).

Let us assume as we did in the previous case that motion of the particles occurs within a tube whose shape may be regarded as unvarying during a relatively short time-interval.

If the rate of flow is shown by—U, the potential energy of a particle of unit mass located in the wave-profile at P is

$$g(h + y)$$

while the kinetic energy is

$$\frac{1}{2} U^2$$

Now according to the law of conservation of energy, each increment of kinetic energy gives rise to an equivalent decrement of potential energy and vice versa, which is mathematically expressed by:

$$-d \frac{U^2}{2} = d \left[g(h + y) \right]$$

whence

$$U \frac{d_U}{dy} = - g$$

This differential equation cannot be integrated since it only expresses the condition which must be fulfilled by the motion of a particle of unit mass located on the wave-surface.

It therefore becomes necessary to lay down a new condition which must be fulfilled by the motion of all particles in the same cross-section of the wave. Let us assume for the purpose that the profile of the wave remains rigid and undeformed during a relatively brief interval of time. We are then entitled to assume that the amount of flow is constant, and, as in the special case described above, we may put:

$$-\, b\,(h + y)\, U = \text{Constant}$$

whence we readily obtain

$$-\frac{d_U}{dy} = \frac{U}{h+y}$$

By substituting this expression in (3 d), we get

$$U = \sqrt{g\,(h+y)} \tag{3 e}$$

This is the value for the speed of propagation of the wave *with respect to the water,* since U includes the current whose speed is represented by *y'*.

The expression (3 e) may hence be written as

$$v - v' = \sqrt{g\,(h+y)} \tag{3 f}$$

in which v is the speed of propagation *with respect to the bottom of the canal.*

The expression (3 f) representing the algebraic difference between the two speeds, it is clear that the expressions defining v' and v must be of the same nature as (3 f). v' may hence be represented by an expression of the form

$$v' = A\sqrt{g\,(h + y)} + B \tag{3 g}$$

where A and B are constants to be determined. In order to eliminate B directly, we need only express it in terms of A, as in the case of $y = O$ (absence of tide), we of course get $v' = O$, whence $B = -A\sqrt{gh}$.

By substituting this value in (3 g), we get

$$v' = A\sqrt{g\,(h+y)} - A\sqrt{gh} \tag{3 h}$$

and by substituting in (3 f)

$$v = (A+1)\sqrt{g\,(h+y)} - A\sqrt{gh} \tag{3 i}$$

The constant A may at present be determined in consideration of the special case of pure progressive waves, which, as we previously noted, are distinguished by the fact that the speed of propagation is identical at all points on their surface. This being true, this speed may be obtained for points where $y = O$, thus changing (3 i) into

$$v_o = \sqrt{gh} \tag{3 j}$$

This expression will only be valid for points on the wave surface at which y is very small compared with *h*, i.e. at greater depths. If $A\sqrt{gh}$ is brought out in expression (3 h), we get

$$v' = A\sqrt{gh}\left[\,(1 + y/h)^{\frac{1}{2}} - 1\,\right]$$

Let us apply the binomial theorem and retain the term of the first order in y/h. In accordance with (3 j) we shall have

$$v' = v_o \frac{Ay}{2h}$$

Finally by equating this expression to (3 c), we immediately obtain $A=2$, and (3 i) becomes

$$v' = 2\sqrt{g(h+y)} - 2\sqrt{gh} \qquad (3\,k)$$

which, by substitution in (3 f) supplies the expression

$$v = 3\sqrt{g(h+y)} - 2\sqrt{gh} \qquad (3\,m)$$

a result arrived at by De Saint-Venant by means of the differential equations of hydrodynamics.

In the usual cases, the relationship y/h is small enough to allow (3 m) to take the form

$$v = \sqrt{gh}\left[\, 3\left(1 + \frac{y}{h}\right)^{\frac{1}{2}} - 2 \,\right] \qquad (3\,n)$$

and enable the series development of $(1+y/h)$ following the powers of $\frac{y}{h}$, terms up to the second order being retained in the result. We thus get:

$$v = v_o\left(1 + \frac{3y}{2h} - \frac{3y^2}{8h^2}\right) \qquad (3\,o)$$

Examination of this expression shows that the principal term in y/h is positive, indicating that at the crest of the wave (y max. positive), the speed of propagation is maximum, while in the trough (y max. negative) the speed is minimum. Under these conditions, it is clear that for any point in the canal penetrated by a progressive wave High Water will *gain* and Low Water *lose*, i.e. the time of rise will be less than the time of fall.

A typical instance of this interesting phenomenon occurs in Macapa harbour, in the channel north of the mouth of the Amazon (Fig. 3 b), in which the interval between HW and LW is about 8 hours, whereas the LW to HW interval is around 4 hours.

MACAPA

MAREGRAM - AUGUST 1952

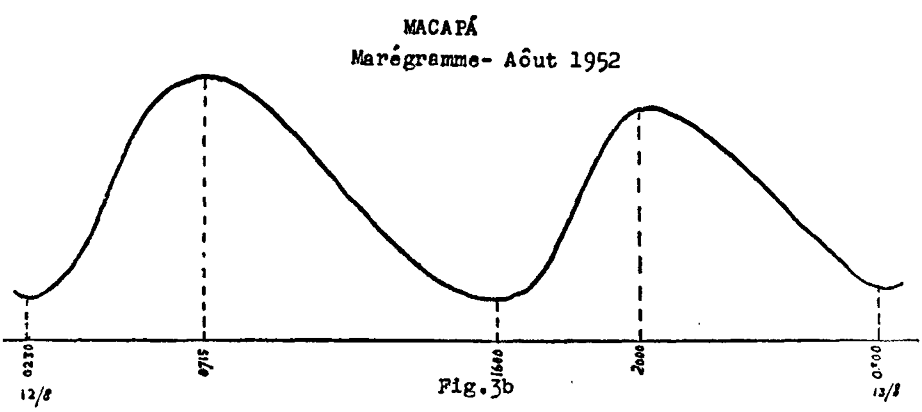


Fig.3b

4. — DERIVATION OF SHALLOW-WATER COMPONENTS

Let us assume that at the mouth of the canal the tidal elevation with respect to mean level is given at time t by

$$y = \Sigma R \cos r$$

If such is the elevation of a point of the progressive wave being propagated within a canal, and if v is the speed of propagation at that point of the wave, it will readily appear that such elevation will occur at a spot x kilometres away from the entrance bar, x/v hours later. Thus, if q generically expresses the speed of any component, it will readily be understood that at time t the elevation of the tide at that point is given by he formula

$$y = \Sigma R \cos \left(a - q \frac{x}{v}\right) \qquad (4\,a)$$

But from (3 a) we derive:

$$\frac{1}{v} = \frac{1}{v_o}\left(1 + \frac{3y}{2h} - \frac{3y^2}{8h^2}\right)^{-1}$$

By applying the binomial theorem and retaining as previously only terms up to the second order, we get:

$$\frac{1}{v} = \frac{1}{v_o} - \frac{3\,y}{2\,v_o\,h} + \frac{21\,y^2}{8\,v_o\,h^2}$$

By only taking one component of the summation expressed by (4 a), and substituting the value of 1/v, we get:

$$y_c = R \cos \left[r - q\frac{x}{v_o} + q\left(\frac{3\,xy}{2\,v_o\,h} - \frac{21\,xy^2}{8\,v_o\,h^2}\right)\right]$$

$$= R \cos \left(r - q \frac{x}{v_o}\right) \cos q \left(\frac{3\,xy}{2\,v_o h} - \frac{21\,xy^2}{8\,v_o\,h^2}\right) -$$

$$- R \sin \left(r - q \frac{x}{v_o}\right) \sin q \left(\frac{3\,xy}{2\,v_o\,h} - \frac{21\,xy^2}{8\,v_o\,h^2}\right) \qquad (4\,b)$$

But if we wish to retain the y/h terms only as far as the second order, we may put:

$$\cos q \left(\frac{3\,xy}{2\,v_o\,h} - \frac{21\,xy^2}{8\,v_o\,h^2}\right) = 1 - q^2 \frac{9\,x^2\,y^2}{8\,v^2\,h^2}$$

$$\sin q \left(\frac{3\,xy}{2\,v_o\,h} - \frac{21\,xy^2}{8\,v_o\,h^2}\right) = q \left(\frac{3\,xy}{2\,v_o\,h} - \frac{21\,xy^2}{8\,v_o\,h^2}\right)$$

whence, by substitution in (4 b):

$$y_c = R \cos \left(r - q \frac{x}{v_o}\right) - q \frac{3\,xy}{2\,v_o\,h} R \sin \left(r - q \frac{x}{v_o}\right) -$$

$$- q^2 \frac{3\,xy^2}{8\,v^2\,h^2} \left[3xR \cos \left(r - q \frac{x}{v_o}\right) - 7 \frac{v_o}{q} R \sin \left(r - q \frac{x}{v_o}\right)\right]$$

or putting

$$3\,x = k \cos \theta \tag{4 c}$$

$$\frac{7\,v_0}{q} = k \sin \theta \tag{4 d}$$

we get:

$$\begin{aligned} y_0 = {} & R \cos \left(r - q \frac{x}{v_0}\right) - \\ & - \frac{3\,xy}{2\,v_0 h}\, qR \sin \left(r - q \frac{x}{v_0}\right) - \\ & - \frac{3\,kxy^2}{8\,v_0{}^2 h^2}\, q^2 R \cos \left(r - q \frac{x}{v_0} + \theta\right) \end{aligned} \tag{4 e}$$

in which k and θ are taken from (4 c) and (4 d) which supply:

$$k = \sqrt{9\,x^2 + \frac{4\,9\,v_0{}^2}{q2}} \tag{4 f}$$

$$\theta = \text{arc tan} \frac{7\,v_0}{3\,qx} \tag{4 g}$$

q and v_0 are constants, x being the distance to the canal entrance. It is logical, therefore, that k and θ should be constant for each cross-section of the canal.

The expression (4 e) may undergo yet another small transformation, as, since only the semi-diurnal components are included in the y/h term of the second order, we can substitute the hourly speed of M_2 for the value of q, in radians, which is nearly equal to $1/2$. Thus, if we consider all the components, (4 e) will be transformed as

$$\begin{aligned} y = {} & \Sigma\, R \cos \left(r - q \frac{x}{v_0}\right) - \\ & - \frac{3\,xy}{2\,v_0\,h}\, \Sigma\, qR \sin \left(r - q \frac{x}{v_0}\right) - \\ & - \frac{3\,kxy^2}{32\,v_0{}^2 h^2}\, \Sigma\, R \cos \left(r - q \frac{x}{v_0} + \theta\right) \end{aligned}$$

The value of y appearing in the second member of this formula may be replaced with no appreciable error by the approximate value given by the first term of the second member; by putting:

$$\Sigma\, R \cos \left(r - q \frac{x}{v_0}\right) = y_0 \tag{4 h}$$

$$\Sigma\, qR \sin \left(r - q \frac{x}{v_0}\right) = z \tag{4 i}$$

$$\Sigma\, R \cos \left(r - q \frac{x}{v^0} + \theta\right) = w \tag{4 j}$$

we may write:

$$y = y_0 + \frac{3\,x}{4\,v_0\,h} (-2\,y_0\, z) + \frac{3\,kx}{32\,v_0{}^2 h^2} (-y_0{}^2\, w) \tag{4 k}$$

By now examining the joint action of two components of semi-amplitude A and B, and whose respective phases are a and b, from (4 h)-(4 j) we derive:

$$y_o = A \cos a + B \cos b$$
$$—z = q\ \ A \cos (a + 90°) + q\ \ B \cos (b + 90°)$$
$$—w = A \cos (a + \theta + 180°) + B \cos (b + \theta + 180°)$$

or again, by taking into consideration the general trigonometric expressions:

$$\cos p \cos q = \frac{1}{2} \cos (p + q) + \frac{1}{2} \cos (p — q) \qquad (4\ l)$$

$$\cos ^2p = \frac{1}{2} + \frac{1}{2} \cos 2p \qquad (4\ m)$$

we get:

$$y^2 = \frac{A^2}{2} + \frac{B^2}{2} + \frac{A^2}{2} \cos 2a + \frac{B^2}{2} \cos 2b + A\ B \cos (a + b)$$
$$+ AB \cos (a — b) \qquad (4\ n)$$

By obtaining the products ($—2y_o\ z$) and ($—y_o{}^2\ w$), and applying the general expression (4l), we obtain a series of new harmonic components arising from the joint action of A and B. Table 4-I supplies the phase angle expressions and the cosine factors of such angles (semi-amplitudes). The speeds of the new components are obtained by substituting q_a and q_b in degrees respectively for a and b.

TABLE 4-I

$—2\ y_o\ z$		$—y_o{}^2\ w$	
Angle	Cosine factor	Angle	Cosine factor
$2a + 90°$	$q_a\ A^2$	$a + \theta + 180°$	$A^{3/2} + AB^2$
$a + b + 90°$	$(q_a + q_b)\ AB$	$a — \theta — 180°$	$(A^{3/2} + AB^2/2$
$2b + 90°$	$q_b\ B^2$	$b + \theta + 180°$	$B^{3/2} + A^2B$
$a — b + 90°$	$(q_a — q_b)\ AB$	$b — \theta — 180°$	$(B^{3/2} + A^2B)/2$
		$3a + \theta + 180°$	$A^{3/4}$
		$2a + b + \theta + 180°$	$3\ A^2B/4$
		$2b + a + \theta + 180°$	$3\ AB^2/4$
		$3b + \theta + 180°$	$B^{3/4}$
		$2a — b + \theta + 180°$	$A^2B/2$
		$2a — b — \theta — 180°$	$A^2B/4$
		$2a — a + \theta + 180°$	$AB^2/^2$
		$2b — a — \theta — 180°$	$AB^{2/4}$

As the arguments of the components in brackets merely differ insofar as the constant part is concerned, if we set them down in the general form:

$$P \cos [\alpha + (\theta + 180°)]$$
$$Q \cos [\alpha — (\theta + 180°)]$$

we may combine them in a single expression represented by:

$$R \cos (\alpha + \theta') = P \cos [\alpha + (\theta + 180°)] + Q \cos [\alpha — (\theta + 180°)]$$
$$= P \cos (\theta + 180°) \cos \alpha — P \sin (\theta + 180°) \sin \alpha +$$
$$+ Q \cos (\theta + 180°) \cos \alpha + \sin (\theta + 180°) \sin \alpha$$

whence

$$R \cos(\alpha + \theta') = (P+Q)\cos(\theta + 180^\circ)\cos\alpha - (P-Q)\sin(\theta + 180^\circ)\sin\alpha$$

In order that the values of R and θ' may satisfy this equation, we need only put:

$$(P + Q)\cos(\theta + 180^\circ) = R\cos\theta'$$
$$(P - Q)\sin(\theta + 180^\circ) = R\sin\theta'$$

whence we easily derive:

$$R = \sqrt{2(P^2 + Q^2)}$$

$$\theta' = \text{arc tan}\,\frac{(P - Q)\sin\theta}{(P + Q)\cos\theta}$$

Let us again examine Table 4-I. We note that for the components in brackets we invariably obtain the identity:

$$P = 2Q$$

and

$$R = \frac{P\sqrt{10}}{2} \qquad (4\,o)$$

$$\theta' = \text{arc tan}\,\frac{\sin\theta}{3\cos\theta} \qquad (4\,p)$$

Observing that q has been taken as equal to 1/2, from the expression (4 g) we derive:

$$\theta' = \text{arc tan}\,\frac{14\,v_o}{9\,x} \qquad (4\,q)$$

It is of interest to note that both the value of θ' and that of θ solely depend on distance x from the canal entrance.

The expression (4 o) enables computation of the cosine factors of all the bracketed terms in Table 4-I, whose new arguments will all take the form $(\alpha + \theta')$. In this way we obtain Table 4-II:

TABLE 4-II

$-2y_o z$		$-y_o^2 w$	
Angle	Cosine factor	Angle	Cosine factor
$2a + 90^\circ$	$q_a A^2$	$a + \theta'$	$(A^{3/2} + AB^2)\sqrt{10}/2$
$a + b + 90^\circ$	$(q_a + q_b) AB$	$b + \theta'$	$(B^{3/2} + A^2B)\sqrt{10}/2$
$2b + 90^\circ$	$q_b B^2$	$3a + \theta + 180^\circ$	$A^{3/4}$
$a - b + 90^\circ$	$(q_a - q_b) AB$	$2b + a + \theta + 180^\circ$	$3\,AB^{2/4}$
		$3b + \theta + 180^\circ$	$B^{3/4}$
		$2a - b + \theta'$	$A^2B\sqrt{10}/4$
		$2b - a + \theta'$	$AB^2\sqrt{10}/4$

The introduction of an additional component of semi-amplitude C and phase c would cause the introduction in $-2\,y_0\,z$ and $-y_0{}^2\,w$ of A and C terms and B and C terms, which would be identical with the A and B terms already known, and in addition to these, of ABC terms in $-y_0{}^2\,w$ as follows:

TABLE 4-II (*continued*)

$-y_0{}^2\,w$	
Angle	Cosine factor
$a + b + c + \theta + 180°$	$3\,ABC/2$
$a + b - c + \theta'$	$ABC\sqrt{10}/2$
$a + c - b + \theta'$	$ABC\sqrt{10}/2$
$b + c - a + \theta'$	$ABC\sqrt{10}/2$

By using Table 4-II, we can tabulate all the shallow-water components normally obtained through the analysis of a year's observations.

In the case of simultaneous action of M_2 and S_2, we would have to replace A, a, q, B, b and q_b by the following values:

M_2	S_2
$A = fH$	$B = H$
$a = V + u - g$	$b = V - g$
$qa = 0.5059$ rad./h	$qb = 0.5236$ rad./h

In practice, however, the g values for shallow-water components usually are not equivalent to the results of g operations for M_2 and S_2 with the constants 90°, θ' and $(\theta + 180°)$ owing to the substitution for a and b of their values. We shall therefore restrict ourselves to determination of the astronomical arguments of the new components. We shall thus take:

$$a = \text{astr. arg. } M_2 = M_2$$
$$b = \text{astr. arg. } S_2 = S_2$$

For similar reasons, we shall obtain the semi-amplitudes with respect to the new components by putting:

$$A = f(M_2)\,.\,0.908$$
$$B = 0.430$$

which are the equilibrium-tide coefficients.

We shall follow this general pattern in deriving the joint effects of any other components when the purpose to be attained is the deduction of relative theoretical influences of shallow-water constants.

We should finally mention that in the following table terms whose arguments are a, b, etc. added to θ' are not included, since they represent disturbances in the basic components themselves and cannot be separated from them. If we consider all the semi-diurnal components simultaneously, we get:

Affected component	Arguments	Cosine factor
A	$a + \theta'$	$\left(\frac{A}{2} \Sigma R^2 - \frac{A^3}{4}\right)\sqrt{10}$
B	$b + \theta'$	$\left(\frac{B}{2} \Sigma R^2 - \frac{B^3}{4}\right)\sqrt{10}$

. . . .

. . . .

which shows that the shallow depth may alter the semi-amplitudes of the basic components.

The foregoing indications will readily show how the following table was drawn up, in which the symbols of the actual shallow-water components explain the combinations producing them.

TABLE 4-III

Shallow-Water Components

General factor : $3x/4v_0h$			General factor : $\frac{3x}{32\,v_0^2\,h^2}\sqrt{9x^2+196v_0^2}$		
Symb	Astr. arg.	Cos. fact. f	Symb.	Astr. arg.	Cos. fact. f
M_4	$2\,M_2$	0.40 $f(M_2)^2$	$2MS_6$	$2M_2+S_2$	0.26 $[f(M_2)]^2$
MS_4	$M_2 + S_2$	0.40 $f(M_2)^2$	M_6	$3M_2$	0.19 $[f(M_2)]^3$
MN_4	$M_2 + N_2$	0.16 $f(M_2)^2$	$2SM_6$	$2S_2 + M_2$	0.12 $[f(M_2)]_2$
MK_4	$M_2 + K_2$	0.11 $f(M_2) . f(K_2)$	$2MN_6$	$2M_2+N_2$	0.11 $[f(M_2)]^3$
S_4	$2\,S_2$	0.09 1	MSN_6	$M_2+S_2+N_2$	0.10 $[f(M_2)]^2$
SN_2	$S_2 + N_2$	0.08 $f(M_2)$	MSK_6	$M_2+S_2+K_2$	0.07 $f(M_2) . f(K_2)$
			$2MK_6$	$2M_2+K_2$	0.07 $f(M_2) . f(K_2)$
MK_3	$M_2 + K_1$	0.37 $f(M_2) . f(K_1)$	S_6	$3S_2$	0.02
MO_3	$M_2 + O_1$	0.26 $f(M_2) . f(O_1)$			
SK_3	$S_2 + K_1$	0.17 $f(K_1)$	$2MS_2$	$2M_2+S_2$	0.26 $[f(M_2)]^2$
SO_3	$S_2 + O_1$	0.12 $f(O_1)$	$2SM_2$	$2S_2 + M_2$	0.13 $f(M_2)$
M_3	1,5 M_2	0.01 $f(M_2)$	MNS_2	$M_2+N_2—S_2$	0.11 $[f(M_2)]^2$
			MSN_2	$M_2+S_2+N_2$	0.11 $[f(M_2)]^2$
OP_2	$O_1 + P_1$	0.03 $f(O_1)$			
OK_2	$K_1 + J_1$	0.01 $f(K_1) . f(J_1)$			
MP_1	$M_2 — P_1$	0.04 $f(M_2)$			
SO_1	$S_2 — O_1$	0.04 $f(O_1)$			
MS_f	$M_2 — S_2$	0.01 $f(M_2)$			

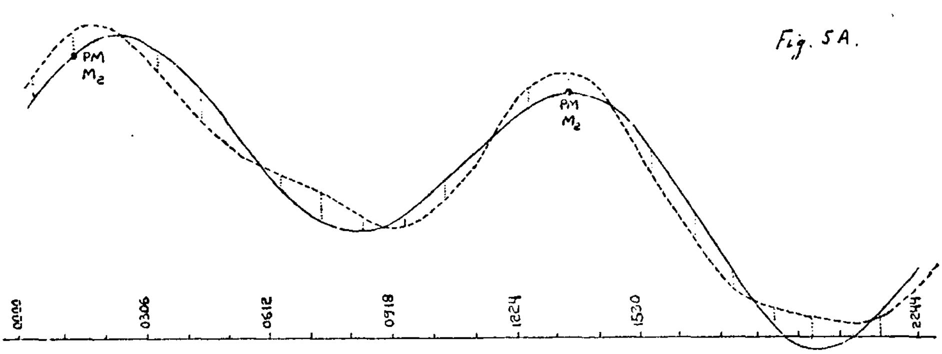
The large number of components in Table 4-III added to the already considerable number of astronomical components is proof of the inherent difficulty of high-quality prediction. This conclusion is further strengthened by the fact that the number of shallow-water components derived herein is still inadequate in certain cases, as with regard to some ports eighth-diurnal components and even others of a higher order appear.

These difficulties may be circumvented by mechanical means through the agency of a tide-predicting machine equipped with all the components required for correct prediction. The German Hydrographic Institute machine is a case in point. Since the cost of such an instrument is relatively high, Dr. DOODSON attempted to solve the problem at the Tidal Institute by a method of correction of times of High and Low Water obtained by means of incomplete tide-predicting machines, based on the mathematical analysis of time and height differences between the observed and predicted curve. Chapter XV of the Admiralty Manual of Tides gives a brief description of the procedure. The method is barely outlined here, since our purpose is the application of knowledge so far acquired to the computation of total corrections applicable to approximate predictions. The following paragraphs will deal with this question.

5. — GENERAL OUTLINE OF TOTAL CORRECTIONS

Let us assume that a prediction has been made accounting only for astronomical constants, and that the result of such prediction is shown in Figure 5A by a continuous line.

We shall proceed to show that it is possible to compute fixed corrections in terms of harmonic constants of the shallow-water components derived from M_2 and S_2 for times preceding and following the time of High Water of M_2 shown on the curve.



After computing such corrections, which are fixed for each place, let us mark on the astronomical tide-curve, starting from High Water of M_2, the points corresponding to intervals of one lunar hour (0102 mean hours). From these points we may add — positively or negatively — the computed corrections. We thus obtain a new set of points, as shown by the pecked line in Figure 5A, and which upon being joined together will supply the tidal curve corrected for shallow-water effect, represented by the components resulting from the joint action of M_2 and S_2.

It should be noted that these corrections lead to remarkably accurate results, especially if it is considered that they are to be applied to prediction methods used by the navigator.

6. — DETERMINATION OF PHASES OF COMPONENTS

Since the total corrections are taken with respect to the time of High Water of M_2, the first step in the computation of such corrections consists in expressing the phases of the shallow-water components resulting from the combined action of M_2 and S_2 in terms of such time.

We learned in Section 4 that the joint action of two components, of semi-amplitude A and B and phase a and b respectively, gives rise to a series of new components, whose semi-amplitudes and phases may easily be derived from the data supplied in Table 4-II. If we therefore merely consider the combined action of M_2 and S_2, the respective phases of which we shall denote by *m* and *s*, we get:

$$m = Vm - g(M_2) = a \qquad (6\,a)$$
$$s = Vs - g(S_2) = b \qquad (6\,b)$$

in which V generically represents the uniformly variable part of the astronomical argument; we neglect therein the nodal part u as regards M_2, since it is invariably of small value and will only slightly affect the accuracy of the corrections.

But since the angular constants 90°, θ' and (θ + 180°) appearing in Table 4-II seldon coincide with those determined by analysis, which happen to be the ones which concern us, we shall replace them by others designated by C_n. Furthermore, phase lags g of the shallow-water components are supplied directly by analysis, and their astronomical arguments are those mentioned in Table 4-III.

Tables 4-II and 4-III enable the following identities to be established for the various components:

TABLE 6-I

Components	Phases
MS_f	$m - s + C_1 = V_m - V_s - g(MS_f)$
$2\,MS_2$ (μ_2)	$2m - s + C_2 = 2\,V_m - V_s - g(2\,MS_2)$
$2\,SM_2$	$2s - m + C_3 = 2\,V_s - V_m - g(2\,SM_2)$
MS_4	$m + s + C_4 = V_m + V_s - g(MS_4)$
$2\,MS_6$	$2m + s + C_5 = 2\,V_m + V_s - g(2\,MS_6)$
$2\,SM_6$	$2s + m + C_6 = 2\,V_s + V_m - g(2\,SM_6)$
M_n	$\frac{n}{2}\,m + C_7 = \frac{n}{2}\,V_m - g(M_n)$

In order to obtain the values of constants C_n we need only take the values of *m*, *s*, and V_s at the time of High Water of M_2, conditioned by the expression

$$m = V_m - g(M_2) = O$$

whence

$$V_m = g(M_2)$$

By substitution of this value for V_m in Table 6-I, and of the value given in (6 b) for *s*, we get the values for C_n appearing in Table 6-II.

TABLE 6-II

Components	C_n
MS_f	$g(M_2) - g(S_2) - g(MS_f)$
$2MS_2$ (μ_2)	$2g(M_2) - g(S_2) - g(2MS_2)$
$2SM_2$	$2g(S_2) - g(M_2) - g(2SM_2)$
MS_4	$g(M_2) + g(S_2) - g(MS_4)$
$2MS_6$	$2g(M_2) + g(S_2) - g(2MS_6)$
$2SM_6$	$2g(S_2) + g(M_2) - g(2SM_6)$
M_n	$\frac{n}{2}g(M_2) - g(M_n)$

Let us now put $m = O$ in Table 6-I, and by substitution of the values given in Table 6-II for C_n, we obtain the phases of the various components at the time of High Water of M_2. Table 6-III supplies these values as well as their hourly variation per mean lunar hour.

TABLE 6-III

Components	Phase when m = O	Angular speed in degrees per lunar hour
MS_f	$g(M_2) - g(S_2) - g(MS_f) - s$	$-1°.0515$
$2MS_2$ (μ_2)	$2g(M_2) - g(S_2) - g(2MS_2) - s$	28 .9485
$2SM_2$	$2g(S_2) - g(M_2) - g(2SM_2) + 2s$	32 .1030
MS_4	$g(M_2) + g(S_2) - g(MS_4) + s$	61 .0515
$2MS_6$	$2g(M_2) + g(S_2) - g(2MS_6) + s$	91 .0515
$2SM_6$	$2g(S_2) + g(M_2) - g(2SM_6) + 2s$	92 .1030
M_n	$\frac{n}{2}g(M_2) - g(M_n)$	$15°n$

We now must express s in terms of civil time T of High Water of M_2, which presents no difficulty, since at 0 hour the astronomical argument of S_2 is always nil, and the phase of this component at such time is accordingly equal to $-g(S_2)$. Since it increases by 30° per mean solar hour, at time T of High Water of M_2 it will be equal to $30° \, T - g(S_2)$. Substituting this value in the preceding table we ultimately get:

TABLE 6-IV

Components	Phase at time T of HW of M_2	Angular Speed per Lunar Hour
MS_f	$g(M_2) - g(MS_f) - 30° T$	$-1°.0515$
$2 MS_2$ (μ_2)	$2g(M_2) - g(2 MS_2) - 30° T$	28 .9485
$2 SM_2$	$-g(M_2) - g(2 SM_2) + 60° T$	32 .1030
MS_4	$g(M_2) - g(MS_4) + 30° T$	61 .0515
$2 MS_6$	$2g(M_2) - g(2 MS_6) + 30° T$	91 .0515
$2 SM_6$	$2g(S_2) - g(2 SM_6) + 60° T$	92 .1030
M_n	$\frac{n}{2} g(M_2) - g(M_n)$	$15° n$

Let us now examine the use of the expressions determined. If we know the value of $-m$ at 0 hour *civil* time, the phase of M_2 at *civil* time T is equal to $29° T - m$, and since High Water of any component occurs when the phase cancels out, at the time of High Water of M_2 we shall have $29° T = m$, whence $T = m/29°$. Knowing the value of T, by means of Table 6-IV we may compute the phases of all the components listed for this particular time.

Thus, if H and r respectively represent the mean semi-amplitudes and the phases of these components, the correction to be applied to the height of the tide, computed without regard to the effect of the shallow depth, at lunar time t reckoned from High Water of M_2, is:

$$\Sigma H \cos (\eta t - r) \qquad (6 c)$$

in which η is the phase variation in degrees per mean *lunar* hour.

If each prediction required such elaborate computation, no mariner would willingly attempt it, and tables have therefore been drawn up which are invariable for each place and supply overall corrections of hourly heights in terms of civil time T of High Water of M_2, and of the lunar time t, computed before and after such instant.

Thus, a time T will correspond to a lunar time *zero*. In England, where the happy idea of the corrections originated, values of T are computed for the semi-diurnal tide at syzygy, intermediate tides, and the tide at quadrature.

Since the time of High Water of S_2 invariably occurs at the same civil time given by the relationship $g(S_2)/30°$, and since during the tide at syzygy the M_2 and S_2 High Waters occur simultaneously, we shall necessarily get : $T = g(S_2)/30°$.

When component S_2, at the time of High Water, is 90° ahead of M_2, High Water will occur $90°/29° = 0306$ hours later (intermediate tide), and at quadrature and the other intermediate tide, delays of M_2 High Water will respectively be $180°/29° = 0612$ hours and $270°/29° = 0918$ hours later.

To sum up, we shall then get the following four values for T:

T for tide at syzygy = $g(S_2) / 30°$
T for first intermediate tide = $g(S_2) / 30° + 0306$
T for tide at quadrature = $g(S_2) / 30° + 0612$
T for second intermediate tide = $g(S_2) / 30° + 0918$

Using the T-values thus determined, and by means of the expressions in Table 6-IV, we compute the phases r of the various components implicit in (6 c). By means of this expression the corrections may then be computed readily by using the KELVIN tide-predicting machine as shown in the following section.

The table of shallow-water corrections given below is reproduced from Part II of the Admiralty Tide Tables

Luni-hourly Intervals	No. 4150 Chittagong SL. HW (1)			
	0213	0520	0826	1132
— 6	+ 1.1	— 0.3	— 0.7	— 0.1
— 5	— 0.1	— 0.8	— 0.9	— 1.0
— 4	+ 0.1	— 0.3	— 0.8	— 0.9
— 3	+ 1.0	+ 0.5	— 0.5	— 0.3
— 2	+ 0.9	+ 0.9	— 0.3	0
— 1	+ 0.4	+ 0.8	— 0.1	+ 0.2
AT	0	+ 0.4	— 0.1	+ 0.3
+ 1	— 0.5	— 0.1	— 0.2	+ 0.1
+ 2	— 0.7	— 0.5	— 0.3	— 0.1
+ 3	0	— 0.4	— 0.4	+ 0.1
+ 4	+ 1.2	+ 0.1	— 0.4	+ 0.6
+ 5	+ 1.9	+ 0.2	— 0.5	+ 0.6

The Brazilian Navy Hydrographic Office does not base its choice of the values of T on the standard followed by the British Admiralty. It believes that T should be more appropriately assigned values from 0 to 12 hours, even though this means a considerable increase in the size of the tables. Interpolation by inspection is thus rendered far easier for the operator entering the table with any time of High Water of M_2. Moreover, upon examination of Table 6-IV, it may be seen that if T covers all values between 0 and 12 hours, all the phases of the components will cover at least one cycle. Hence, as regards this set of T-values, the phases of the various components will cover practically all the possible relative values.

The following table, drawn up on this basis, gives results for the Brazilian port of Florianopolis:

Civil Time HW M_2	Lunar intervals from HW of M_2											
	—6	—5	— 4	— 3	—2	— 1	0	+1	+ 3	+ 3	+ 4	+5
0	+10	+ 3	— 5	— 6	+ 1	+ 7	+ 7	0	— 8	—11	— 5	+ 6
1	+12	+ 5	— 4	— 7	0	+ 7	+ 8	0	— 9	—13	— 6	+ 7
2	+14	+ 5	— 7	— 9	— 1	+ 8		+ 2	— 9	—13	— 5	+ 9
3	+13	+ 3	—10	—12	— 2	+10	+12	+ 3	— 8	—12	—4	+ 9
4	+12	0	—14	—13	0	+13	+15	+ 5	— 8	—12	— 2	+ 9
5	+10	— 3	—15	—13	+ 2	+15	+15	+ 3	6 —	—12	—3	
6	+10	— 3	—14	—11	+4	+14	+12	0	—11		0	+11
7	+11	— 3	—13	—10	+ 3	+11		— 2	—11	— 9	+3	+13
8	+11	— 2	—12	— 9	+ 1	+ 8	+ 5	— 3	— 9	— 6	+ 5	+13
9	+10	— 2	—11	— 9	0	+ 6	+ 4	— 2	— 7	— 5	+ 5	+12
10	+ 9	—1	—10	— 8	0	— 4	+ 5	— 1	— 6	— 5	+2	+ 9
11	+ 8	0	— 7	— 7	0	+ 7	+ 6	0	— 6	— 8		+ 6
12	+10	— 3	— 5	— 6	+ 1	+ 7	+ 7	0	— 8		— 5	

Each line of the above table supplies corrections for tidal heights corresponding to lunar times preceding and following High Water of M_2, which occurs at the civil time indicated in the first column.

7. — MACHINE COMPUTATION OF SHALLOW-WATER CORRECTIONS

As a result of theoretical explanation, use of the Kelvin machine becomes so instinctive that a mere outline of the procedure should suffice.

(*a*) Compute phases of components by successively introducing in the expressions of Table 6-IV values of T from 0 to 11;

(*b*) Set the Kelvin machine at zero, adjust the *mean* semi-amplitudes H of the components, then phases of the latter for T = 0:360° (formulae of Table 6-IV);
(*c*) Adjust limb of M_2 to 0-graduation;

(*d*) On height-scale read off height of stylus with respect to mean level, whence the correction for the time of M_2 High Water is obtained;

(*e*) The machine is operated forward, and is stopped when the M_2 limb reads 30°; a reading of the height-scale in this position supplies the correction corresponding to the first lunar hour which follows M_2 High Water;

(*f*) The operation is repeated for the indications of the M_2 limb equivalent to 60°, 90°, 120° and 150°, and a whole series of corrections is thus obtained for the five lunar hours following M_2 High Water;

(*g*) The machine is now operated backward, and by reading the height-scale when the M_2 limb successively shows 330°, 300°, 270°, 210° and 180°, corrections are obtained for the six lunar hours preceding M_2 High Water;

(*h*) The settings of the shallow-water component phases are varied for the values successively corresponding to T = 1, 2, ... 11 by repeating in each case the operations described in (*a*) to (*g*), without further change. The corrections for T = 12 are the same as for T = 0.

We may now note that corrections are repeated for M_2 High Water civil times exactly twelve hours apart. But since, during any particular day, the two M_2 High Waters are separated by an interval of 1225 hours, it will readily be apparent that the correction series applied by taking the first High Water as reference will not be identical with that corresponding to the second High Water, whose time should be subtracted from 1200 to supply a new entry to the tables and thus enable a new series of corrections to be obtained. For purposes of routine operation, the second series of corrections is considered as being practically equal to the first.

It should at last be explained that when there occurs a component M , in which n is an odd number and which is of considerable amplitude, two series of corrections are necessary. As an example, let us assume that the component M_3 is of relatively large semi-amplitude and that at the time of first M_2 High Water the two components are in conjunction. Then at the time of second M_2 High Water, both components will be in opposition, and thus it will not be possible to use the first correction series for the second M_2 High Water. New values of T between 12 and 24 hours will thereupon have to be tabulated, in addition to the values applying to the general case analyzed above.