LENGTH OF THE GEODESIC DETERMINED FROM MERCATOR COORDINATES

by Ole MØLLER, Head of Computation Division of Kongelige Sokort-Arkiv, Denmark

A formula, the chief part of which is a series, is presented and discussed. It has already been published by D.H. Sadler, but was then developed to a decisive, lesser number of terms.

The following is a discussion of its application in the computation of hyperbolic lattices.

It is well-known that the complicated problem of determining the length of the geodesic over long ranges has always impelled computers to form, from the long established theory, simpler procedures of computation. It is a contribution to this that shall be given here, limited to the problem as it is often met with in hydrographic work, namely: to compute the distance on the earth from the Mercator grid coordinates of the points involved.

The formula we are about to discuss was first published by D.M. Sadler in « The Computation of Decca Lattices » (1), page 12 (13). The development was here given to the third degree (in our notation this means that f_2 of formula (1) was included), and the accuracy was not such as to encourage its use throughout the Decca computations; nor was this the actual aim of the formula at that time.

Working with the problem of setting up a method for fully automatic computation of Decca on digital electronic calculators the writer searched for a method involving only a few, simple operations which would give the distance on the earth between two points, using nothing but their Mercator coordinates (except the latitude of the « starting » point). The result was the formula of D.H. Sadler where the development was brought further, in fact to the sixth degree (in (1), f_5). The deduction has been done independently and has followed a different method, the basic idea of which has not been indicated by the writer but has been taken from an unpublished and very elegant deduction of the Givry correction (2) to the second order given by G. Elfving (now professor at Helsinki) when be served in the Geodetic Institute of Denmark and happened to be on board a Danish survey vessel. A description of the method will not be given here. It is only pointed out that Elfving used Euler's equation from the calculus of variation to determine, in the Mercator plane, a curve which corresponds to the shortest line on the earth connecting two points. The quantities involved are, besides the coordinates of the two points, the latitude and angle of direction at one point. The formula presented here is deduced from this curve simply by evaluating its length on the earth and then performing an elimination of the direction angle. As Elfving only needed third order terms — to get second order in the Givry correction — his determination of the curve had to be extended to sixth order terms.

When two points are given in the following way:

first point: latitude φ, northing o, easting o

second point: northing x, easting y — in units of equator radius

it is found, by the above-described method, that their distance may be computed from:

$$d = \cos\varphi \sqrt{x^2+y^2} \left\{ \begin{array}{l} k_{00} \\ + k_{10}x \\ + k_{20}x^2 + k_{02}y^2 \\ + k_{30}x^3 + k_{12}xy^2 \\ + k_{40}x^4 + k_{22}x^2y^2 + k_{04}y^4 \\ + k_{50}x^5 + k_{32}x^3y^2 + k_{14}xy^4 \end{array} \right\} \qquad (1)$$

where:

$$k_{00} = 1 + \frac{\sin^2\varphi}{2} e^2 + \frac{3}{8} \sin^4\varphi . e^4 + \frac{5}{16} \sin^6\varphi . e^6$$

$$k_{10} = - \frac{\sin\varphi}{2} \left(1 + \frac{\sin^2\varphi}{2} e^2 + \frac{3}{8} \sin^4\varphi . e^4\right)$$

$$k_{20} = + \frac{1}{6} \left(1 - 2\cos^2\varphi + \frac{1 - 3\cos^2\varphi}{2} e^2 + \frac{3 - 12\cos^2\varphi + 3\cos^4\varphi - 2\cos^6\varphi}{8} e^4\right)$$

$$k_{02} = - \frac{\sin^2\varphi}{24} \left(1 + \frac{\sin^2\varphi}{2} e^2 + \frac{3}{8} \sin^4\varphi . e^4\right)$$

$$k_{30} = - \frac{\sin\varphi}{24} \left(1 - 6\cos^2\varphi + \frac{1 - 7\cos^2\varphi - 12\cos^4\varphi}{2} e^2\right)$$

$$k_{12} = + \frac{\sin\varphi}{48} \left(1 - 3\cos^2\varphi + \frac{1 - 4\cos^2\varphi - \cos^4\varphi}{2} e^2\right)$$

$$k_{40} = + \frac{1}{120} \left(1 - 20\cos^2\varphi + 24\cos^4\varphi + \frac{1 - 21\cos^2\varphi - 72\cos^4\varphi + 120\cos^6\varphi}{2} e^2\right)$$

$$k_{22} = - \frac{1}{1440} \left(7 - 90\cos^2\varphi + 99\cos^4\varphi + \frac{7 - 97\cos^2\varphi - 107\cos^4\varphi + 261\cos^6\varphi}{2} e^2\right)$$

$$k_{04} = + \frac{1}{1920} \left(1 - 10\cos^2\varphi + 9\cos^4\varphi + \frac{1 - 11\cos^2\varphi + 3\cos^4\varphi + 7\cos^6\varphi}{2} e^2\right)$$

$$k_{50} = - \frac{\sin\varphi}{11520} \left(16 - 960\cos^2\varphi + 1920\cos^4\varphi\right)$$

$$k_{32} = + \frac{\sin\varphi}{11520}(8 - 420\cos^2\varphi + 780\cos^4\varphi)$$

$$k_{14} = - \frac{\sin\varphi}{11520}(3 - 90\cos^2\varphi + 135\cos^4\varphi)$$

(e: excentricity of the earth).

When referring to the different terms it is briefly written:

$$d = f_0 + f_1 + f_2 + f_3 + f_4 + f_5$$

where:

$$f_0 = \cos\varphi\sqrt{x^2+y^2}.\ k_{00}$$
$$f_1 = \cos\varphi\sqrt{x^2+y^2}.\ k_{10}.\ x$$
$$f_2 = \cos\varphi\sqrt{x^2+y^2}.\ (k_{20}x^2 + k_{02}y^2)$$
$$\vdots$$

To show the practical value of the formula, a table is given, in which some max. values of the terms:

$$f_3 = \frac{\cos\varphi\ \sin\varphi}{48}\sqrt{x^2+y^2}\left\{(1 - 3\cos^2\varphi + \frac{1 - 4\cos^2\varphi - \cos^4\varphi}{2}e^2)\,xy^2 - 2\,(1 - 6\cos^2\varphi + \frac{1 - 7\cos^2\varphi - 12\cos^4\varphi}{2}e^2)\,x^3\right\}$$

$$f_4 = \cos\varphi\sqrt{x^2+y^2}\left\{\frac{1}{1920}(1 - 10\cos^2\varphi + \dots)\,y^4 - \frac{1}{1440}(7 - 90\cos^2\varphi\ \dots)\,x^2y^2 + \frac{1}{120}(1 - 20\cos^2\varphi + \dots)\,x^4\right\}$$

$$f_5 = \frac{\cos\varphi\ \sin\varphi}{11520}\sqrt{x^2+y^2}\left\{(3 - 90\cos^2\varphi + \dots)\,xy^4 - (8 - 420\cos^2\varphi + \dots)\,x^3y^2 + (16 - 960\cos^2\varphi + \dots)\,x^5\right\}$$

are shown.

The are max. values, in the sense that, in relation to a point at the latitude put down in the table, the value of the term f_i for any point within the ranges of 100, 500 and 1000 km. respectively, is numerically less than or equal to the quantity F_i of the table.

The figures in the table are computed from the above expressions in the following way:

1) e terms are suppressed;

2) polar coordinates (r, θ) are used, e.g.:

$$f_4 = \frac{\cos\varphi\,.\,r^5}{5760}\left\{(3 - 30\cos^2\varphi + 27\cos^4\varphi)\sin^4\theta - (28 - 360\cos^2\varphi + 396\cos^4\varphi)\cos^2\theta\sin^2\theta + (48 - 960\cos^2\varphi + 1152\cos^4\varphi)\cos^4\theta\right\}$$

Introducting the metre as unit, we put:

$$a\,.\,f_4 = F_4 = r^5\,.\,K_4(\theta)\quad (a = \text{equator radius})$$

and $K_4(\theta)$ max. is determined. For the values of the table the max. is attained at $\theta = 0°$; however in some narrow φ-regions this is not the case.

3) The length of the south-north Mercator chord corresponding to lengths on the earth of 100, 500, 1000 km. is determined: $\underline{R}$ (the max. Mercator chord for fixed distance).

4) Finally $\underline{R}^{i+1} \cdot \underline{K}_i(\theta)_{max}$ is found and furnishes the max. value of $\underline{F}_i$ for the latitude concerned.

The question of the advantages of formula (1) as compared to other procedures is of course subject to personal taste and habit. However, as other methods most frequently in use go through geographicals (φ, λ) which, in the connections especially thought of here, are only intermediates, (1) should be compared to the complex of a transformation (x, y) (φ, λ) and of the procedure in question. It is therefore the writer's opinion that (1) in many cases may present a useful solution of the problem.

We shall now discuss a concrete application.

As previously mentioned it is intended to use (1) for computations of hyperbolic lattices on the chart. In the Danish Hydrographic Office the Standard Method of the Nautical Almanac Office(1) has been used, and it is felt that (1) here is a real simplification.

First the accuracy has to be looked at. It should at once be stated that more terms could of course be produced, and, before their behaviour has been investigated the method as such should not be discarded from accuracy arguments. But except for the automatic computation, it is the relatively few terms, when stopping at f_4, or even before, that make (1) a useful procedure. (A reduction of the degree by means of Chebychev polynomials could possibly be taken into consideration). However, the above table shows that stopping at $\underline{f}_4$, and hence using f_5 as an estimate of the error, we may reach nearly 70° latitude with 500-km distances and still keep metre accuracy, while 1000-km distances present errors of $\sim$ 5 metres from about 50° latitude. An accuracy of 1 — 5 metres corresponds roughly to $\frac{1}{100}$ lane of the Decca System, and it is only in quite specific cases that charts on scales large enough to show clearly $\frac{1}{100}$ of a lane are prepared for areas beyond the range of 500 km. It should in this connection be noticed that the error diminishes considerably when avoiding bearings in the neighbourhood of 0° and 180°, which means that, if the north-south component could be kept less than $\sim$ 500 km, greater distances could easily be reached. It is therefore assumed that the accuracy of $\underline{d} = f_0 + f_1 + f_2 + f_3 + f_4$ (actually showing nine separate terms) would be acceptable for ordinary Decca computations and meet every practical demand of published charts even up to 70° latitude.

The ranges and accuracy here regarded seem to be in full accordance with the corresponding data quoted by M. Dupuy[3].

When using the Standard Method, a considerable number of distances have to be computed. They all have one end point in one of the stations. Hence the troublesome coefficients, $\underline{k}_{ij}$ merely appear as initial values which are gone through at once, and a great part of the computations then simply consist in multiplying the polynome by a square root. The advantage is obvious when we

φ	K_3 m k_4 k_5	100 km		500 km		1000 km	
		R in units of equator radius	F_3 m. F_4 " F_5 "	R	F_3 m F_4 " F_5 "	R	F_3 m F_4 " F_5 "
0°	0 +256 767 0	.015 71	0 +.000 25 0	.078 51	0 +.77 0	.157 46	0 24.9 0
10°	+219 021 +218 676 - 84 356	.015 97	+.014 3 +.000 23 -.000 0014	.080 26	+9.09 + .73 - .023	.162 15	+152. +24.6 -1.54
20°	+394 544 +102 545 -118 404	.016 73	+.031 +.000 14 -.000 003	.084 74	+20.4 +.45 -.044	.172 50	+350. +15.7 -3.13
30°	+402 781 - 23 016 - 90 146	.018 21	+.044 4 -.000 047 -.000 004	.092 68	+29.8 -.158 -.058	.190 45	+530 -5.78 -4.33
40°	+329 906 -100 646 - 31 034	.020 63	+.060 -.000 38 -.000 003	.105 83	+41.4 -1.34 - .044	.220 15	+776. -52.2 -3.54
50°	+193 558 -108 184 + 14 417	.024 61	+.071 -.000 98 +.000 0033	.127 94	+51.9 - 3.72 + .064	.271 29	+1050 -160 + 5.76
60°	+ 57 541 - 66 442 + 24 935	.031 77	+.059 -.002 2 +.000 026	.168 25	+46.2 -9.0 +.57	.366 70	+1041. -162. +60.5
70°	- 25 466 - 18 382 + 12 462	.046 75	-.122 -.004 2 +.000 131	.257 64	-113. -20.9 + 3.66	.601 07	-3330. -870. +588.
80°	- 36 969 + 3 761 + 1 137	.029 06	-2.94 +.029 +.000 81	.595 66	-4660 +283. +51.2	2.261 0	-966400. +222300. +153000.

think of adapting the method for automatic computation, e.g. on digital, electronic calculators. But even in the case of more ordinary equipment, the advantage seems clear. It should also be noted that in plotting the chart by means of modern equipment, as for instance with a coordinatograph, the linear grid coordinates are unquestionably preferable to the geographical.

Now, in the Standard Method the computation of distances plays a great role. However, quite different methods are also in use, as for instance the very elegant Grid Method[4] used in the Nautical Almanac Office. Here the need for distances is considerably reduced. Whether or not (1) presents a help in the successive approximations here in use has not been investigated by the writer, and, as it is understood, the question of computation of distances should not here be an essential one. The Grid Method has also the advantage of avoiding inverse interpolation (a chief characteristic of the Standard Method) and of being especially adaptable near the stations. On the other hand, the grid used on the sheet is less convenient. — Another very elegant method, some main features of which are the same as in the Grid Method, is used by The French National Geographic Institute [3]. Here the computation of distances has been completely avoided. But then a transformation of specific variables ($\underline{p}$, $\underline{q}$) to the grid coordinates ($\underline{x}$, $\underline{y}$) has to be performed for a set of points appropriately chosen in the area concerned. As in the Standard Method modified by (1), the grid is the Cartesian grid of the chart (sheet).

Methods like these seem preferable to the Standard Method especially because of the inverse interpolation being avoided. However, regarding the adaptation for fully automatic computation, it is difficult to judge — in the writer's opinion anyhow — whether or not the Standard Method modified by (1) will possibly prove competitive. It is suggested (the code has nearly been accomplished) that a procedure of successive approximations be used directly from the method. Of course it must be expected that the rough method of successive approximations will be looked at more thoroughly and possibly replaced by more appropriate methods, but it is thought that a first decision could be made even from the foregoing.

Conclusion.

The reason for publishing the formula in question is mainly that of giving an extremely direct method for computation of distances from the Mercator Grid coordinates. The range and accuracy taken together are fairly good.

Its application to Decca computations improves to some extent the procedure of the Standard Method, and as far as automatic computation is concerned it is intended to make a comparison between the value of this as opposed to more recent procedures.

REFERENCES

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