THE CONCORDANCE METHOD AND HARMONIC ANALYSIS BY APPROXIMATE CONSTANTS

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The reduction of soundings in a hydrographic survey of vast proportions, on the basis of tidal observations carried out at various tide-staffs, necessitates the comparison of ranges with reference to time at such tide-staffs ; from this point of view the concordance method gives good results at the expense of a minimum of effort. But when a sounding datum must be selected and the tidal characteristics must be derived that will enable its prediction, it is obvious that analysis of the phenomenon requires greater care.

In this connection, harmonic analysis is unquestionably an extremely effective method. Without postulating the physical existence of the various « waves », it is but natural to seek, within the spectrum of the periodical phenomenon constituted by the tide, constituents whose periods result from the breaking-up of the luni-solar tide-generating potential into factors of mean time. The application of the method in its usual form however involves a large amount of computation and implies the existence of a fairly long period of continuous observations.

The analytical method suggested here is much more flexible than the usual harmonic method. It may be adapted to the length of the observational period, whatever this may be, and may likewise be adapted to the amount of time the observer is able to spare for this work.

This original method, which is described in the latter part of this paper, derives from the fact that whereas harmonic analyzers are rare, harmonic predictors are fairly common. It appears natural, therefore, to obtain an approximate artificial curve with a predictor, and then study the differences between the artificial and natural curves in their relationship with the constants used to « shape » the artificial curve.

In the first part of this study, the principles of the « Concordance Method » have been reviewed, various forms of « correlation surfaces » are referred to, and it is shown that such surfaces taper to a « point », which is important when the method is used to select sounding datum.

I. — SIMILAR TIDES. CONCORDANCE

The height of water at a port A, due to the tide, may be written:

$$y^A = N^A + \sum_i A_i \cos (q_i t - \alpha_i)$$

where:

N^A is the mean level;

A_i and α_i are the harmonic constants of wave i at the port;

q_i is the velocity of wave i expressed in units of arc per unit of mean time.

Two ports are considered in which the harmonic constants are:
A_i and B_i for the moduli of waves i,
α_i and B_i for the phases of waves i, and we assume that we have:

$$\frac{A_i}{B_i} = K = \text{a constant not dependent on } i\text{ ;}$$

$\alpha_i - \beta_i = kq_i$ where k is a time not dependent on i.

Then, if we construct curves C_A and C_B representing the height of water at each port plotted against time, and providing k is not too large, so that, during the time it represents, the variation of the astronomical factors « f » of the waves may be neglected, one curve may be derived from the other by transference (consisting of a translation parallel to the axis for heights and another parallel to the time-axis) accompanied by a change in scale on the axis for heights. In this case the tides at the two ports are said to be *similar*.

A. — In order to define this comparison of two tides, which forms the basis for the « Concordance Method », we shall examine the case in which the tide at both ports is due to a single wave. The curves showing heights of water against time are as represented in Figure 1, which is produced by a translation parallel to the time-axis and equivalent to T, the period common to the two tides. The delay in the tide of B with respect to that of A is determined by the difference in the times of passage at the mean levels (which in this simple case are the means of the heights of HW and LW, also called half-tide level); this delay represents $\frac{\alpha - \beta}{q}$ and may also be read as the difference in the times of the two HW and LW. The range of the tide at A is obtained by taking half the difference of the HW and LW heights at A, and the range at B by taking the half-difference of the HW and LW heights at B. The ranges may be connected by their difference or by their ratio. In the Concordance Method, the ratio is determined, and the reason therefor will be shown later on. The simple graphical method consists in plotting on two rectangular axes the heights of HW and LW at A on the one hand, and those of B on the other, as shown in Figure 2. Observational errors cause a certain amount of scatter, and a straight line D is drawn as smoothly as possible through the centres of the spots A_1 and B_1. This is a « straight line of concordance », and its slope gives the range ratio.

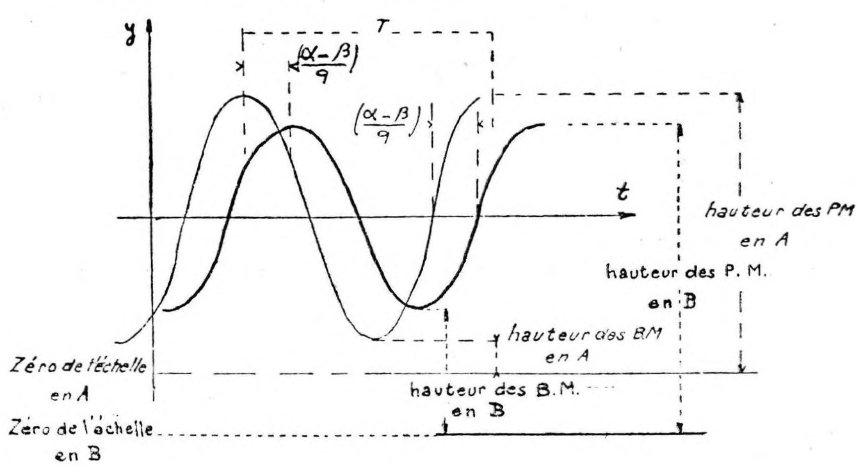


Fig. 1.

It should be noted that the « concordant » heights might just as well have been taken a constant time t_0 after (or before) the respective HWs, and a similar graph would have given two spots A_2 and B_2 centred on the same line D. Heights are taken at HW and LW for the following reasons:

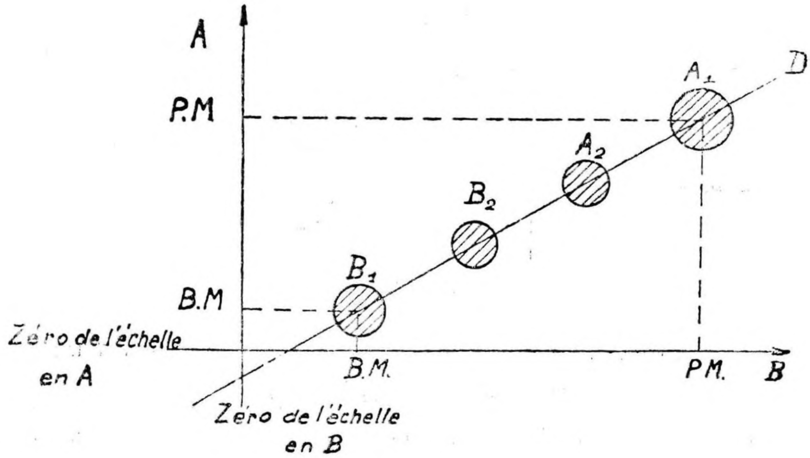


Fig. 2.

a) Spots A_1 and B_1 are the farthest apart, and determine the slope of D under optimum conditions;

b) The lag of one tide with respect to the other need not be known in order to plot the corresponding points on the graph;

c) Heights vary only slightly in the neighbourhood of HW and LW.

Thus the « Concordance Method » consists in comparing the two waves (in the present simple case, the two tides) *when the two waves are in the same tidal situation.*

The result of concordance is expressed by two « constants »: the lag of B with respect to A, which is equivalent to

$$\left(\frac{\alpha - \beta}{q}\right) = k$$

and the ratio of range, which is equivalent to

$$\frac{A}{B} = K.$$

B. — We shall now examine the case in which the tide at the two ports consists of several waves. The height of water as plotted against time at A is as illustrated in Figure 3. The height of the water at A due to the wave having i as its index at time $(t+k)$ $\left(\text{where } k = \frac{\alpha_i - \beta_i}{q_i}\right)$ is:

$$y^A_i = A_i \cos\left(q_i[t+k] - \alpha_i\right) = A_i \cos\left(q_i t - [\alpha_i - q_i k]\right)$$
$$= A_i \cos\left(q_i t - \beta_i\right)$$

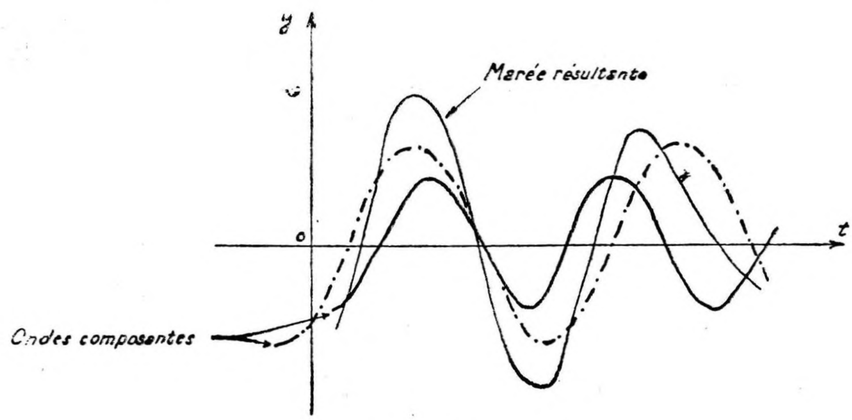


Fig. 3.

The height of the water at B due to the same wave at time t is:

$$y^B_i = B_i \cos (q_i t - \beta_i)$$

This results in the derivation of the curve representing y^B_i plotted against time from the curve representing y^A_i plotted against time by means of a translation parallel to the time-axis and equivalent to k, and scale multiplication along the axis for heights equivalent to $\frac{B_i}{A_i} = \frac{1}{K}$.

If « similar tide » conditions are achieved, k and K are independent of i, all the waves are subjected to the same translation and the same scale multiplication, and the two tidal curves are derived from each other by the same transformation.

In order to illustrate these conclusions (Figure 4), a vector A_i with a modulus of value $\vec{A}_i$ and with a polar angle, with respect to the axis for heights, of value $(q_i t - \alpha_i)$ is termed the « vector of wave i at port A », in which the height of water at A, due to wave A_i, is the projection of $\vec{A}_i$ on the axis for heights, i.e. y^A_i. Similarly, the « vector of the tide at port A » is defined by the geometrical summation:

$$\sum \vec{A}_i$$

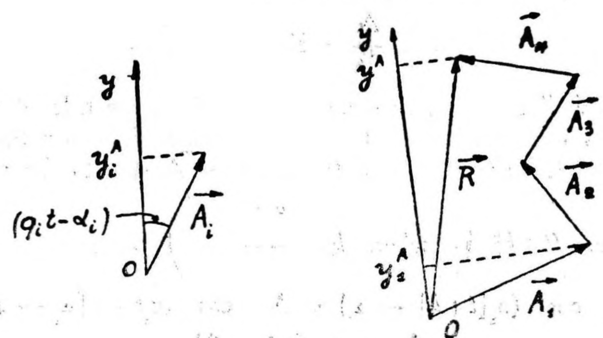


Fig. 4

« Similar tide » conditions involve the similarity of the polygonal lines:

$OA_1A_2A_3A_4$ considered at time $t + k$;

$OB_1B_2B_3B_4$ considered at time t,

and the result is that the ratio $\dfrac{y^A(t+k)}{y^B(t)} = K$ is independent of t.

If a concordance graph is constructed for this case (Figure 5), a series of points distributed along D is found, instead of two points A_1 and B_1 as in the preceding paragraph. Actually, an elongated spot divided by line D is found.

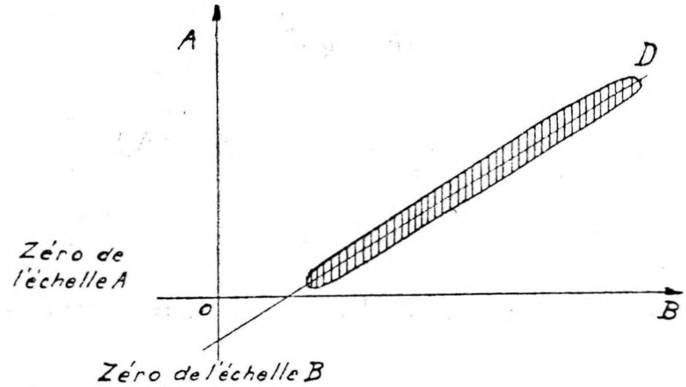


Fig. 5.

C. — It is of interest to investigate the natural conditions for which the two tides at A and B will be similar. A group of progressive waves with an identical *direction of propagation* is considered (Figure 6). For one of these waves, the height of water at A at time t is written:

$$y_i^A = A_i \cos (q_i t - \alpha_i).$$

At B we write:

$$y_i^B = B_i \cos (q_i t - \beta_i) = \frac{A_i}{K} \cos \left\{ q_i t - \left(\frac{2\pi d}{L_i} + \alpha_i \right) \right\},$$

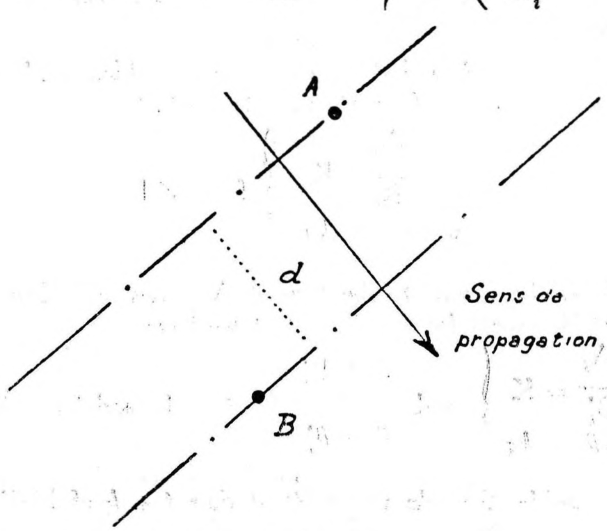


Fig. 6

L_i being the wavelength of the i wave and d being the projection of the distance AB on the direction of propagation. The velocity of propagation V being the same for all these waves, which are mass waves, we get :

$$q_i L_i = 2\pi V$$

and therefore :

$$\frac{2\pi d}{L_i} + \alpha_i = q_i \frac{d}{V} + \alpha_i = \beta_i$$

whence

$$\alpha_i - \beta_i = q_i \frac{d}{V}$$

and the similar tide conditions with respect to phase are satisfied. If the depth between A and B is constant, we get $K = 1$, ($B_i = A_i$), thus fulfilling the condition with respect to range. If the depth decreases from A to B (without however becoming so small at B as to produce overtides or compound tides), it can be shown, considering the conservation of energy of each progressive wave, that the ratio $\frac{A_i}{B_i} = K$ is the same for all waves (here K is smaller than 1).

A group of stationary waves with parellel nodal lines will now be considered. Each may be regarded as the sum of two progressive waves of like amplitude propagated in opposite directions, and by a simple process of thought, it will be seen that this pattern supplies approximately similar tides only, and even then the speeds of the waves must be nearly the same, i.e. the latter must be of the same type (diurnal or semi-diurnal) and their nodal lines must be near one another.

II. — DISSIMILAR TIDES. CORRELATIONS

If the conditions $\frac{A_i}{B_i} = K =$ constant independent of i, $\alpha_i - \beta_i = kq_1$ where k is independent of i, are not fulfilled, the tides at ports A and B are termed « dissimilar tides ».

A. — Two ports A and B are considered in which similar tidal conditions obtain for all waves save one (say of index 1), so that :

$$\left.\begin{aligned} \frac{A_i}{B_i} &= K \\ \alpha_i - \beta_i &= kq_i \end{aligned}\right\} \text{for } i \neq 1$$

Let tide B' be similar to the tide at A, whose constituents are identical with the tide at B, except for $i = 1$, so that we have :

$$\left.\begin{aligned} \frac{A_i}{B'_i} &= K \\ \alpha_i - \beta_i &= kq_i \end{aligned}\right\} \text{ and } \begin{aligned} B_i &= B'_i \\ \beta_i &= \beta'_i \end{aligned} \quad \text{for } i \neq 1 \quad \text{and } \overrightarrow{B_1} = \overrightarrow{B'_1} + \overrightarrow{b_1}$$

Let us consider the tide vector $\overrightarrow{R^A}$ at time $t + k$ of HW at A, and the « concordant » vector $\overrightarrow{R^{B'}}$ at time t of HW of B' (Figure 7). Both vectors are

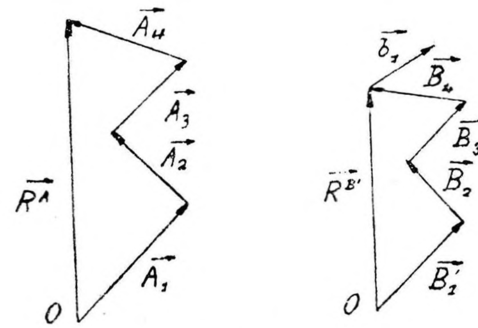


Fig. 7.

generally obtainable for any shape of the similar polygonal lines $OA_1A_2A_3A_4$ or $OB'_1B_2B_3B_4$, i.e. for any direction of $\vec{B_1}$ or of $\vec{b_1}$, since the triangle (B_1, b_1, B'_1) is of unvarying shape. This results in the following: if a single HW height $y^{B'}$ of the B' tide corresponds to a HW height y^A at A, then an infinite number of heights y^B will correspond to this same height y^A, between:

$$(y^{B'} - b_1) \quad \text{and} \quad (y^{B'} + b_1).$$

There is no longer concordance but correlation. As long as the modulus of $\vec{R^{B'}}$ is small enough so that all possible distortions of the polygonal line $OB'_1B_2B_3B_4$ enable $\vec{B'_1}$ to assume all possible directions, the area of correlation is limited by two straight lines derived from line D and which are in concordance as between A and B', following a translation parallel to OB and equivalent to $\pm\, b_1$. It may further be stated that a segment of a straight line parallel to OB, of length $2b_1$, and centred on D', corresponds to a point of axis OA. But if the modulus of $\vec{R^{3'}}$ exceeds the length

$$\sum_{i}^{1} B'_i - B_1$$

then the possible distortions of the polygonal line $OB'_1B_2B_3B_4$ no longer enable $\vec{B'_1}$ or $\vec{b_1}$ to assume all possible directions. Hence a segment of line parallel to OB, whose length is smaller than $2b_1$ and which is no longer centred on D', corresponds to a point on axis OA. In order that $\vec{R^A}$ and $\vec{R^{B'}}$ may assume their maximum value, line $OB'_1B_2B_3B_4$ must reduce to a straight line whose length is $\sum_i B'_1$, the direction of $\vec{B'_1}$, therefore of $\vec{b_1}$, is then determined, and a single HW height y^B at B corresponds to the maximum HW height y^A at A (y^B incidentally is not necessarily the maximum height at B). Similarly, a single LW height y^B at B corresponds to the minimum LW height y^A et A (which is not necessarily the minimum height at B). Thus, provided the modulus of any one of the constituent waves be no larger than the sum of the moduli of the other waves, a correlation area is obtained of the type of the hatched area in Figure 8. A line D may be drawn dividing the area as accurately as possible, and this line may serve as the line of concordance as between the tide at A and the tide at B, for the purpose of obtaining an order of magnitude of the tide at B from the tide at A.

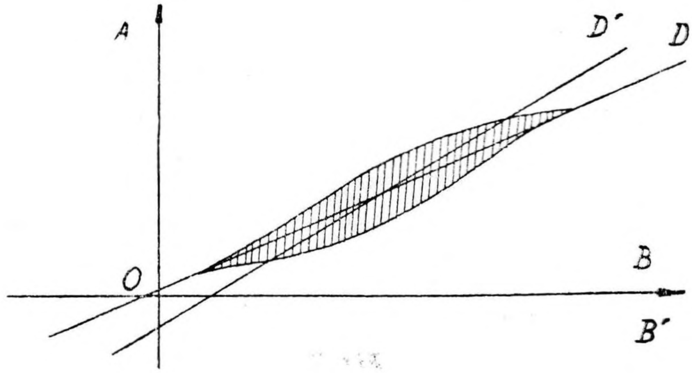


Fig. 8.

Let us now assume that the modulus of vector $\overrightarrow{B'_1}$ is larger than $\sum_{i \neq 1} B'_i$; then the minimum value of the modulus of the tide vector $R^{B'}$ is limited by the length $B'_1 - \sum_{i \neq 1} B'_i$, and in the concordance between A and B', use is made of two segments only of line D' instead of the whole line. There thus exists a height y^A of *high water* at A, and hence a height $y^{B'}$ of *high water* at B which is smaller than all the others, and well-defined directions of the vectors $\overrightarrow{B'_1}$ and $\overrightarrow{b_1}$ correspond to this height. Therefore, a single HW height $y^{B'}$ at B corresponds to this minimum HW height y^A at A. Moreover, the width of the correlation area, reckoned parallel to OB, is always smaller than $2b_1$. A correlation area is obtained which is divided into two parts and which has the appearance of the hatched area in Figure 9. Two

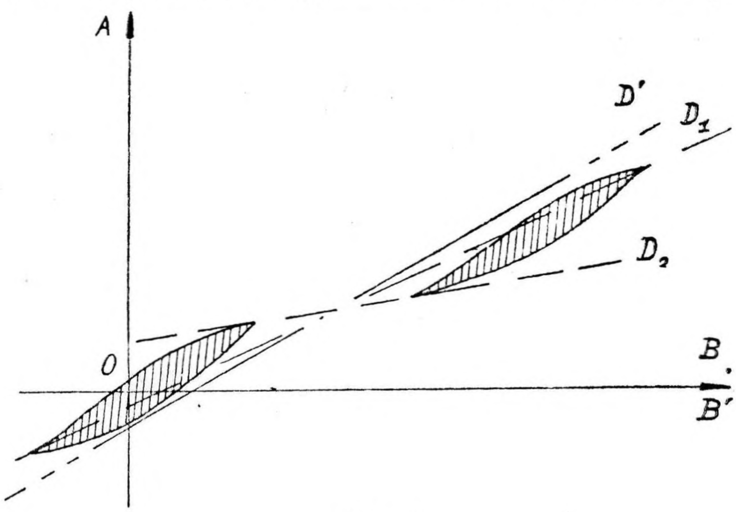


Fig. 9.

lines of concordance D_1 and D_2 may then be drawn, of which one will be used for the long-range tides, and the other for the short-range tides, in order to obtain by means of the tide at A an order of magnitude for the tide at B.

B. — In the case of a short observational period, two separate patterns are frequently obtained, even when the modulus of one of the constituents is not larger than the sum of the moduli of the others. In this case, the relative position of the *n* vectors (of indices 1 to *n*, say), whose speeds are adjacent, remains practically unchanged during the entire period of observation; it is said that such waves of indices less than *n* « do not separate ». The resultant of these vectors has an approximately constant modulus and phase during the observational period, and behaves as an actual tidal constituent, whose modulus may be larger than the sum sum of the moduli of the « separate » waves with indices larger than *n*. Thus, in a correlation area such as the hatched section in Figure 10, a short period of observation will only give the two cross-hatched areas.

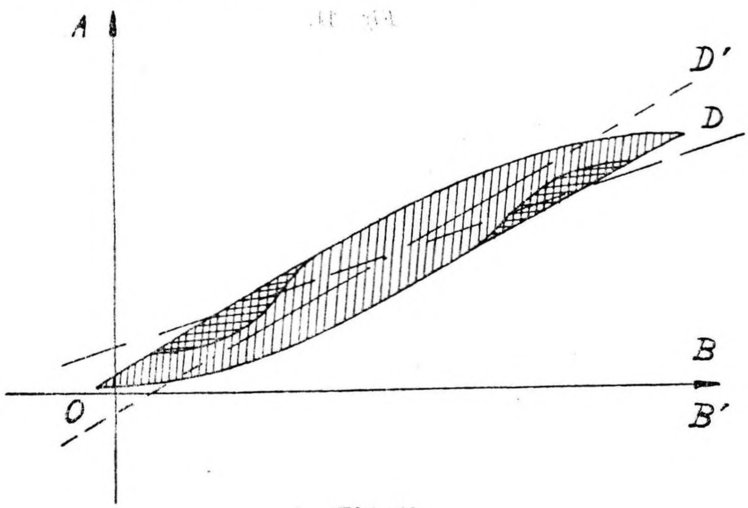


Fig. 10.

The concordance that may be derived from the graph is shown by the straight line D. This concordance is entirely adequate for expressing the tide at B in terms of that at A *during the period of observation,* and enables the easy calculation of formulae for the reduction of soundings in the various areas of a hydrographic survey. But the concordance should be used with caution in determining sounding datum and in investigating the characteristics of the tide at B. If a new short period of observations is used simultaneously for A and B, other parts of the correlation area and a line D different from the first will be found.

This type of figure is especially encountered when the tides are of a single type (generally semi-diurnal), which is why concordance is described as « good » when they are of such type. *It is clear that this quality is only apparent.*

C. — There is a case of dissimilar tides where the correlation area is reduced to a line, i.e. the case where the two tides at A and B consist of two waves only (Figure 11). Knowledge of $\overrightarrow{R^A}$ at time $t + k$ supplies $\overrightarrow{R^{3'}}$ at time t, as in the general case, but here only two directions are possible for vector $\overrightarrow{B'_1}$, and therefore for the complementary vector $\overrightarrow{b_1}$, and finally only two vectors R^B at the time of HW at B. Instead of infinity as in the general case, a curve is obtained like the one shown in Figure 12. Each branch of the curve corresponds to the case where $\overrightarrow{B'_1}$ is on one side or the other of $\overrightarrow{R^{B'}}$.

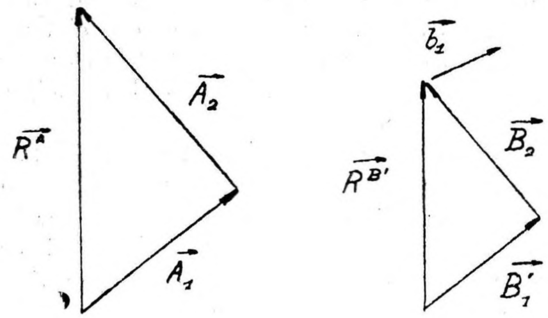


Fig. 11.

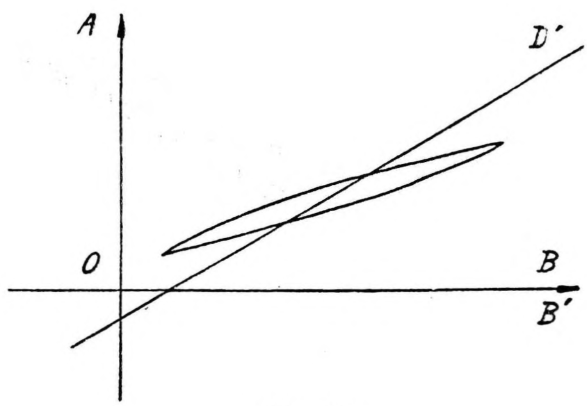


Fig. 12.

If one of the waves is a solar wave, it suffices to know whether the time of HW at A precedes of follows a given time in order to ascertain without ambiguity the height of the corresponding HW at B.

D. — Let us now examine two dissimilar tides at A and B, consisting of only three waves, and let us assume that the heights of the corresponding high waters at A and B are recorded, *only as regards those which occur at A at a given time of one of the waves.* For the sake of clarity, it will be assumed that one of the three waves is a solar wave, and that the time selected is the mean time of HW at A. It will readily be seen that the previous case obtains, and that the height of HW at B is determined without ambiguity. It is possible to visualize the construction of the various curves corresponding to the various times taken for HW at A, the above given time being taken as reference, but it is more convenient to tabulate these results.

III. — CONCLUSIONS AS REGARDS CONCORDANCE METHOD

The preceding account has been limited to a cursory description of height concordance. Concordance in time has similar characteristics.

The above indications are sufficient for the purpose of showing both the simplicity of application of the concordance method and the limits within which it may be used. The method is particularly suited to the investigation of corrections that should be applied to soundings in hydrographic surveying, but as regards the precise determination of tidal characteristics and prediction, it can only supply qualitative results.

IV. — HARMONIC ANALYSIS BY MEANS OF APPROXIMATE HARMONIC CONSTANTS

A) *Separation of a wave.*

The height of water due to the tide at a port A may be written:

$$y^A = N^A + \sum_i A_i \cos (t_i - \alpha_i)$$

where:

N^A is the mean level;

A_i and α_i are the harmonic constants of the wave of index i;

$t_i = q_i t$ is the « time of wave i ».

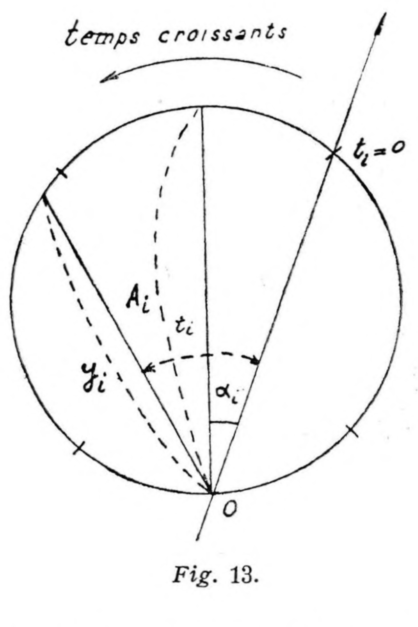


Fig. 13.

We shall for the time being consider the tide due to a single wave (A_i, α_i), and propose to determine the phase α_i and modulus A_i. It is first assumed that the mean level is known, and the curve for heights y_i is plotted in polar coordinates above this mean level against time t. This height is expressed by:

$$y_i = A_i \cos (t_i - \alpha_i)$$

and the curve obtained is a circle. Figure 13 shows that the plotting of this circle supplies A_i (its diameter) and α_i.

It will now be assumed that mean level is only known approximately (say determined by inspection). The curve then obtained is a spiral. But if construction is continued during a period equivalent to 24 h_i, $h_i = \frac{15}{q_i} h_m$, where h_m is the mean time), for each polar angle, we get two points corresponding to the two generally different values of y_i, and the centre of these two points is located on the circle previously described. The circle may thus be plotted and supplies the elements A_i and α_i.

We shall then examine an additional wave (A_j α_j q_j). The height of water at time y is written :

$$y_{i+j} = y_i + y_j = A_i \cos (t_i - \alpha_i) + A_j \cos (t_j - \alpha_j)$$

Retaining the same representation as above, it will be seen in Figure 14 that y_i is algebraically increased by the projection on this segment of vector $\overrightarrow{A_j}$, the angle of projection being $(t_j - \alpha_j)$.

If the height y_{i+j} is again taken at a time $12\ h_i$ later (i.e. at time $t_i + 12\ h_i$ of the i wave), segment OY is in the same location but vector $\overrightarrow{A_j}$ will have rotated

$$\Delta q \left(\frac{q_j - q_i}{q_i} \right) \pi.$$

It will therefore be realized that by repeatedly taking heights for y_{i+j} under these conditions, segment y_j will sometimes be positive in length and sometimes negative, except for the harmonics of wave i whose frequency will be an odd multiple of the fundamental frequency (see IV-E). By averaging the y_{i+j} heights, we get :

$$y_i + \mathcal{R}_j^i$$

where $\mathcal{R}_j^i$ is a residual which will be examined later. It will readily be realized that this residual decreases as the number n of heights taken increases. The residual moreover cancels out during the course of operations, but generally not at time t_i of the i wave, so that this property can but with difficulty be used.

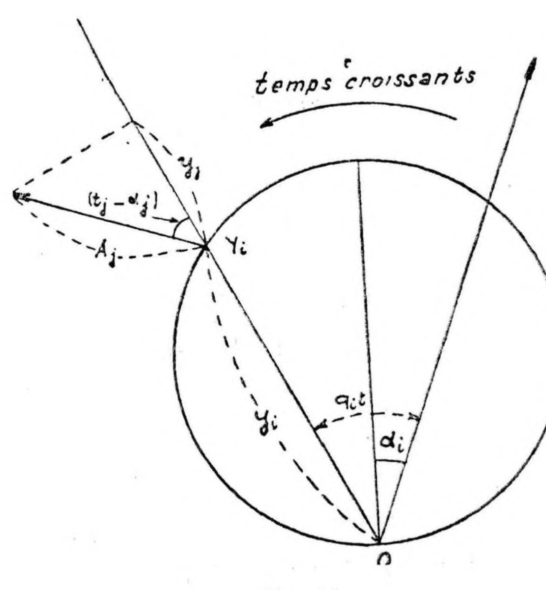


Fig. 14.

A tide will finally be considered which is composed of several waves, and from which it is proposed to derive the harmonic constants of the i wave. Application of the process to both the times of wave $(t_i)_1$ and $(t_i)_2$ will enable the determination, provided a sufficient number of observations are available, of $(y_i)_1$ and $(y_i)_2$, and the construction of the circle passing through the origin and the two points:

$$(Y_i)_1 \{ (t)_1 \ ; \ (y_i)_1 \}$$
$$(Y_i)_2 \{ (t_i)_2 \ ; \ (y_i)_2 \}$$

In practice it will be well to construct a certain number of points $(Y_i)_p$ to take care of the errors in observations which have been reduced already in the plot (by hand or tide-gauge) of the height-curve against time. If the duration of observations is of no great length, we get:

$$(y_i)_p + \sum_j (\mathcal{R}_i^j)_p \quad \text{au lieu de} \quad (y_i)_p$$

and for a given duration of observations $(\mathcal{R}_j^i)$ depends on the time $(t_i)_p$ considered. It will be seen that even if such times $(t_i)_p$ are fairly numerous and are evenly distributed over the 12 hours of the wave, the circle which most nearly fits the points $(y_i)_p$ thus obtained is not the same circle as that determined above and which supplies the constants sought for.

If the i wave is a solar wave, and if times $(t_i)_p$ one hour of mean time apart are taken, it will be seen that normal harmonic analysis is the process actually being used, and that the preceding considerations are a method of describing the theory.

B) *Study of residual.*

Vector $\overrightarrow{Aj}$ is examined in its various positions during the n observations of heights as previously defined. Such positions are distinguished by:

$$\overrightarrow{A_j^1} \quad \overrightarrow{A_j^2} \quad \dots \quad \overrightarrow{A_j^n}$$

The residual $(\mathcal{R}_j^i)^n$ at the end of n measurements will be the projection on axis OY of the resultant of n vectors $\overrightarrow{A_j^m}$ ($m = 1 \dots n$). The extremity of the resultant will be the barycentre ω_j^n of the n extremities of the n vectors $\overrightarrow{A_j^m}$. The extremity of vector $\overrightarrow{A_j^m}$ may be replaced by the centre of gravity of the circular arc of centre Y_i and radius:

$$R = \frac{\beta}{\sin \beta} A_j^m$$

placed as shown by Figure 15 with recpect to vector $\overrightarrow{A_j^m}$.

Thus the point ω_j^n will be the centre of gravity of a circular arc of angle $2n\beta$ at its centre. The locus of point ω_j^n has been plotted in Figure 16. Angle 2β is equivalent to:

$$2\,\beta^\circ = \left(\frac{q_j - q_i}{q_i} \right) \times 180^\circ.$$

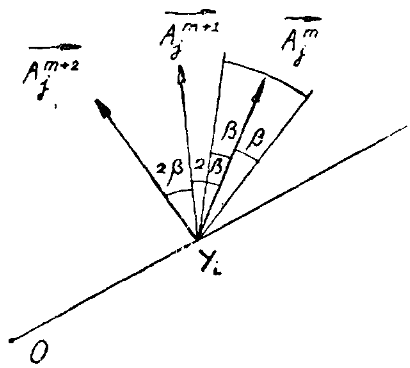


Fig. 15.

In Figure 16, vector $\overrightarrow{A^1_j}$ corresponding to the first height measurement may be drawn, followed by axis OO' containing OY_i, provided angle $(q_j t - \alpha_i)$ is known for this period. The number n of consecutive heights plotted from observations determines arc $2n\beta$ and locates vector $\overrightarrow{Y\,\omega^n_j}$, whose projection on axis OO' supplies $(\mathcal{R}^1_j)^n$. This operation may be repeated for all j waves in the tide, and in particular it is possible to ascertain the time of initial measurements, which, for a given n value, supplies a zero-residual for one of the j waves (say the most important one), or one may ascertain the smallest n value which for a given time of the beginning of observations cancels the residual for a given wave. OO' need only be approximately normal to $\overrightarrow{Y_i\,\omega^n_j}$. It will be noted that points ω^n_j are located on the spiral of the figure but do not occupy all points thereof, as n is an integer. In particular, if the heights for p times t_i of the wave are measured and the value n is selected, the residuals corresponding to each of these times will generally not be very different (provided the tide is of the single type), since the projection of a point of the spiral on axes OO' located within an angle smaller than 2β will be involved. The conclusion is that multiplication of the times t_i (such as hourly measurements) does not reduce the residual, as anticipated in the foregoing section.

C) *Indirect use of approximate constants.*

Thus the determination of points ω^n_j enables the calculation of the residuals ^i_j left by all the j waves on the wave i to be extracted, and therefore the separation of all waves whose difference in phase-lag does not vary exactly by 2π during the observational period. But the placing of point ω^n_j presupposes a knowledge of the harmonic constants of wave j. These constants are not known, but their approximate value may be estimated, with the result that separation of the waves, whose phase-lag difference does not vary exactly by 2π during the period of observation, will not be perfect but will be the best that can be derived from such period.

The use of approximate constants for the computation of such residuals has been termed by us « *indirect* utilization », and is applied as described below (see Figure 16):

Locus of point ω^n plotted against $2n\beta$

The outer solid black curve corresponds to $2n\beta$ between 0 and 2π.

The dot-and-dash curve corresponds to $2n\beta$ between 2π and 4π.

The inner solid black curve corresponds to $2n\beta$ between 4π and 6π.

For higher values of angle $2n\beta$, the « loop » corresponding to $2n\beta$ between $2k\pi$ and $2(k+1)\pi$ closely approximates a circle. The diameter of this circle ρ and the polar angle of this diameter are given plotted against k by the curves shown below.

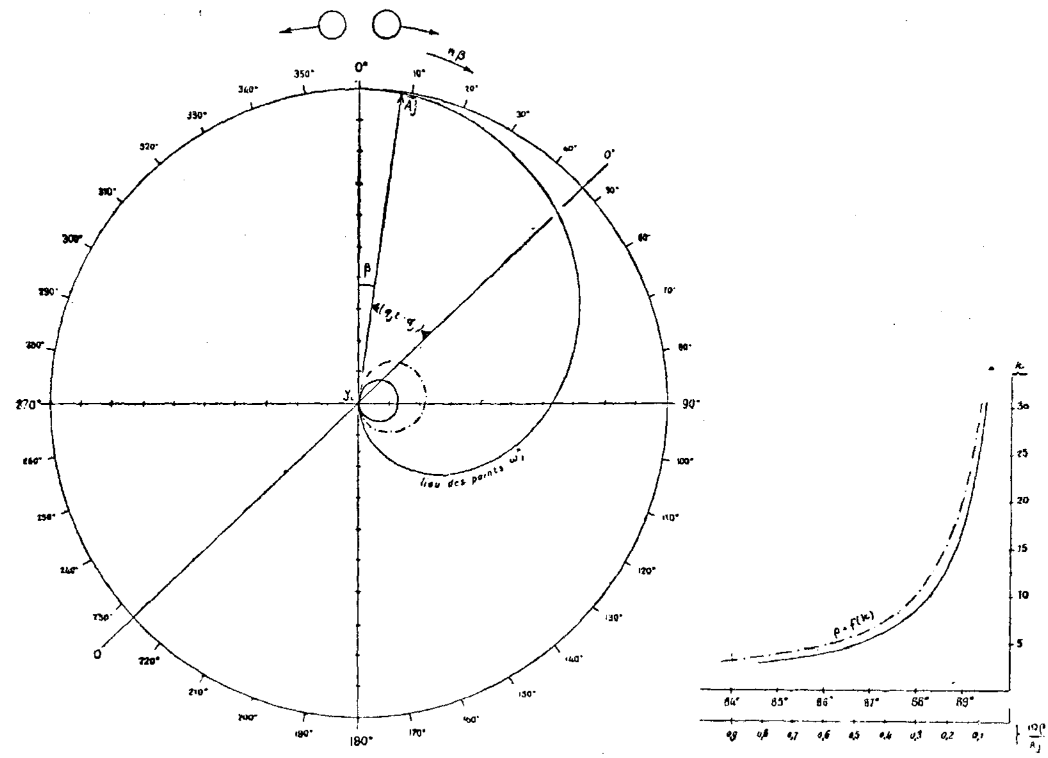


Fig. 16.

a) Vector A^1_j corresponding to the first n observation is placed on the graph in such a way as to produce an angle β with respect to axis $Y_i 0°$, and so that its extremity is located on the spiral. The scale of lengths is thence determined on the graph, as well as the positive and negative directions of the rotations indicated by + and — in the circles with arrows: thus if $\beta > 0$, the + direction is in the right-hand circle, and if $\beta < 0$ the + direction is in the left-hand circle.

b) The extremity of vector $Y_i \omega^n_j$ is on the spiral and its polar angle is $n\beta$ reckoned in the direction of arrow $n\beta$.

c) Angle $(q_j t - \alpha_j)$ enabling the positioning of axis OY_i is plotted with due regard to the direction of circulation on the circle (see *a)* above), and may easily be computed with the Nautical Almanac as regards the eight principal waves. If m^1_j is the mean time of the first of the n observations of the i wave and if m^M_j is the mean time nearest HW of wave j, then the ratio $\dfrac{q_j t - \alpha_j}{m^1_i - m^M_j}$ is in the vicinity of:

15° for the diurnal waves;

30° for the semi-diurnal waves;

60° for the quarter-diurnal waves.

d) The length of the projection of vector $\overrightarrow{Y_i \omega^n_j}$ on axis OY_i, measured on the same scale as $\overrightarrow{A^1_j}$, supplies the value of $(\mathcal{R}^i_j)^n$

D) *Direct utilization of approximate constants.*

1. If the moduli of the j waves are small, the residuals left by these waves on an i wave are small and may be neglected with regard to the modulus of that wave. It is assumed that curve C_a of an artificial tide is available, whose constants are the approximate constants, and that the heights as described in section A are taken, not from mean level, but from curve C_a. This operation will enable the extraction of a wave whose vector is a_i, whose constants are $(a_i, \varepsilon_i, q_i)$, and such that (Figure 17):

$$\overrightarrow{A_i} \text{ approx.} + \overrightarrow{a_i} = \overrightarrow{A_i} \text{ observed.}$$

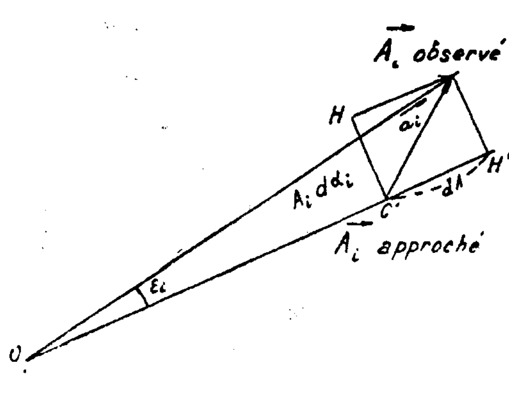


Fig. 17

If the period of observation is short, there will be residuals r^i_j left in wave a_i by the a waves, but if the moduli of such a_j waves are small, the residuals will be negligible with regard to A_i.

If the constants of the artificial tide are suitably approximate, the observed height is written:

$$dy = \sum dA_i \cos (q_i t - \alpha_i) + A_i d\alpha_i \sin (q_i t - \alpha_i)$$

and if the dy values are taken at times t_i and $t_i + 12\ h_i$, the mean of the dy values over a long period represents that which, in each, relates to a_i, and therefore:

$$dy_{\text{mean}} = a_i \cos (q_i t - \alpha_i) + A_i \varepsilon_i \sin (q_i t - \alpha_i).$$

Rectangular axes are now used in a_i and $A_i \varepsilon_i$; the equation above is that of a straight line whose distance to the origin is precisely dy_{mean}. The straight lines corresponding to the various times t_i of the wave, i.e. to the various values of $(q_i t - \alpha_i)$, all intersect at an identical point I, whose coordinates supply:

$$dA_i = a_i = \overline{0'H'}$$

$$A_i d\alpha_i = A_i \varepsilon_i = \overline{0'H}$$

Among the times t_i to be selected, those which render $(q_i t - \alpha_i)$ equivalent to 0 (HW or LW of the approximate wave) or to $\pi/2$ (half-tide of the approximate wave) may advantageously be taken. In the first case, dy_{mean} is very nearly equal to a_i, and in the second, ε_i closely approximates $\frac{dy_{\text{mean}}}{A_i}$. The two corresponding straight lines are of course those which are parallel to the axes in Figure 18.

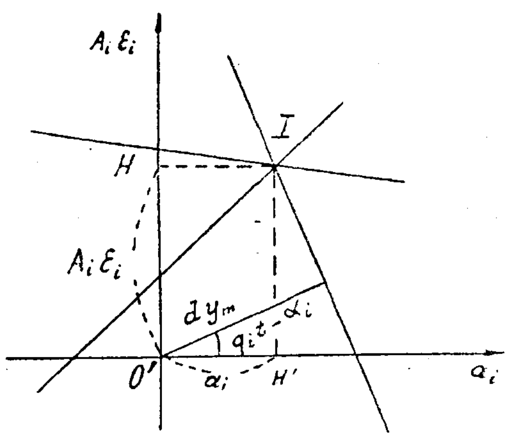


Fig. 18.

A graph is obtained which has the appearance of all graphs that are illustrative of approximation methods. It may however be remarked that here the straight lines are not tangent to the loci of point I, but are the loci themselves, so that, apart from the elimination of the residuals, the approximate constants may be fairly different from the actual constants, without change to the graph. Only the interpretation will be different. Figure 17 shows that:

$$A_i = \frac{A_i \text{ approx.} + \overline{0'H'}}{\cos \varepsilon_i} \qquad \tan \varepsilon_i = \frac{\overline{0'H}}{A_i \text{ approx.} + \overline{0'H'}}$$

II. In this form, harmonic analysis is closely related to concordance between the actual tide and the approximate artificial tide, but such concordance is established separately for each wave.

The practical application of the method of computation raises no difficulty. When surveying, it will suffice to propose approximate constants to the Central Hydrographic Office and request to be supplied with a curve obtained with the predicting machine for the period of observations. The times t are easily obtained by means of a rule graduated for the speed of each wave, and the artificial curve carries reference marks showing HW for the waves every five or six days in order to obviate the accumulation of errors due to plotting with the graduated rule. When computations are carried out at the Hydrographic Office, the approximate tide need not be recorded; it will be sufficient to read the height of water on the dial at each passage of the wave-pointer in front of a graduation of the ring at the base.

E) *Particular case of « odd harmonics ».*

The case here involves the harmonics A_j of a wave A_i whose frequency is an odd multiple of the frequency of A_i. We have previously noted in section IV-A that, in these waves, the residual $\mathcal{R}^i_j$ did not decrease in accordance with the number of observations as described. It may be added that the procedure indicated in section IV-C for determining the residual is impracticable: as an example, wave M_6, an overtide due to the superimposing in shallow-water of waves M_2 and M_4, may be taken. In this case $\beta = 180°$, and it will be seen that vector $\overrightarrow{A^1_j}$ cannot be placed in Figure 16. Retaining this example, we shall see how M_6 may be separated from M_2.

A wave A_j is taken, and it is assumed that the heights Y_j it produces are plotted in polar coordinates as in section IV-A, but this time *by taking absolute values of* Y_j. The diagrammatic representation (Figure 19) now consists of two circles, one corresponding to the polar angles between 270°, 0° and 90°, and the other to the polar angles between 90°, 180°, and 270°. Two segments $(OY)_1$ and $(OY)_3$ separated by an angle of 270° yield by geometric construction a point m; similarly, by geometric construction, two segments $(OY)_2$ and $(OY)_4$ separated by an angle of 270° supply a point M, $OM = Om = A_j$.

The graph of wave M_2 is now considered, which is similar to Figure 17 and such that the straight lines I and III are at right angles, i.e. correspond to two series of measurements separated by three hours of M_2; as is the case for the pair of lines II and IV (Figure 20). It will be assumed for the time being that the points M_2 representing wave M_2 is known; then M_2H_1 and M_2H_3 represent segments of group $(OY)_1$ $(OY)_3$; whence P represents m and similarly Q represents M. M_2 is therefore at the midpoint of PQ and easily placed. It is not essential, in order to find point M_2, to take groups of heights of wave M_2 separated by three hours of the wave, but this practice considerably facilitates the investigation of M_6 and is to be recommended. The length $M_2P = M_2Q = A_j$ supplies the modulus of M_6. Finding the phase is simple: let us assume, for purposes of clarity, that the line I corresponds to the HW of M_2; consequently line III corresponds to the process down to mean level (the case illustrated by the figure), and we have the following pair of equations:

$$\overline{M_2H_1} = A_j \cos (3q_it - \alpha_j) \text{ for } (q_it - \alpha_i)=0 \quad \text{hence } \cos (3\alpha_i - \alpha_j) = \frac{\overline{M_2H_1}}{A_j}$$

$$\overline{M_2H_3} = A_j \cos (3q_it - \alpha_j) \text{ for } (q_it - \alpha_j)=90^\circ \quad \text{hence } \sin (3\alpha_i - \alpha_j) = \frac{\overline{M_2H_3}}{A_j}$$

These equations determine α_j without any ambiguity.

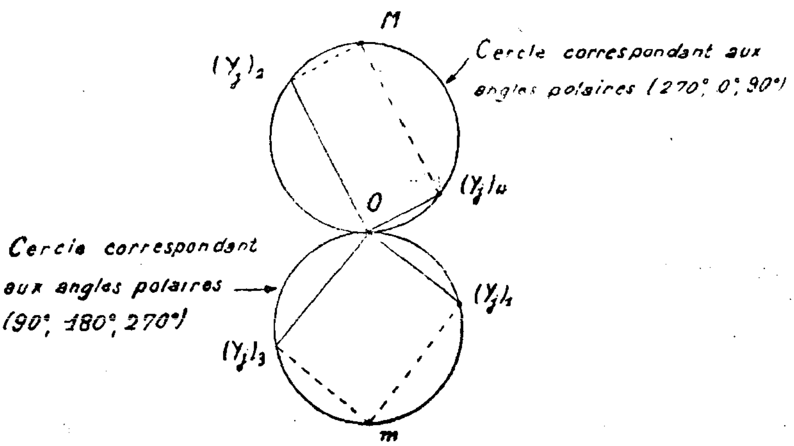


Fig. 19.

Thus, whenever the direct use of approximate constants leads to a graph with lines too far apart to be attributed to inaccuracy of observation, the existence of an odd harmonic must be suspected. It is not advisable to introduce this odd harmonic in the approximate constants, as the cause is local and data for its evaluation are practically non-existent. The precaution should merely be taken of constructing at least two pairs of right-angled segments $(OY)_1$ of the fundamental tide.

F) *Conclusions.*

Thus approximate constants play the same part in the computation of the constants at a port as the reference tide in the concordance method. The number of ports at which the constants are known is quite large already, and the availability of constants for a port near the hydrographic survey is not exceptional. Moreover, in selecting the approximate constants, the conclusions reached in the available theoretical studies of the propagation of the various waves may be used. In the particular case where it is proposed to rectify the former harmonic constants of a port by means of recent observations, the former constants may be taken as approximate constants.

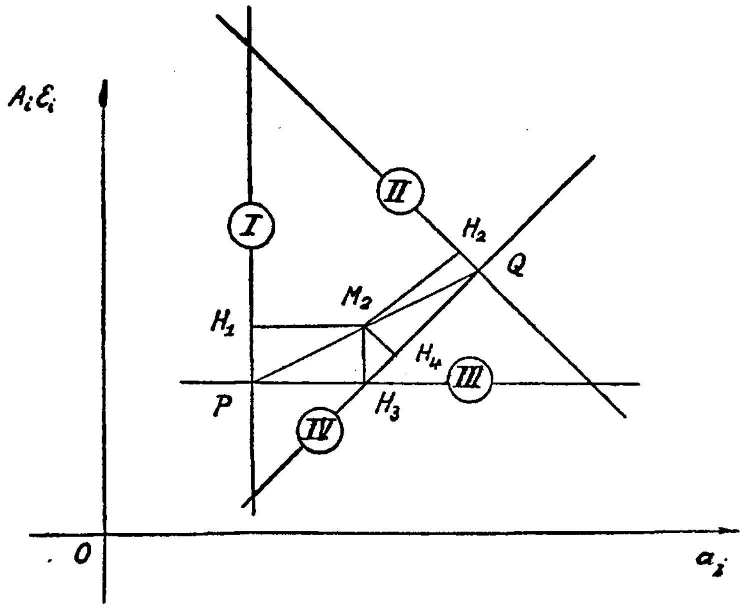


Fig. 20.

The essential feature of the presently described method finally consists in the elimination or calculation of the residuals, which is of undoubted advantage: if we consider the curve in Figure 16, we note that the residual $\mathcal{R}_j^t$ may still be equivalent to 0.06 A, whereas the j wave has gained $10\,\pi$ over the i wave. In order to eliminate this residual, conventional analysis makes use of a period of observation whose length is defined according to the speeds q_i and q_j, with the result that the entire observed period may not be used, or that no period of adequate length may be found in the observations.

Another feature of the proposed method is the taking of heights at times t_i of the wave, whereas the conventional method uses heights at the mean times. This practice has the following advantages:

a) As the periods of the waves being sought are different, the number of heights being equal, « exploration » of the curve for heights is more thorough than in the conventional method.

b) As the observations in relation to each segment OY_i (see Figure 14) are used independently, an error of observation or summation may easily be detected, and the harmonics whose frequencies are odd multiples of the fundamental frequency may easily be separated. Moreover, the number of OY_i segments determined for each wave is adaptable to the accuracy desired and to the *amount of time that may be available for analysis.*

c) The segments OY_i (and consequently the times t_i) may be selected with due regard to optimum conditions for the determination of the unknown values A_i and α_i. In particular, the method of hourly heights expressed in mean time introduces into the mass of equations a large number of equations in which the coefficients of the unknown values are small. These equations are of little use in solving the unknowns, as shown by the considerations forming the basis of the Cauchy-Tisserand method in the resolution of a system of superabundant equations.

d) Use of the time of the wave enables the discovery of a possible odd harmonic, whereas in the conventional method, the existence of such a harmonic must be assumed beforehand.

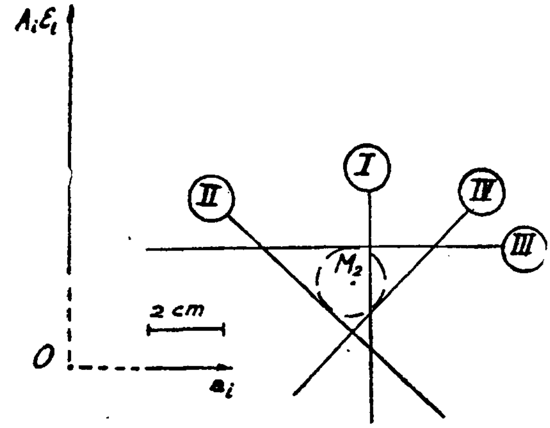
Although the present system may appear to require more time than the conventional method, in actual fact this is not the case, as heights are measured with a curvimeter (or surveyors' tape), which effects the algebraic sums on its own, with the result that the number of readings on the instrument is extremely small.

V. — EXAMPLES

A) As a test, the method of approximate constants was first applied to a height curve corresponding to thirty days of observation. This curve was plotted by the predictor, so that the « observations » are perfect and the results of the calculation may be compared with the exact values. The constants were determined by the « direct method » (see IV-D). The constants of waves M_2 and (S_2+K_2), which had the largest modulus, were determined by four straight lines. The graphs are shown in Figure 21. The other waves were determined by two straight lines parallel to the axes.

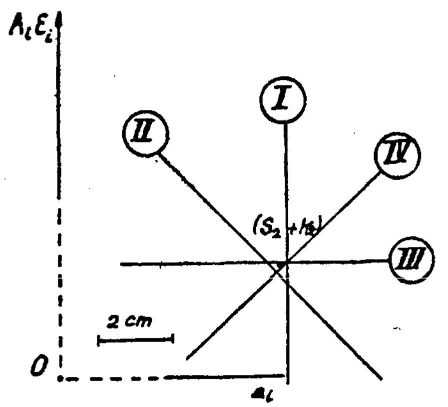
The results are shown in the following table: waves K_2 and S_2 have been separated as described in V-B.

ONDES	CONSTANTES CALCULÉES		CONSTANTES EXACTES	
	A cm	$\alpha°$	A cm	$\alpha°$
M_2'	109,5	10	110,0	10
S_2	44,7	199	45,0	200
K_2	9,9	199	10,0	200
K_1	40,1	159	40,0	160



Graph of wave M_2

(The point representing the approximate constants is outside the figure.)



Graph of wave (S_2+K_2)

(The point representing the approximate constants is outside the figure.)

Fig. 21.

These results were obtained on curves plotted at the scale of 1 :15, i.e. a distance of 1 cm corresponds to 0.6 mm on the curve. This amount is the limit of graphical accuracy.

The following information was obtained during the test :

a) The error in the measurement of height differences with the curvimeter depends on the number of measurements and not on the measured length, i.e. on the scale of heights. Various measurements taken under identical conditions showed that the error over a thirty-day observational period is 0.10 mm per measurement for the semi-diurnal waves, and 0.12 mm per measurement for the diurnal waves. At the scale used here, this means an error with reference to segment OY_1 (Figure 14) of 1.5 mm for the semi-diurnal waves and 1.08 mm for the diurnal waves. The use of the curvimeter is therefore entirely suitable.

b) The largest error is due to the setting of the predictor. This is only possible to within 0.3 mm, which at the scale used results in an error with reference to segment OY_1 of 4.5 mm. This explains the discrepancy of 5 mm noted with regard to wave M_2.

c) The time required to extract a wave from thirty days of observations is 50 minutes for a semi-diurnal wave if one pair of segments OY_1 is used (which decreases to 40 minutes per pair if three pairs are used); and 40 minutes for a diurnal wave if one pair of segments OY_1 is used.

B) *Natural tide.*

The method was then applied to a natural tide. Thirty days' observations at Ziguinchor (on the Casamance River, in French West Africa) were used. This tide was recorded on the scale of 1 :10 on a Brillié recorder by the West African Survey. This set of observations was chosen as they had been analysed elsewhere by the least squares method described by B. Imbert (1) and by the conventional analysis method (2). Two features of the work deserve particular attention: the bringing out of a constituent wave M_6 and the separation of waves K_2, S_2 and K_1, P_1.

M_6 wave (See IV-E). — This wave was not introduced in the artificial tide compared with the actual tide according to the process indicated in IV-D; the graph of wave M_2 therefore supplied a « cocked hat » which is incompatible with accuracy of the observation (Figure 22).

The separating of wave M_6 enables the constants of M_2 to be fixed at:

$$\begin{cases} H = 280 \text{ mm.} \\ g_0 = 17.7°. \end{cases}$$

Those of M_6 are:

$$\begin{cases} H = 10 \text{ mm.} \\ g_0 = 274°. \end{cases}$$

It will be noted in Figure 22 that the distance from M_2 to the representative points, derived from the other methods, is of the order of the modulus of M_6, a wave which the other methods did not separate.

Separation of waves K_2 and S_2. — The wave K_2 not having been introduced in the artificial tide, the residual of K_2 left in S_2 is important since the speeds of these waves are very close to each other. In order to compute the residual, we shall proceed as in IV-C, in which j is attributed to K_2 and i to S_2.

Vector $\overrightarrow{A^i_j}$ is practically coincident with Y 0°, since B = 0.25°; in the present case $n = 114$, therefore $\beta = 28°$, and ω^n_i is located (see Figure 23). Thus, it is seen that residual $(\mathcal{R}^i_j)^n$ on any segment OY_i will be the projection on this segment of vector $\overrightarrow{Y_i \omega^n_i}$; this vector is parallel to vector A_j *taken at the centre of the period of observation,* but of modulus $\left(\frac{\omega^n_i}{A^1_i}\right) A_j$ The ratio $\frac{\omega^n_j}{A^1_j}$ is measured on Figure 23, and found to be equivalent to 0.95. It is now assumed that the tide due to $(S_2 + K_2)$ at Dakar is similar to the tide due to these same

(1) See Information Bulletin of « Comité Central d'Océanographie et d'Etude des Côtes », Year VI, No. 9.

(2) See *Analyse d'une courte période d'observations* (Analysis of a short period of observations), by M. ROLLET DE L'ISLE, « Annales Hydrographiques », 1896.

waves at Ziguinchor, which is a logical hypothesis in view of the nearness of both places and the close equivalence of the speeds (see I-B). Hence:

$$\left(\frac{A_j}{A_i}\right)_{\text{Dakar}} = \left(\frac{A_j}{A_i}\right)_{\text{Ziguinchor}} = 0.34$$

(account being taken of astronomical factor $f_K = 1.18$ for the period of observation).

Therefore:

$$\left(\frac{\omega_j^n}{A_i}\right)_{\text{Dakar}} = \left(\frac{\omega_j^n}{A_i}\right)_{\text{Ziguinchor}} = 0.34 \times 0.95 = 0.323$$

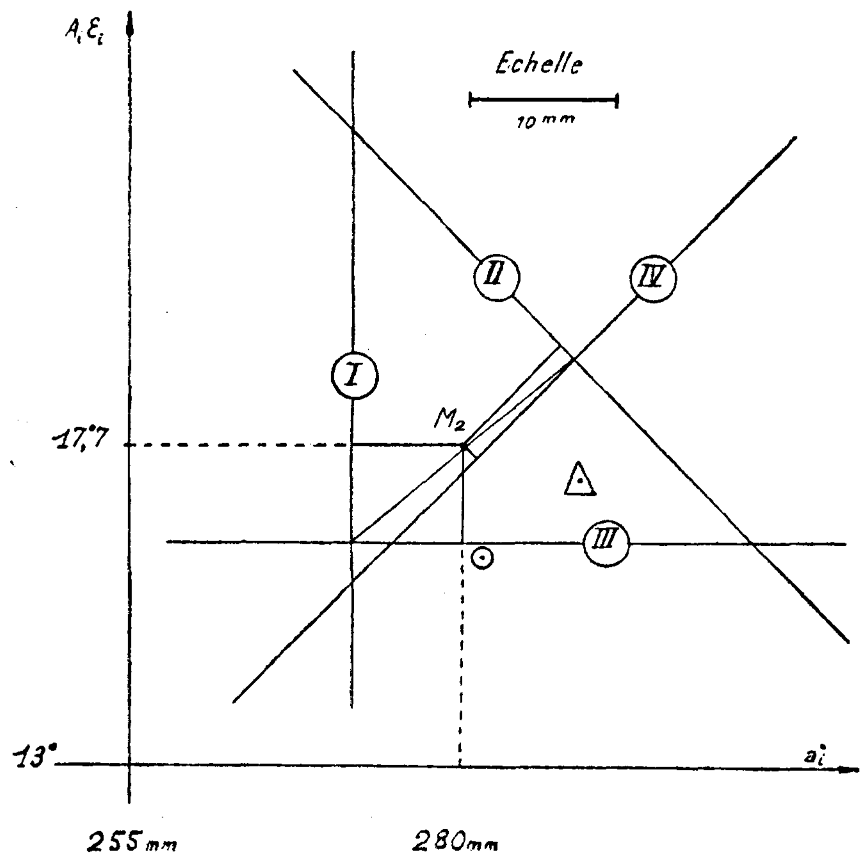


Fig. 22.

Point representing M_2 according to results of Least Squares Method : △ ; of Conventional Method : ⊙

I. Straight line corresponding to HW of M_2.
II. do 3 hours of M_2 following HW.
III. do 6 hours of M_2 following HW.
IV. do 3 hours of M_2 before HW.

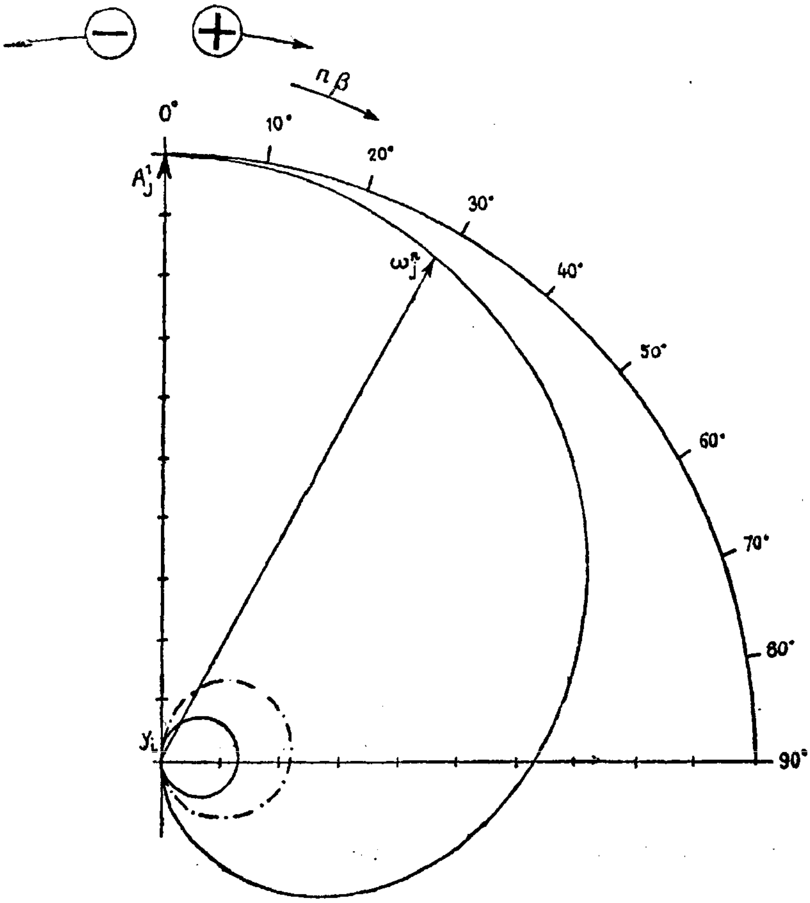


Fig. 23.

It will then be recalled that $\overrightarrow{Y_i \omega_j^n}$ is parallel to vector $\overrightarrow{A_j}$ *taken at the centre of the period of observation,* and the triangle of Figure 24 is constructed for Dakar, for angle $\gamma = 154°$ and the ratio $\frac{AB}{OA} = 0{,}323$ are known.

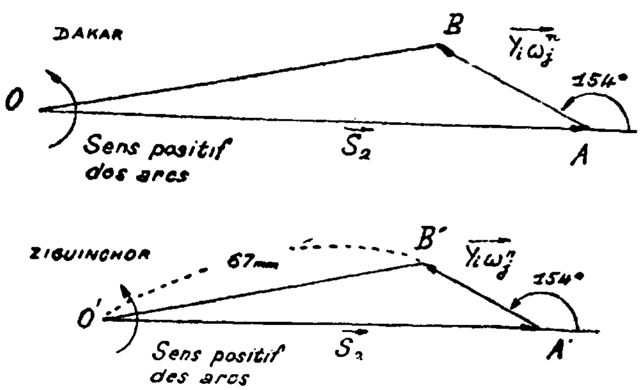


Fig. 24.

The triangle corresponding to Ziguinchor is similar (see I-B). Now O'B' represents the vector resulting from ($S_2 + K_2$), which was obtained from the direct utilization of the approximate constants (IV-D); we found that:

$$H_{S_2 + K_2} = 67 \text{ mm}$$

$$g_{0\,(S_2 + K_2)} = 54^\circ$$

The harmonic constants of S_2 and K_2 are therefore obtained without difficulty, i.e. at Ziguinchor:

$S_2 \begin{cases} H = 91 \text{ mm} \\ g_0 = 66^\circ \end{cases}$ $K_2 \begin{cases} H = 26 \text{ mm} \\ g_0 = 64^\circ \end{cases}$

Separation on the basis of results from the least squares method was not carried out. Separation in accordance to the conventional method give slightly different results, for the following reasons:

a) In the conventional method the ratio $\frac{A_j}{A_i}$ is always taken as being equivalent to 0.273 regardless of local conditions.

b) The residual is calculated by taking $\overrightarrow{A_j}$ at the centre of the observation period, but with its integral value. This practice gives rise to an error that increases with the length of the period of observation; thus it may be seen in Figure 23 that for three months of observations $\frac{\omega_j^n}{A_j^1} = 0.6$.

Separation of waves K_1 and P_1. — These waves were separated by the same method as the one just described. The divergence with respect to the conventional method as regards P_1 may be explained by the fact that this method assumes *a priori* that P_1 and K_1 are equal in phase, whereas they have been given the phase-difference obtaining at Dakar, which is the more logical assumption.

Waves ONDES	Least Squares Method MÉTHODES DES MOINDRES CARRÉS		Conventional Method MÉTHODE CLASSIQUE		Described Method MÉTHOD' DÉCRITE	
	$H^m/_m$	g^o_0	$H^m/_m$	g^o_0	$H^m/_m$	g^o_0
M_2^*	285	17,2	281	16,1	280	17,7
$S_2 + K_2$	63	54,9	67	54,6	67	54,0
S_2	Non séparée		95	69	91	66
K_2	Non séparée		26	69	26	64
N_2	31	5	18	359,5	32	6
K_1	Non séparée		60	61	62	59
P_1^*	Non séparée		20	61	10	41
O_1	14	297,7	18	303,7	15	300
M_4	30	232	29	231,8	31	230
M_6	Non séparée		Non séparée		10	274

Voir dans le texte pour l'explication des divergences.

* See text for explanation of divergences.