NOTE ON METHODS OF DETERMINING MONTHLY AND ANNUAL VALUES OF MEAN WATER LEVEL

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Introduction

A subject which is becoming of increasing interest to an ever-widening cross-section of both scientific and non-scientific workers is that of long period changes in sea level. By way of illustration we may cite investigations into the relationship between sea level and the climate, ranging from variations with periods of days, weeks and months and connected with prevailing winds, air pressure variations and changes in the temperature and salinity of the water masses, to the much slower changes with periods of many centuries which are closely correlated with the melting of polar ice. National levelling networks are always related to values of mean sea level wherever possible, and meticulous observations of sea level may make it possible in the future to link triangulation networks across such bodies of water as the Adriatic, the Straits of Dover, and the North Channel of the Irish Sea. Long records of sea level observations also yield information relating to the rising or sinking of land masses.

It is against this background of an increasing need for accurate sea level data that this paper has been written. For the vast majority of research purposes it becomes necessary to eliminate variations of sea level due to the astronomical tide generating forces, and the success with which various methods achieve this goal is discussed here.

Methods of observing sea level

In general, observations may be obtained visually by means of a tide staff, or automatically by a tide gauge or water level recorder. Unless special precautions are taken, visual observations are of necessity rather inaccurate and cannot be recommended. Modern automatic tide gauges can be obtained at reasonable prices. The recording apparatus is generally housed at the site of the gauge well or pipe, but reliable remote recorders are also in common use, recording via transmission lines, radio signals or visual signals.

Mention might also be made here of the médimarémètre type of recorder. This consists essentially of a well, float and recording mechanism, but the physical response of the gauge is so damped as to reduce considerably the shorter period tidal oscillations. The records thus give approximate values of mean sea level. The degree of approximation depends upon the extent to which the major tidal oscillations (diurnal, semidiurnal, quarter diurnal, etc.) are damped out, that is to say, by the instrumental characteristics. Whilst this avoids tabulation and reduction of many readings per day, it also means that the actual variations in level from hour to hour, which are often of great interest for many other reasons, are lost.

METHODS OF DETERMINING MEAN VALUES

Since 1940 all available monthly and annual values of « mean sea level » have been collected and published by the International Association of Physical Oceanography, with the cooperation of the International Hydrographic Bureau, for many places throughout the world. No uniform system of reduction of the sea level records can be said to exist, since individual contributors to the publications use methods best suited to their local requirements. The following are the ways in which the basic daily mean values have been computed:—

(A) Direct average of 1, 3, 6 or 8 heights per day, at fixed hours

(B) Direct average of 24 or 25 hourly heights

(C) Numerical filters applied to hourly heights

(D) Direct average of high and low water heights

(E) Integration of daily record by planimeter

Before proceeding to compare mathematically the methods enumerated, we may remark that (A) can only be used in regions where tidal variations are small; the Baltic affords an example. (B), (C) and (E) give results which are generally referred to as mean sea level, to distinguish from mean tide level as defined by (D). (E) is a method in which the total area under the tidal curve is measured and divided by the (time) length of record. Theoretically the result satisfies the definition of daily mean sea level, but as the periods of tidal constituents—other than those of solar origin—are not exact sub-multiples of 24 hours, there remains a daily contribution from the tide. The accuracy of such a method depends largely upon the skill of the planimeter operator, and it is doubtful whether the speed of working can compare with other methods which eliminate tidal variations with equal success.

DAILY MEAN VALUES

Let a harmonic tidal constituent be represented by

$$\zeta_H = R \cos (\sigma H + \rho D - \varepsilon) \tag{1}$$

with R the amplitude and ε the phase lag, both constant at any one place

σ the speed in degrees per mean solar hour
ρ the speed in degrees per mean solar day
H is time in hours
D is time in days

Then with the initial value of H=O, any summation of q values at intervals of p hours gives

$$\sum_{M=0}^{q-1} R \cos (p\sigma M + \rho D - \varepsilon) = \frac{\sin pq\sigma/2}{\sin p\sigma/2} R \cos \left(\rho D - \varepsilon + \frac{q-1}{2}\sigma\right) \tag{2}$$

where $M = H/p$

Methods (A), (B). The proportion of the maximum contributions from the major tidal constituents to the daily mean values, i.e. the values of

$$\frac{1}{q}\,\frac{\sin pq\sigma/2}{\sin p\sigma/2}$$

are given in Table 1.

Method (C). In 1928 Doodson devised a system of numerical filters for the purpose of separating the different tidal species (diurnal, semi-diurnal, etc.,) one from another. The filter X_0 has been in general use at the Tidal Institute for determining mean sea level. Its efficiency in reducing contributions from all species other than those of long period is illustrated in Table 1.

When a full year's tidal analysis is not required, however, the labour of tabulating hourly heights and applying the X_0 filter can be greatly reduced by tabulating the heights at 3-hourly intervals and using a filter (Z_0), which will now be described.

Following Doodson's notation, a linear combination such as $\zeta_H + \zeta_{H'}$ may be shown to be of the form

$$J\, R \cos (\sigma H + \rho D - \varepsilon + \tau) \tag{3}$$

From the various combinations given, we may choose only the following :—

Combination with H = 0	J	τ	Eliminates
$\zeta_0+\zeta_3$	$2\cos 3\sigma/2$	$3\sigma/2$	S_4
$\zeta_0+\zeta_6$	$2\cos 3\sigma$	3σ	S_2, S_6
$\zeta_0+\zeta_9$	$2\cos 9\sigma/2$	$9\sigma/2$	S_4
$\zeta_0+\zeta_{12}$	$2\cos 6\sigma$	6σ	S_1, S_3

Using each combination once, we have

$$Z_0 = [\{(\zeta_0+\zeta_3)+(\zeta_6+\zeta_9)\}+\{(\zeta_9+\zeta_{12})+(\zeta_{15}+\zeta_{18})\}] + [\{(\zeta_{12}+\zeta_{15})+(\zeta_{18}+\zeta_{21})\} + \{(\zeta_{21}+\zeta_{24})+(\zeta_{27}+\zeta_{30})\}]$$

$$= \zeta_0+\zeta_3+\zeta_6+2\,(\zeta_9+\zeta_{12}+\zeta_{15}+\zeta_{18}+\zeta_{21})+\zeta_{24}+\zeta_{27}+\zeta_{30}$$

as the filter to be applied.

The effect of this filter upon a constituent such as (1) is given by (3), with

$$J = 16 \cos 3\sigma/2 \cos 3\sigma \cos 9\sigma/2 \cos 6\sigma,\ \tau = 15\sigma$$

Numerical values of J/16 appear in Table 1.

Method (*E*). The result of integrating one day's record by a planimeter will give, for a particular constituent,

$$\int_0^{24} R \cos (\sigma H + \rho D - \varepsilon)\, dH = \frac{R}{\sigma} [\sin (\sigma H + \rho D - \varepsilon)]_0^{24}$$

Now $24\sigma = \rho$, thus the contribution to mean level

$$= \frac{\sin \rho/2}{\rho/2} R \cos (\rho D - \varepsilon + \frac{\rho}{2})$$

Values of $\dfrac{\sin \rho/2}{\rho/2}$ are given in Table 1.

Method (*D*). Methods (B), (C) and (E) give values which are commonly referred to as « mean sea level », as opposed to « mean tide level » which will now be discussed. For obvious reasons it is a more attractive proposition to extract and average 4 high and low water heights per day than to handle 24 to 30 hourly heights per « day », but the accuracy of the results must also be considered. Corrections to the mean tide level of a predominantly semidiurnal tide arising from shallow water constituents have been given in Chapter XI of the *Admiralty Manual of Tides.* The problem dealt with here is of a more general nature.

Consider two harmonic constituents

$$Z = R \cos \sigma t, \qquad z = r \cos (\sigma_1 t - \gamma),$$

the former being predominant.

Let $\sigma_1 = n\sigma + \Omega$, where Ω is small and Ωt may be considered constant over the period of a day.

Then $z = r \cos (n\sigma t - \alpha)$ where $\alpha = \gamma - \Omega t$.

We have taken the origin of time at the first high water of the predominant constituent. Let t_m be the times of high and low waters of the compound tide, given by

$$t_m = \frac{m\pi}{\sigma} + \tau_m \qquad (m = 0,\ 1,\ \text{etc.})$$

τ_m is the time lag of the compound tide on the dominant tide.

Hence M_o, the mean tide level of a predominantly semidiurnal tide, is given by

$$M_o = 1/4 \sum_{m=0}^{3} \{ R \cos (\sigma\tau_m + m\pi) + r \cos (n\sigma\tau_m - mn\pi - \alpha) \} \tag{4}$$

whilst for a predominantly diurnal tide the summation is for $m = 0,\ 1$, and the factor is 1/2 instead of 1/4. We shall assume that $\sigma\tau_m$ and $n\sigma\tau_m$ are both small enough to put

$$\sin (\quad) = (\quad) - 1/6 (\quad)^3$$
$$\cos (\quad) = 1 - 1/2 (\quad)^2$$

τ_m is given by $\dfrac{\delta}{\delta\tau_m} (Z - z) = 0$ (5)

and with the above approximations this is satisfied by

$$\sigma\tau_m = - \frac{nd \sin (mn\pi - \alpha)}{\cos m\pi + n^2 d \cos (mn\pi - \alpha)} \tag{6}$$

to the second order in d^2, where $d = r/R$.

We can now consider 3 major cases most commonly found in nature.

Case 1. $n = 1$. From (6) we find τ_m is constant, and hence (4) gives $M_o = 0$.

This result, which can be deduced by more direct methods, means that if we confine our attentions to one species only, there is no tidal contribution to M_o. This statement applies only to tidal oscillations with periods not greater than a day.

Case 2. $n = 2$. From (6) we find that for the case of a diurnal tide dominating a semidiurnal tide ($\sigma \doteqdot 15°/\text{hr.}$),

$$\left.\begin{aligned}\sigma(\tau_0 + \tau_1) &= \frac{2\,d \sin \alpha}{1 - 16\,d^2 \cos^2 \alpha}(-8\,d \cos \alpha) = -8\,d^2 \sin 2\,\alpha = \mathrm{A}\\ \sigma(\tau_0 - \tau_1) &= \frac{2\,d \sin \alpha}{1 - 16\,d^2 \cos^2 \alpha}(2) \qquad\qquad = 4\,d \sin \alpha = \mathrm{B}\end{aligned}\right\} \quad (7)$$

ignoring higher powers of d than the 2nd.

Then (4) becomes

$$\mathrm{M_o} = \mathrm{R}\left\{-\sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2} + d\cos(\mathrm{A} - \alpha)\cos \mathrm{B}\right\}$$

which reduces to

$$\mathrm{M_o} = \mathrm{R}\,d \cos \alpha \quad (8)$$

In a similar fashion it may be shown that when $\sigma \doteqdot 30°/\text{hr.}$, the case of a semidiurnal tide with a minor quarter-diurnal tide, $\mathrm{M_o}$ is again given by (8).

Case 3. $n = \frac{1}{2}$. From (6) we find that for the case of a semidiurnal tide dominating a diurnal tide ($\sigma \doteqdot 30°/\text{hr.}$),

$$\left.\begin{aligned}\frac{1}{2}\sigma(\tau_0 + \tau_2) &= -\frac{1}{16}d^2 \sin 2\,\alpha = \mathrm{C} & \frac{1}{2}\sigma(\tau_1 + \tau_3) &= \frac{1}{16}d^2 \sin 2\,\alpha = -\mathrm{C}\\ \frac{1}{2}\sigma(\tau_0 - \tau_2) &= \frac{1}{2}d \sin \alpha = \mathrm{D} & \frac{1}{2}\sigma(\tau_1 - \tau_3) &= \frac{1}{2}d \cos \alpha = \mathrm{E}\end{aligned}\right\} \quad (9)$$

Then (4) becomes

$$\mathrm{M_o} = \frac{1}{2}\mathrm{R}\left\{\cos \mathrm{C}\cos \mathrm{D} - \cos \mathrm{C}\cos \mathrm{E} + d\left[-\sin\left(\frac{\mathrm{C}}{2} - \alpha\right)\sin\frac{\mathrm{D}}{2} - \cos\left(\frac{\mathrm{C}}{2} - \alpha\right)\sin\frac{\mathrm{E}}{2}\right]\right\}$$

which reduces to

$$\mathrm{M_o} = -\frac{1}{16}\mathrm{R}\,d^2 \cos 2\,\alpha \quad (10)$$

We may therefore conclude that the contribution to $\mathrm{M_o}$ of a tide compounded of 2 or more species is in general a non-zero quantity. This follows from (8) and (10).

Special cases when $\mathrm{M_0}$ is zero are given by

$$\alpha = \frac{\pi}{2} \text{ or } \frac{3\,\pi}{2} \text{ for } n = 2\ ;\ \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \text{ for } n = \frac{1}{2}$$

The maximum and minimum values of $\mathrm{M_o}$ occur when

$$\alpha = 0 \text{ or } \pi \quad \text{for} \quad n = 2, \quad \text{then}$$

$$[\mathrm{M_o}] = \mathrm{R}\,d = r \quad (11)$$

$$\alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \quad \text{for} \quad n = \frac{1}{2}, \quad \text{then}$$

$$[\mathrm{M_o}] = \frac{1}{16}\mathrm{R}\,d^2 = \frac{1}{16}r\,d \quad (12)$$

The basic assumption employed to obtain these results is that $\sigma\tau_m$ and $n\sigma\tau_m$ are small. Without this mathematical simplification an explicit equation for τ_m can only be obtained through a cumbrous algebraic equation in a circular function of $\sigma\tau_m$.

Restricting considerations to the maximum value of M_0 we have :—
for $n = 2$, $\alpha = 0$ then $n\sigma\tau_m = 0$ and hence (8) is valid for all values of d which give only one tide a day when $\sigma = 15°/\text{hr.}$, and for all values of d which give only two tides a day when $\sigma \doteqdot 30°/\text{hr.}$

$$\text{for } n = \frac{1}{2},\ \alpha = 0,\ \sigma\tau_0, = \sigma\tau_2 = 0,\ \sigma\tau_1 = -\ \sigma\tau_3 = \frac{d}{2}$$

and hence (10) is only valid when $\frac{d}{2}$ is small. A maximum for d is therefore of the order of 0.6.

A numerical solution of (5) has been obtained and used to compile Table 4 of *Admiralty Tide Tables*, Part III, for the combination of a semidiurnal and a diurnal tide. This enables us to verify (8) for $\sigma = 15°/\text{hr.}$, and (10). The values of j for which M_0 is zero, maximum or minimum agree with the values of α given above.

NOTE: In Table 4 of *A.T.T.*

$$j = -\frac{\alpha}{30°/\text{hr}},\ J = \frac{1}{d},\ M_c = \frac{1}{2} r\, \Sigma L \quad \text{for} \quad n = 2$$

$$j = \frac{\alpha}{15°/\text{hr}},\ J = d,\ M_o = \frac{1}{4} R\, \Sigma L \quad \text{for} \quad n = \frac{1}{2}$$

Confirmation of (8) for $n=2$, $\sigma \doteqdot 30°/\text{hr}$ comes from equation 11.7g of the *Admiralty Manual of Tides*. This expresses the contribution of M_0 from a quarter-diurnal tide as

$$A_4 = M_4 \cos (2M_2^0 - M_4^0)$$

$$= r \cos (-\alpha) \qquad \text{in the notation here used.}$$

Table 2 gives the maximum daily values of M_0 for a combined diurnal and semidiurnal tide, for values of

$$\frac{\text{diurnal amplitude}}{\text{semidiurnal amplitude}}$$

up to 10. These come from Table 4 of *A.T.T.* and cover the cases of $n = 2$ ($\sigma = 15°/\text{hr}$) and $n = \frac{1}{2}$, with $\Omega = 0$.

MONTHLY AND ANNUAL MEAN VALUES

Mean Sea Level.

When methods (A), (B), (C) or (E) are used to determine daily mean level, the result may be expressed by

$$JR \cos (\rho D - \varepsilon + \eta) \qquad (13)$$

A monthly summation of such values gives

$$JKR \cos (-\varepsilon + \eta + \eta') \qquad (14)$$

where $K = \dfrac{\sin s\rho/2}{\sin \rho/2}$, $\eta' = \dfrac{s\text{-}1}{2}\rho$ $\quad s$ = number of days in the month.

Values of K/s are given in Table 3 for $s = 28$ to 31.
A yearly average is obtained by putting $s = 365$ or 366.

Mean Tide Level.

From (8) and (10) we see that, to the 2nd order in d, the contribution to M_0 for Cases 2 and 3 are circular functions of α. We can therefore treat M_0 in the same way as mean sea level in so far as monthly and annual means are concerned.

The increment in speed per day of M_0 is given by the variable part of α, which is

$$(\sigma_1 - n\sigma) t \times 24 = (\rho_1 - n\rho) D$$

Examples of the value of K for M_0, obtained by replacing ρ in (14) by $(\rho_1 - n\rho)$ are given in Table 4.

Conclusions

The contribution from a tidal constituent to a monthly or annual value of mean sea level is obtained by multiplying together the appropriate factors from Tables 1 and 3. For a predominantly lunar semidiurnal tide, the maximum contribution (from M_2) to a monthly mean of 30 days using methods (B), (C) and (E) are respectively

0.00055, 0.00010, 0.00001, 0.00039 and 0.00027

times the amplitude of M_2. For a predominantly diurnal tide the corresponding maximum contributions from K_1 are

0.0027, 0.0421, 0.0001, 0.0011 and 0.0014 respectively.

On the basis of these figures, we may conclude that the best all-round method to use is the X_0 filter, but if the amount of labour involved has to be justified, the Z_0 filter is a most valuable method. It will be seen that this filter is also effective for shallow water tides.

More detailed comparison would be profitless here; with a knowledge of the harmonic constants at a given port the above mentioned tables afford criteria for choosing a method of reduction and also indicate the order of error contained in any monthly or annual value.

As shown earlier, the mathematical analysis for tidal contributions to mean tide level is more complicated than that for mean sea level; since we have to deal with the effect of one species upon another, each of which is compounded of a number of constituents, only approximate values of the anomalies in M_0 can be given. Nevertheless, equations (10) and (12), together with the figures given in Tables 2 and 4, provide ample indication that monthly and annual values of M_0 are generally inferior to mean sea level.

By an obvious extension of the method, mean sea level values which may be felt to contain unwanted errors due to insufficient elimination of tidal constituents may be corrected.

Considering annual means, if V_0 be the phase of the corresponding equilibrium constituent at the origin of time, determined by the method of reduction (generally zero hour, January 1st), then its contribution to the annual mean is given by

$$JKR \cos (V_0 - \varepsilon + \eta + \eta'), \text{ from (14)}$$

Apart from a minor variation in η' and K due to the existence of leap years, the principle variation in this contribution from year to year is determined by the changes in V_0. For the principle constituents, the increments per year are approximately

M_2	100°
N_2	12°
K_1	0.2°
O_1	101°

TABLE 1

Maximum contributions from harmonic constituents to daily mean level as obtained by methods (A), (B), (C) and (E), expressed as coefficients of constituent amplitude.

From	K_1	O_1	M_2	N_2	M_3	M_4	M_6	Msf
Method								
(A) p = 8, q = 3	—.0033	.0885	—.0783	—.1169	.9840	.1749	.9403	.9939
p = 4, q = 6	—.0029	.0782	—.0415	—.0639	.0524	—.0765	—.9341	.9939
p = 3, q = 8	—.0028	.0769	—.0384	—.0592	.0432	—.0517	—.0978	.9958
(B) p = 1, q = 24	—.0027	.0753	—.0353	—.0544	.0353	—.0355	—.0361	.9883
p = 1, q = 25	—.0426	.0328	.0064	—.0128	—.0065	.0066	.0070	.9879
(C) X_o filter	.0001	.0030	—.0006	.0017	—.0091	.0026	—.0020	.9849
Z_o filter	—.0011	.0352	.0249	.0364	—.0415	.0082	—.0835	.9893
(E) Integration	.0014	—.0362	—.0171	—.0262	—.0169	—.0167	—.0161	.9932

TABLE 2

Maximum values of contributions to mean tide level for mixed semidiurnal and diurnal tides, expressed as coefficients of semidiurnal amplitude.

Diurnal ampl. / Semidiurnal ampl	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	3.0	4.0 and above
Contribution to M_o	.00	.005	.01	.02	.04	.065	.09	.12	.16	.205	.25	.56	1.00

TABLE 3

Factors by which contributions to daily mean sea level values are reduced by monthly and annual averaging processes

Contribution from :	K_1	O_1	M_2	N_2	M_3	M_4	M_6	Msf
Month of 28 days	.9903	—.0138	—.0541	.0302	.0534	—.0524	—.0497	—.0541
29 days	.9896	.0214	—.0184	—.0056	.0185	—.0187	—.0192	—.0184
30 days	.9888	.0532	.0157	—.0384	—.0158	.0160	.0165	.0157
31 days	.9884	.0804	.0470	—.0652	—.0465	.0457	.0437	.0470
Year of 365 days	.0007	—.0096	.0100	—.0009	—.0022	—.0065	.0022	.0129
366 days	—.0021	—.0076	.0080	.0019	—.0047	—.0064	.0042	.0129

TABLE 4

Factors by which contributions to daily mean tide level values are reduced by monthly and annual averaging processes.

Period of principal tide equal to that of		K_1	O_1	M_2	S_2
Period of minor tide equal to that of:					
M_2	month of 30 days........	.084	.084		
	year of 365 days........	—.009	—.009		
S_2	month of 30 days........	.951	.051		
	year of 365 days........	—.001	—.006		
K_1	month of 30 days........			—.088	.989
	year of 365 days........			—.022	.001
O_1	month of 30 days........			—.088	.053
	year of 365 days........			—.022	—.010
M_4	month of 30 days........			1.000	.016
	year of 365 days........			1.000	—.006
MS_4	month of 30 days........			.016	.016
	year of 365 days........			.010	.010

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