A NEW METHOD OF COMPUTATION OF EQUAL-ALTITUDE OBSERVATIONS

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The development of computing machines and their increased applications will cause great changes to take place in the numerical computation methods that have gradually been perfected for solving numerous practical problems in the observational sciences field.

It is no longer sufficient to use the machines for operations heretofore carried out by logarithms; computing procedures must be completely revised and adapted to the new possibilities offered by the machines. The reduction of equal-altitude observations is an especially simple and characteristic instance of this development.

EQUAL ALTITUDE METHOD

We shall begin with a rapid review of the principle of the equal altitude method, which is used for the simultaneous, accurate determination of latitude and local time by observations of the same type. Suggested by GAUSS in 1808 to remedy imperfections of the sextant, it was not extensively applied until the early years of the twentieth century, when instruments specially adapted to use of the method were devised, with particular reference to the equilateral prismatic astrolabe of CLAUDE and DRIENCOURT.

The method consists in noting the times of transit of stars at a strictly constant altitude, which need not be known with accuracy. This altitude is considered as a supplementary unknown quantity, added to the two actual unknown values of latitude and local time, or latitude and longitude, if the observation times may have been connected with the international meridian by means of radio time signals.

GAUSS was satisfied to note the passage of three stars, which was theoretically sufficient, since the number of observations was equivalent to the number of unknowns. Since that time, however, the custom has arisen, as in all the observational sciences, of taking a much larger number of observations than required in theory for determining the unknowns, in order to free the results from random errors affecting the measurements and the data. Owing to the method of observation, which consists in matching the direct image of the star with its image reflected in a mercury bath, the conventional prismatic astrolabe enables only one sight per star to be obtained, with an accuracy appreciably under that afforded by the excellent maintenance of instrumental altitude. Many stars must hence be observed in order to decrease the effect of random error in pointing, and precise determinations made with the instrument include series of thirty or forty stars or more. When an instrument such as DANJON's impersonal astrolabe is used, enabling several sights of the same star, an accurate result is obtained by observing a much smaller number of stars. Eight or ten stars per series should however be observed, as random errors also exist in relation to the stars' positions.

The known elements of the problem are the coordinates of the star observed, the declination δ, right ascension α, and time t of the observation expressed in sidereal time of the international meridian; these elements reduce to two, as only the hour angle H intervenes, referred to the international meridian and given by $H = t - \alpha$. We shall designate the three unknowns by φ : latitude, by G : west longitude, and by h : constant altitude of observation.

The observational equation expressing the known and unknown values is supplied by the basic formula of spherical trigonometry :

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos (H - G) \qquad (1)$$

Each star observed leads to a similar equation in which the known values δ and H each time assume individual values. The unknown values are obtained by solving the system formed by the observational equations.

GAUSS, who, as we know, considered only three observations, indicated a relatively simple trigonometric method for solving the three-equation, three-unknown system to which he was led. In 1812, DELAMBRE suggested another, which may be somewhat simpler.

GAUSS also showed that the elegant solution discovered by CAGNOLI to the problem of determining the position of a solar spot and of the heliacal equator by means of three heliocentric observations of the spot could be applied to the equal-altitude problem.

But when there are more than three observations, the number of observational equations is larger than required, and owing to the errors affecting the known values, the system to be solved is incompatible. The method of least squares must then be applied for the most probable solution of the system to be obtained.

METHOD OF LEAST SQUARES

We know that the least squares method, which requires that the observational equations appear in linear form, enables us to take advantage of all the observations made by determining values for the unknowns which, without strictly satisfying each equation, best satisfy the system as a whole.

This consists in deducing from the observational equations a system of normal equations equivalent in number to the number of unknowns, and that are linear. The solution of the system of normal equations supplies the solution to the problem.

To form the normal equation relating to one of the unknown values, all the coefficients and the constant term of each observational equation are multiplied by the coefficient of the unknown being considered, and all the equations thus transformed are then added member by member. In the normal equation thus obtained, the coefficient relating to the unknown in question is the sum of the squares of the coefficients of this unknown in the observational equations, and consequently is of high value. The other coefficients of the normal equation, which are formed from the coefficients of any sign of the observational equations in the usual manner, are generally of much lower value.

If for instance the observational equations are written as follows, X, Y and Z designating the unknown values :

$$
\begin{aligned}
a_1\,X + b_1\,Y + c_1\,Z &= d_1 \\
a_2\,X + b_2\,Y + c_2\,Z &= d_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots& \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&
\end{aligned}
$$

the system of normal equations is the following, the summations being designated by the symbol [ ]

$$
\begin{aligned}
[aa]\,X + [ab]\,Y + [ac]\,Z &= [ad] \\
[ab]\,X + [bb]\,Y + [bc]\,Z &= [bd] \\
[ac]\,X + [bc]\,Y + [cc]\,Z &= [cd]
\end{aligned}
$$

The quality of the observations is characterized by the the size of the residuals R_1, R_2 ... obtained by substituting the solution of the system of normal equations in each of the observational equations. It is moreover shown that $[R^2]$ is minimum in relation to the sum of the squares of the residuals supplied by any other solution; from this property derives the name of the method.

CONVENTIONAL METHOD OF REDUCING EQUAL-ALTITUDE OBSERVATIONS

In 1832 Knorre, at Nicolaief, then in 1835 Anger, at Dantzig, hit upon the idea of applying the least squares method to the reduction of equal-altitude observations. In order to make the observational equation linear, they had recourse to the general method, which consisted in selecting an approximate solution, φ_0, G_0 and h_0, and in taking as new unknown values the differences $\Delta\varphi$, ΔG, Δh between the approximate values of the unknowns and the values sought. If the approximate solution is sufficiently close to the result, the products and squares of the new unknowns may be regarded as negligible, and the change in variables is carried out by simple differentiation. This renders the observational equation linear with respect to these unknowns, such as in the case of equal-altitude observations :

$$\Delta\varphi \cos Z - \Delta G \cos \varphi_0 \sin Z - \Delta h = h_0 - h_1 \qquad (2)$$

where Z designates the azimuth of the star at the time of observation, and h_1 the value of the altitude of observation computed strictly by means of equation (1) in which approximate values are given to latitude and longitude; h_1 must be computed to an accuracy at least equivalent to that expected from the observations, i.e. with 6 or 7 significant figures.

As, by assumption, the unknown values are small, since they are corrections to approximate values of the initial unknowns, the relative errors that may be tolerated in their determination are fairly large. Their coefficients, whether in the observational equations or in the normal equations formed by the least squares method, need not necessarily be known with high accuracy. Generally two or three significant figures will suffice, which enables use of CRELLE's multiplication table or the slide rule to form the usual products of the coefficients of the observational equations in order to obtain those of the normal equations.

The choice of the approximate solution, which forms the basis of the computation, is sometimes a lengthy one. Certainly astronomical observations of equal altitudes require, to be carried out, foreknowledge of data concerning the setting, azimuth and time of passage of the stars to be observed. The computation of these data is also based on an approximate position of the station, but in the case of this determination, an approximation of one minute of arc in the latitude and altitude of observation, and of a few seconds of time in longitude, is amply sufficient. When the observations are reduced, however, the approximate solution must be to within a few seconds of arc and a few tenths of second of time. To attain this accuracy, it is usually necessary to obtain an initial solution, with roughly approximate data, by means of four or five carefully observed stars well distributed in azimuth. The result of this computation supplies a sufficiently accurate solution to be used as a starting point for the reduction of the series by the least squares method.

To sum up : when the observation series consists of more than three stars, which is generally the case, the principal operations to be carried out are the following :

— a broadly approximate solution for a group of four or five observations in order to obtain a sufficiently accurate approximate solution;

— computation with seven significant figures of the value of h_1 supplied by equation (1), for each observed star, in order to form the second members of the linear observational equations;

— computation with three significant figures of the coefficients of the new unknowns in these equations, i.e. sin Z and cos Z, since the unknown taken is actually $\Delta G \cos \varphi_0$ and not ΔG;

— formation of the squares and usual products of the coefficients of each equation in order to obtain by summation the coefficients of the normal equations with three significant figures;

— solution of the normal equation system, also with three significant figures.

Simplifying by use of diagram

In 1890, by an ingenious extension of the position line introduced in 1875 by MARCQ DE SAINT-HILAIRE in computing position at sea, PERRIN devised a graphic method avoiding the formation and solution of normal equations. In this method, a straight line corresponding to each observational equation is drawn on a large-scale diagram in a direction perpendicular to that of the star. Its distance from the origin of the diagram representing the approximate position of the station is equal to the known term $h_0 - h_1$ of the observational equation. The straight lines relating to the various observations closely envelop a circle : the centre, determined to the greatest possible accuracy, supplies the geographical position of the station in reference to the approximate position, and the radius represents the correction required for the approximate altitude. When a large number of stars are observed, the diagram can be simplified by replacing the sets of straight lines corresponding to adjacent azimuths by mean values of groups of these lines.

The graphic solution, which is faster than solving by the least squares method, offers the added advantage of a synoptic view of the observations and brings out abnormal discrepancies. Many geodesists, however, consider it inadequate and use it only as an adjunct to the least squares method, in order to check results supplied by the latter.

NEW METHOD OF REDUCING EQUAL-ALTITUDE OBSERVATIONS

Actually it is unnecessary to resort to an approximate solution in order to make the observational equation linear and subject the numerous observational equations to treatment by the least squares method. The equal-altitude equation can be put directly in linear form in terms of three auxiliary unknowns X, Y, Z, which are themselves simple functions of the unknowns.

If we put :

$$X = \frac{\cos\varphi \cos G}{\sin h} \qquad Y = \frac{\cos\varphi \sin G}{\sin h} \qquad Z = \frac{\sin\varphi}{\sin h} \qquad (3),$$

relations from which we inversely derive * :

$$\tan G = \frac{Y}{X} \qquad \tan\varphi = \frac{Z}{X}\cos G \qquad \sin h = \frac{1}{Z}\sin\varphi \qquad (4),$$

the observational equation (1) is written :

$$X \cos\delta \cos H + Y \cos\delta \sin H + Z \sin\delta = 1 \qquad (5)$$

When only three observations of stars have been made, it suffices to solve in X, Y, Z the system of three linear equations with three unknowns formed by the three observational equations. This type of solution is far more direct and far easier to obtain than the trigonometrical solutions given by GAUSS and his contemporaries, but offers no additional advantage insofar as length of computation is concerned. It is curious to note, however, that it was not indicated at the time.

In the case of extra numbers of observations, the system of observational equations put in the form under (5) may thus be dealt with directly by the least squares method. The unknowns, however, unlike the case of the approximate values, must here be determined with a very small relative error and hence be obtained with six or seven significant figures. This means that the same accuracy conditions apply to their coefficients in the normal equations and the observational equations. The computation of the normal equation coefficients by the formation and addition of the usual products of the observational equation coefficients therefore becomes an arduous process by ordinary computation methods, as does the solving of the normal equation system. The existence of rapid, powerful calculating machines, however, now makes this easy, and the type of solution considered thus seems to be more advantageous than the conventional method. For if the formation and solution of normal equations with coefficients having

(*) Of course, as in the conventional reduction method, the value 0.021 s sin h (i.e. 0.018 s for the equilateral prismatic astrolabe) must be subtracted from the west longitude G in order to allow for the effect of diurnal aberration.

seven significant figures is longer than with three, the laborious search for an approximate solution is however avoided, as well as the computation with seven significant figures of the values of h_1 by means of the relation (1) for each star observed.

It should nevertheless be noted that the proposed method also includes the relatively brief calculation of the coefficients $\cos\delta \cos H$, $\cos\delta \sin H$ and $\sin\delta$, of the observational equation put under form (5), as well as the change in variables defined by the relations in (4) and enabling the actual unknowns φ, G and h to be derived from the auxiliary unknowns X, Y and Z supplied by the solution of the normal equation system.

The operations to be carried out may be summarized as follows :

- computation of the coefficients of each observational equation (5) with seven significant figures;
- formation of squares and usual products of such coefficients in order to obtain by summation the coefficients of the normal equations with seven significant figures;
- solution of normal equation system with seven significant figures to obtain auxiliary unknowns X, Y, Z;
- passage from unknowns X, Y, Z to unknowns G, h by relations under (4).

An example was worked out under the direction of Ingénieur Hydrographe en Chef P. MANNEVY, and a series of twelve stars observed with an SOM geodetic-type prismatic astrolabe was dealt with by this method and the conventional method. The computations, carried out to seven significant figures, gave the same tenth of a second of arc for latitude, a difference of a hundredth of a second of time for longitude, and of .15″ for the instrumental altitude.

Interpretation of residuals

When X, Y, Z, in an observational equation (5), are repaced by their values supplied by the solution of the normal equations, a residual :

$$R = X \cos\delta \cos H + Y \sin\delta \sin H + \sin\delta - 1$$

is obtained, and its significance easily determined.

By solving for h equation (1) for the values found for φ and G, we get an altitude h_2 defined by :

$$\sin h_2 = \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos (H - G).$$

The residual is hence written

$$R = \frac{\sin h_2 - \sin h}{\sin h} = (h_2 - h) \cot h$$

whence we derive, expressing $h_2 - h$ in seconds of an arc

$$h_2 - h = \frac{R \tan h}{\sin 1''} \tag{6}$$

As the altitude h_2 represents the actual altitude of the observation at the position of the station and the observed instant of passage, and h designates the constant instrumental altitude, the difference $h_2 - h$ represents the error made in the altitude of observation. It corresponds on the solution diagram to the difference existing between the position line relating to the star involved and the circle enveloping the position lines.

In the case of the CLAUDE and DRIENCOURT prismatic astrolabe we get, in seconds :

$$h_2 - h = 3.57 \cdot 10^5 \,\mathrm{R}$$

In the example mentioned above the differences $h_2 - h$ obtained by the two methods showed a maximum of .15″ for eleven stars, and as much as .2″ for a single star.

Consideration of refraction variations

If measurements of temperature and atmospheric pressure show that refraction has varied to an appreciable extent during the observation series, the variation may easily be allowed for.

The unknown value then taken is h, the observational altitude at a given instant, say the mean instant of the series, and $h + \Delta h$ to designate the altitude at the time of an observation. As Δh is very small, we may write, expressing this quantity in seconds of an arc :

$$\sin (h + \Delta h) = \sin h \,(1 + \Delta h \cot h \sin 1'')$$

so that the second member of the observational equation (5) must no longer be taken equal to 1 but to $1 + \Delta h \cot h \sin 1''$, an expression in which Δh is computed by means of the observed refraction.

In observations with the equilateral prismatic astrolabe, the coefficient of Δh is equal to :

$$\frac{1}{3.57} \cdot 10^{-5}, \text{ or } 2.8 \cdot 10^{-6}$$

REMARKS

I. The possibility of directly applying the least squares method to prismatic astrolabe observations by computing machine was indicated in 1954 by a member of the U.S. Navy Hydrographic Office. But the author of this proposal, in order to put the observational equation in linear form, used a stereographic projection of the given altitude circle on the plane of the equator, with the result that the observational equation is written :

$$\mathrm{X} \cos \mathrm{H} \tan \left(45° - \frac{\delta}{2}\right) + \mathrm{Y} \sin \mathrm{H} \tan \left(45° - \frac{\delta}{2}\right) + \mathrm{Z} = \tan^2 \left(45° - \frac{\delta}{2}\right)$$

The auxiliary unknowns X Y Z implicitly designate the values :

$$\mathrm{X} = \frac{2 \cos \varphi \cos \mathrm{G}}{\sin \varphi + \sin h} \qquad \mathrm{Y} = \frac{2 \cos \varphi \cos \mathrm{G}}{\sin \varphi + \sin h} \qquad Z = \frac{\sin \varphi - \sin h}{\sin \varphi + \sin h}$$

Computations are hence slightly more complicated than in the method indicated above; it is moreover difficult to allow for variations in refraction and to interpret residuals.

II. The computation method indicated herein essentially applies to equal-altitude observations at temporary observatories. But large astronomical observatories which make permanent use of Danjon's impersonal prismatic astrolabe may use to advantage the conventional reduction method, computing $h_0 - h_1$ by a differential method by means of a table of hour angles, carefully obtained for an accurate latitude of the station and for adjacent values of declination.

III. Few standard geodetic or geodetic astronomy problems outside of the reduction of equal altitude observations offer an observational equation that may be rendered linear without recourse to an approximate solution.

In geodesy, the determination of a target by means of directed sights originating from known positions meets this requirement, since each sight is likened to a straight line.

Similarly, in geodetic astronomy, we have the determination of latitude and azimuth by observation of a single unknown star, a problem discussed by the author in the *International Hydrographic Review,* November 1954, and the solution of which analytically is identical to that of the equal-altitude problem.

Reference to this article in the *Review* shows that the observational equation is :

$$\sin \delta = \sin \varphi \sin h + \cos \varphi \cos h \cos (V + L)$$

a relationship in which the given values are the altitude of observation h and the reading L of the horizontal limb of the instrument. The three unknowns are the declination δ of the observed star, the latitude φ of the station and the azimuth V of the zero of the limb.

By analogy with the process followed in the equal-altitude method, we put :

$$X = \frac{\cos \varphi \cos V}{\sin \delta} \qquad Y = \frac{\cos \varphi \cos V}{\sin \delta} \qquad Z = \frac{\sin \varphi}{\sin \delta}$$

and the observational equation takes the following linear form :

$$X \cos h \cos L - Y \cos h \sin L + Z \sin h = 1$$

BIBLIOGRAPHY

An extensive bibliography on the equal-altitude method will be found in the article : « Instruments for Observing Equal Altitudes in Astronomy », by Mme E. CHANDON and A. GOUGENHEIM, *Hydrographic Review,* Vol. 12, No. 1, May 1935.

The following articles may be added :

A. H. KERRICK. Precise astronomic positions from projected star positions analytically processed. *International Hydrographic Review,* Vol. 31, No. 2, November 1954.

A. GOUGENHEIM. Une nouvelle méthode de réduction des observations de hauteurs égales (A new method for reducing equal-altitude observations). *Comptes rendus de l'Académie des Sciences,* tome 246, No. 13, session of 31 March 1958, p. 1976.