A METHOD FOR ASCERTAINING PERIODS FROM A CONTINUOUS SERIES OF OBSERVATIONAL DATA

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In studying a periodical phenomenon, or more generally an oscillation phenomenon consisting of periodic elements, knowledge of the periods of the various oscillations is of interest.

For if the values of the existing periods are known, the observed series can then be broken down by selecting its elementary components according to a filtering process or by any other method.

Any of the operations connected with tidal analysis are facilitated by prior knowledge of the periods of the various components; in further investigations only the phases and amplitudes need then be determined.

A less straightforward oceanographic problem is investigation of seiches. The period of this process can occasionally be determined beforehand, but only approximately and theoretically: values thus obtained may be remote from those actually observed; hence data obtained through experience must be used for more accurate research and appropriate marigrams constructed.

The harmonic constants of the local tide, derived from a lengthy series of observations, must first be known as accurately as possible. The best theoretical marigram for the locality can then be formed for the entire time considered. The difference between the observed marigram and theoretical marigram is generally an unevenly oscillating function showing the effects of seiches or other wave processes. This function is subjected to appropriate computations permitting evaluation of the existing periods.

It is a well-known fact that a linear combination (e.g. with constant and symmetrically equivalent coefficients) applied to a series of equidistant and adjacent sets of observed values can result, depending on the value of the coefficients and number of terms used, in more or less extensive damping and even in the elimination of certain components.

Thus if we have a series of observations consisting of $S + 1$ values (from 0 to S), and we work out, between a certain set of $2p + 1$ of these values, the linear combination

$$2\,\lambda_0\, y_0 + \lambda_1(y_{-1} + y_{+1}) + \lambda_2(y_{-2} + y_{+2}) + \dots + \lambda_p(y_{-p} + y_{+p})$$

in which y_0 indicates a generic value of the series and y_{-1}, y_{-2} the adjacent values on the left spaced 1,2 .... abscissa units apart, and y_{+1}, y_{+2} .... similar values on the right, we substitute for y_0 the transformed value $\overline{y}_0$. Proceeding in succession for each ordinate (y_0) to the extent of the $s + 1 - 2p$ values, we get a transformation series with respect to the original: in this, if the original series can be represented by the formula

$$y = A + \sum_{1}^{k} r\, a_r \cos(\omega_r\, t + \varphi_r)$$

some or all of the r frequencies may cancel out. The condition for cancellation of frequencies ω_1, ω_2, ω_3, ω_r, ω_k may be represented by the system :

$$\begin{array}{l}
\lambda_0 + \lambda_1 \cos \omega_1 + \lambda_2 \cos 2\, \omega_1 + \lambda_3 \cos 3\, \omega_1 + \dots + \lambda_p \cos p\, \omega_1 = 0 \\
\lambda_0 + \lambda_1 \cos \omega_2 + \lambda_2 \cos 2\, \omega_2 + \lambda_3 \cos 3\, \omega_2 + \dots + \lambda_p \cos p\, \omega_2 = 0 \\
\lambda_0 + \lambda_1 \cos \omega_3 + \lambda_2 \cos 2\, \omega_3 + \lambda_3 \cos 3\, \omega_3 + \dots + \lambda_p \cos p\, \omega_3 = 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\lambda_0 + \lambda_1 \cos \omega_r + \lambda_2 \cos 2\, \omega_r + \lambda_3 \cos 3\, \omega_r + \dots + \lambda_p \cos p\, \omega_r = 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\lambda_0 + \lambda_1 \cos \omega_k + \lambda_2 \cos 2\, \omega_k + \lambda_3 \cos 3\, \omega_k + \dots + \lambda_p \cos p\, \omega_k = 0
\end{array}$$

This may be resolved in λ if $p = k$ (only the λ_0 value is arbitrary and may be taken as equal to 1). In any case, the number of frequencies k to provide for (or to cancel) determines a minimum length for the series S which must be at least of such a value that $S \geqslant 2k$. Given any series S, for any generic value of y_0 of this series we can deduce the values:

$$2\, y_0\,,\ (y_{+1} + y_{-1}),\ (y_{+2} + y_{-2}),\ \dots\ (y_{+p} + y_{-p})$$

which, to simplify, we shall designate by A_0, A_1, A_2, ... A_p. Since we may suppose that the values of series S are determined from a sum of the sinusoids we can look for the coefficients λ_0, λ_1, λ_2 ...... λ_p which cancel out the combination:

$$\lambda_0\, A_0 + \lambda_1\, A_1 + \lambda_2\, A_2 + \dots + \lambda_p\, A_p$$

and extrapolate the condition to sucessive groups of ordinates, in such a way as to establish the system:

$$\begin{array}{l}
\lambda_0\, A_{10} + \lambda_1\, A_{11} + \lambda_2\, A_{12} + \dots \lambda_p\, A_{1p} = 0 \\
\lambda_0\, A_{20} + \lambda_1\, A_{21} + \lambda_2\, A_{22} + \dots \lambda_p\, A_{2p} = 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
\lambda_0\, A_{p0} + \lambda_1\, A_{p1} + \lambda_2\, A_{p2} + \dots \lambda_p\, A_{pp} = 0
\end{array}$$

These coefficients, if p is greater than the number of the components existing in the series S, are the same as those found true for all the ω_r frequencies existing in the series S :

$$\lambda_0 + \lambda_1 \cos \omega + \lambda_2 \cos 2\, \omega + \dots + \lambda_p \cos p\, \omega = 0\,.$$

After having determined the values of the different λ coefficients one may procede to the solving of the above equation (it is often sufficient to procede by graphical means after having established a few values for ω) by determining with the aid of zeros, the values of the frequencies existing in S.

With regard to all generic values for y_0 of S (over the length $S + 1 - 2p$) one may also, as a check, calculate the differences

$$(y_{+1} - y_{-1}),\ (y_{+2} - y_{-2}),\ \dots\ (y_{+p} - y_{-p})$$

which one will represent by $\overline{A}_1$, $\overline{A}_2$, ... $\overline{A}p$, and determine, as always, for the consecutive p values of y_0 the value of the coefficients which cancel out the quantity:

$$\overline{\lambda}_1 \overline{A}_1 + \overline{\lambda}_2 \overline{A}_2 + \dots + \overline{\lambda}_p \overline{A}_p .$$

These coefficients are the same as those which cancel out the function:

$$\overline{\lambda}_1 \sin \omega + \overline{\lambda}_2 \sin 2\,\omega + \dots + \overline{\lambda}_p \sin p\,\omega$$

and correspond to the existing frequencies, the values of which must correspond, apart from errors of approximation, to those previously found. According to the method used, if a certain number of sinusoidal components continue to exist in the whole of the series S, by extrapolating the calculations to consecutive groups of ordinates in such a way as to cover the whole length of the interval, one will always find the same solution (i.e. the same frequency).

If all the components do not continue to exist one will again find equality in the values of ω only for certain parts of series S. In this case one may also find, by taking into account the whole series of data, the values of the frequency of a certain number of ficticious components (i.e. without any physical reality) of which the sum, in the portion of S analysed, approaches as near as possible the tendency of the observed series.

In a similar manner, if the length of S is too small in relation to the large number of components in question, one may determine some ficticious components which approximate as far as possible the observed series in the part under study. It can be proved that all the components are analysed, indirectly as well as directly, when one sees that, although increasing the value of p, the number of zeros in the resolving equation does not vary and corresponds to the same frequencies.

Knowing the values of the periods of the components, it will always be possible to re-establish them and to determine the amplitude and the phase.

For example by means of the systems of the type indicated, in calculating in such a way that all the existing frequencies except one cancel one another out, and by means of linear combinations on the diagram, one may reconstitute each component wave one after the other. The components must be referred to the scale, care being taken not to omit the reduction in amplitude which is implicit in the linear combination.

In the case of incomplete series, each one having only one component, one may ultimately use the method of least squares (considering that it is always possible to make errors of determination), in order to establish numerically, instead of graphically, the values of the amplitude and phase of each of the components by means of which using the procedure just described one may obtain the value of the period.