ACCURACY OF A FIX BY TWO BEARINGS ON AN ARTIFICIAL SATELLITE

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The possibilities of using artificial satellites as navigational aids are quite numerous. A satellite may serve as a reference to fix the position of a point on the earth, i.e. the point above which the satellite is to be found at a given moment. From this standpoint, artificial satellites constitute a class of terrestrial objects visible at great distances. Bearings taken on such objects are used in navigation either to fix a position or to measure distances. Satellites may also be considered as stars, the elevation of which above the horizon one can measure. Measurement of the height of a celestial body gives a circle as a position line, the radius of which is the zenithal distance and the centre the projection of the star on the earth's surface.

There is a simple relation between the zenithal distance of a satellite at no great height and the rectilinear distance, which can be proved quite simply (fig. 1)

$$z_i^2 \sim \frac{e_i^2 - h^2}{r^2}$$

where e_i is a measured distance, z_i its spherical distance, r the radius of the earth and h the height of the satellite above the surface of the earth.

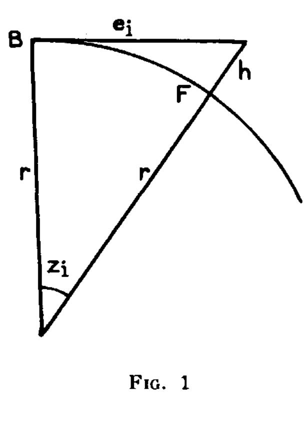

FIG. 1

Among the possible methods of using artificial satellites only the method of position fixing by two ranges will be discussed here and particularly the accuracy of the fix obtained.

For this purpose, let us assume a satellite describing a circular orbit around the earth, the simplest possible form of planetary motion. The data for some circular orbits with a number of revolutions rounded off to the nearest convenient number are given in the following table :

Height above earth's surface (km)	Velocity (km/h)	No. of revolutions (min)	Distance of substellar point if satellite is on horizon
320	28 300	90	17 2/3°
805	27 300	100	27 1/2
1 610	25 300	120	37
35 400	10 900	1 440	81

The last line relates to the orbit of the satellite to be recommended for many reasons and because it always remains over the same meridian and thus has the same duration of revolution as the earth itself. Since this satellite cannot yet be realized, we shall consider a satellite at a height of approximately 1 600 km taking two hours to make one revolution. It travels about 7 km or 4 nautical miles per second. Measuring the distance by electronic devices with an accuracy of 1 n. m. will be possible, although the synchronisation of the measurements will be a difficult task. The calculation of z_i using the formula given above is not difficult, an error in the measured distance e_i influences z_i by almost the same value.

The discussion of the accuracy of the fix can start with the accepted formula for astronomical navigation. The error in fix established by two astronomical position lines is given by

$$d = \frac{1}{\sin(\mathrm{Az}_1 - \mathrm{Az}_2)} \sqrt{d_1^2 + d_2^2 - 2 d_1 d_2 \cos(\mathrm{Az}_1 - \mathrm{Az}_2)}$$

if the errors in the two altitude observations are d_1 and d_2 and the two azimuths Az_1 and Az_2. The formula is a special case of the general formula for all types of position lines

$$\sigma = \frac{1}{\sin \alpha} \sqrt{\sigma_1^2 + \sigma_2^2 + 2 \cos \alpha \, k_{12} \, \sigma_1 \sigma_2}$$

for two variations σ_1 and σ_2, the position lines intersecting to give the angle α, with a correlation coefficient k_{12}. For astronomical observations, as for the distance measuring of satellites, $k_{12} = 0$, so that we can establish :

$$\sigma = \frac{1}{\sin \alpha} \sqrt{\sigma_1{}^2 + \sigma_2{}^2} \qquad \mathrm{Az}_1 - \mathrm{Az}_2 = \alpha$$

On condition that $\sigma_1 = \sigma_2 = 1$ n. m. we can discuss this formula, which gives a diagram of circles of equal σ from which the length AC between the two positions of the satellites or their substellar points is seen under the same angle α. But we shall study the accuracy of this more geometrically on the sphere.

The projection of the orbit of the satellite on the surface of the earth may be a great circle. This hypothesis offers two advantages in studying the accuracy. First, it facilitates the evaluation of such observations : second, it enables us to calculate the error in a fix, if the orbit is not exactly known. With this information concerning the orbit it is indicated mathematically that the substellar point is known at any moment from its

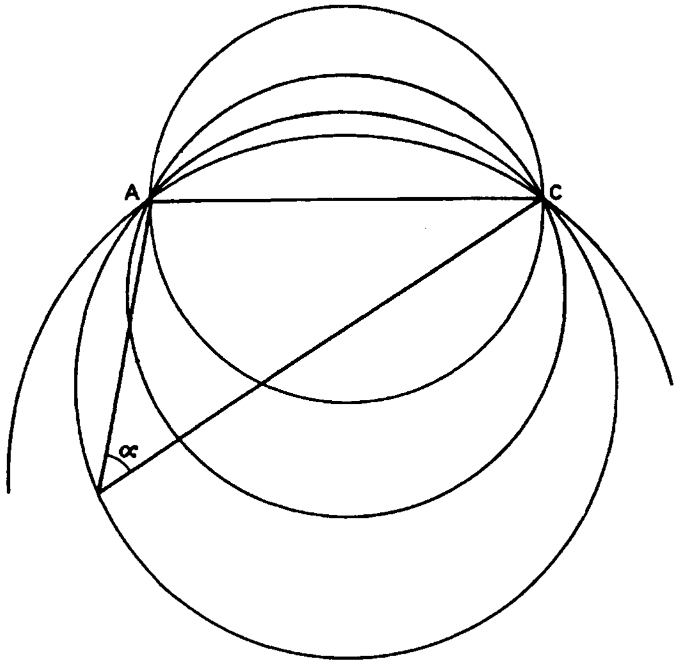

FIG. 2

geographical coordinates. Since, in the first place, the period of revolution is known, the part of the orbit between two points is given exactly. The precession of the orbit, which means that the projection of the orbit is not exactly a great circle, is not worth taking into consideration because it is always possible to draw a great circle between two points on a sphere.

Let us consider first of all the spherical triangle formed by the two basic points F_1, F_2 and the point B of observation and resolve it without reference to the position on the sphere. It is possible, for instance, to establish the position of the point of observation in relation to the point F_1 and to calculate Λ, as if z_1 and Λ were polar coordinates with respect to F_1; this method is useful for the study of the error, because the error in the spherical distance dz_1 (which was assumed to be the same as de_1) and the error $d\Lambda$ (the effect of which is $z_1 d\Lambda$) give the error of the fix by Pythagoras' theorem.

After having fixed the point of observation with respect to F_1 and F_2 it is necessary to find its position on the sphere. For this purpose, it is most practical to establish the position of the great circle on the sphere. A great circle is fixed on the sphere by the geographical coordinates φ_s and λ_s of its vertex. Each point of the great circle can be calculated according to the formulae :

$$\sin \varphi_i = \sin \varphi_s \cos E_i$$
$$\tan (\lambda_i - \lambda_s) = \tan E_i \sec \varphi_s \qquad (1)$$
$$\tan v_i = \operatorname{cotan} \varphi_s \operatorname{cosec} E_i$$

where $E_i = \dfrac{2\pi}{\text{revolut.}} (t_i - t_s)$, t_s being the time between passing the vertex

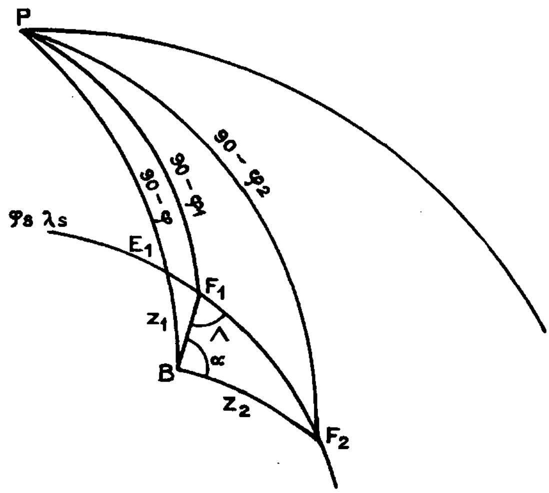

FIG. 3

t_s and the the time of observation t_i. The coordinates are derived from those of F_1 by the formulae :

$$\cos \Lambda = \frac{\cos z_2 - \cos (E_2 - E_1) \cos z_1}{\sin (E_2 - E_1) \sin z_1}$$

$$\cos (90^\circ - \varphi) = \sin \varphi_1 \sin z_1 + \cos \varphi_1 \cos z_1 \cos (180^\circ - v_1 + \Lambda) \qquad (2)$$

$$\sin (\lambda - \lambda_1) = \frac{\sin z_1 \sin (180^\circ - v_1 + \Lambda)}{\sin (90^\circ - \varphi)}$$

The trigonometrical formulae are so complicated that the evaluation must be carried out by mechanical aids in order to save time. Consideration of the geometrical data, the great circle and it position on the sphere, provides without difficulty the effect of an error in position. The formulae for φ, λ are differentiated by assuming Λ and E_i to be constant, the former because the evaluation of the spherical triangle F_1F_2B is a peculiar problem discussed later, the latter because the length of $E_2 - E_1$ is always well known. φ_1 and λ_1 are erroneous if φ_s and λ_s are in error, as will be seen from the formula (1). The calculation gives :

$$d\varphi = \cos (\lambda - \lambda_s)\, d\, \varphi_s$$

$$d\lambda = d\lambda_s + d\varphi_s \tan \varphi \sin (\lambda - \lambda_s)$$

The order of magnitude of the effect of an error in the position of the orbit is easy to obtain and is of the same order as that of the error in the position of the vertex.

The effects of measuring errors in the relative position of B in relation to F_1 and F_2, which will now be studied, are greater and more serious. Mathematically, they are obtained by differentiating the formula for cos Λ (group 2) to dz_1 and dz_2. The distance F_1F_2, as stated above, is always known exactly. The formula becomes convenient for calculations by introducing the angle α, from which F_1F_2 is seen from the point of observation. But this auxiliary angle is only introduced to facilitate the calculations, and one must have a clear idea of the effects of these errors without having recourse to that angle.

The following formulae :

$$\cos \alpha = \frac{\cos (E_2 - E_1) - \cos z_1 \cos z_2}{\sin z_1 \sin z_2}$$

$$d\Lambda = \frac{dz_2 - \cos \alpha \, dz_1}{\sin z_1 \sin \alpha} \qquad d = \sqrt{dz_1^2 + z_1^2 \, d^2 \Lambda} \tag{3}$$

have been calculated for a series in pairs of values $z_1 z_2$ selected in such a manner that the two terms in the numerator in the second formula were always added, which corresponds to the most unfavourable combination. Least errors result, of course, if $\alpha = 90°$. The formulae immediately show that for this value of α, one obtains the same values of d for different values of $E_2 - E_1$. It is sufficient therefore to demonstrate the value of d as a function of $E_2 - E_1$ with the aid of a simple graph without scale, which will indicate by the intensity of the black trace the magnitude of the error. Such a graph is important for practical evaluation because it shows the error of a fix resulting from the distance first measured, if the second distance is known. The diagram shows one distance as abscissa, the other

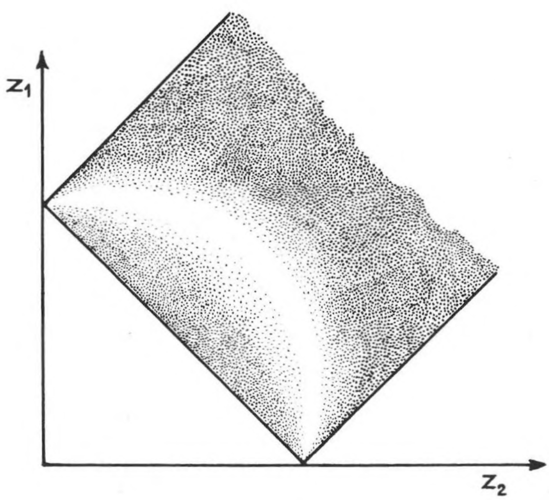

FIG. 4

as ordinate (fig. 4). The minimum values of errors are arranged in an arc of 90°, joining the two points, which have one distance of zero and the other exactly equal to the value $E_2 - E_1$. The line joining these two points and the perpendiculars to it at these two points are lines for which d is very large, because $\alpha = 0°$. If $z_1 + z_2$ approximates $E_2 - E_1$, and if the same applies to $z_2 - z_1$ or $z_1 - z_2$, the errors are greatest. This is the case if the point of observation is near the projection of the orbit. For a point exactly on this line, the two circles of the measured distances do not intersect, but touch one another. To study the effects of the errors in this special case, the errors resulting from perpendiculars to the orbit of 60 and 30 miles were investigated. The error is greatest if the point of observation is in the middle of the base F_1F_2 and if the base is large. For a base 60 nautical miles long between F_1 and F_2 and with a perpendicular of 30

n. m., the error increases up to 100 n. m. for $dz_1 = dz_2 = 1$ n. m. . Observations shortly after the rising or before the setting of the satellite give bad results. Near the orbit it is even doubtful which side of the orbit includes the point of observation. Under these circumstances and according to the formula (cos is indeterminate) it is also uncertain for greater perpendicular distances, but dead reckoning will provide an answer. The case of a position near the orbit is striking for the observer because the variation of z is similar to that of E. Thus, the unfavourable cases are fortunately recognized right away. Resolving this uncertainty is possible by taking a bearing on the satellite. However, such a bearing will be difficult to obtain optically and, in certain critical cases, if obtained by radio aids, it may not be accurate enough. Only a radar bearing may be sufficient.

The distance reduces the accuracy in as much as the favourable circle with F_1F_2 as diameter is only usable to determine the point of observation, as long as the baseline is not too large, otherwise the satellite may be under the horizon. The best results will be obtained if one distance is measured when the satellite is nearest to the point of observation. This will be advantageous not only from the point of view of method but also instrumentally.

The calculations, only briefly studied here, must be made by mechanical or electronic calculating aids. There are no difficulties in this, as such computers have been developed for astronomical navigation. But it seems certain that considerable technical facilities must be provided in order to reach a reasonable degree of accuracy in fixes by measuring the distance of a satellite, e.g. registering the distance automatically by means of a chronometer of high precision on board the ship or aircraft.