THE TIENSTRA METHOD OF RESECTION

by A. L. Allan, M.A., A.R.I.C.S.
Lecturer in Land Surveying, South West Essex Technical College
London, E. 17, England

There are many solutions to the problem of resection, of various complexity : some give the desired coordinates directly, others give the values of vital angles or bearings; but the method described here, which gives the coordinates of the resected point directly, must surely be the most elegant mathematically and at the same time, the most straightforward to compute. Indeed, in many cases a slide rule solution is sufficient.

I claim no originality for the formulae, which I understand are due to the late Professor Tienstra, and which were brought to my notice via the Northern Nigeria Survey Dept. The suggested computation layout and the summarized proof are mine, neither of which is the last word on the subject.

The justification for writing this note is that I feel that this method is quite the best available and that all surveyors who do not know it will find it indeed a boon.

Formulae

Refer to diagram (1). The lettering of the figure is important, especially in the case of the outside fixation. If we choose a clockwise convention, a is the clockwise angle between the directions PB and PC, etc.

The coordinates of the new point P are given by :

$$E_P = \frac{n_1 E_A + n_2 E_B + n_3 E_C}{+ n_1 + n_2 + n_3}; \qquad N_P = \frac{n_1 N_A + n_2 N_B + n_3 N_C}{+ n_1 + n_2 + n_3}$$

Where

$$\frac{1}{n_1} = \text{Cot A} - \text{Cot } a$$

$$\frac{1}{n_2} = \text{Cot B} - \text{Cot } b$$

$$\frac{1}{n_3} = \text{Cot C} - \text{Cot } c$$

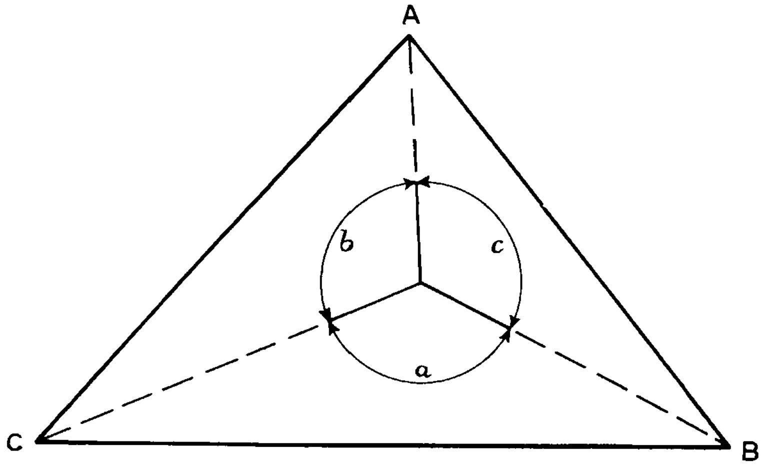

FIG. 1

Some advantages

The method has the following advantages :

(1) The formulae are easy to remember, being of the centre of gravity type.

(2) Because Eastings are in terms only of Eastings, and Northings of Northings, an arbitrary datum for computation can be chosen for each by inspection, to give the most convenient solution. (Often one Easting and/or Northing can be zero, and almost always when Logs are used.)

(3) The factors Cot A — Cot a etc., indicate the strength of the resection; particularly in the case of the danger circle when they tend to zero. This is at once evident and the computer is on his guard.

(4) The formulae lend themselves readily to a consideration of the errors in a fixation.

Example

The following example illustrates the method and gives a suggested layout for computation.

Field Diagram

If the angles between the fixed points are not known at once, they have to be computed from bearings in the usual way; even so, the method is still very quick.

There is also the check on the use of the computing machine that :

$$n_1 (E_P - E_A) + n_2 (E_P - E_B) + n_3 (E_P - E_C) = 0$$

and similarly for the Northings.

Angle		Cotangent	n	Easting	Northing
A	31 34 52	+ 1.626679	n_1	356 442.74	148 778.96
a	87 14 09	+ .048282		354 000.00	144 328.37
	$1/n_1$	+ 1.578397	+ .633554	+ 2 442.74	+ 4 450.59
B	131 19 32	— .879312	n_2	356 788.89	144 328.37
b	227 34 54	+ .913712	— .557717	354 000.00	144 328.37
	$1/n_2$	— 1.793024		+ 2 788.89	zero
C	17 05 36	+ 3.251890	n_3	351 240.55	138 628.57
c	45 10 57	+ .993650		354 000.00	144 328.37
	$1/n_3$	+ 2.258240	+ .442823	— 2 759.45	— 5 699.80
	180 00 00 360 00 00	$n_1+n_2+n_3=$	+ .518660	— 2 371.02 354 000.00	+ 570.09 144 328.37
		Coordinates of P.		351 628.98	144 898.46

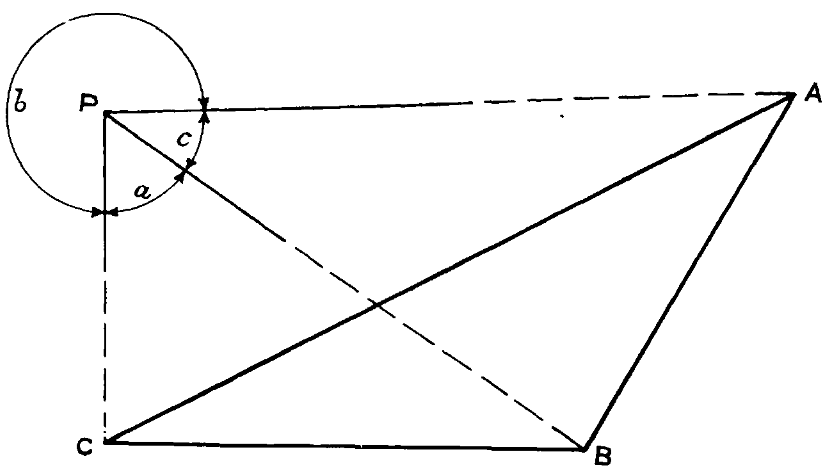

FIG. 2

Proof of formula

(A) *Two theorems in trigonometry* which will be well known to many are required for this proof. They are :

Refer to fig. (3). In triangle ABC, CD meets AB in D, making $\widehat{CDB} = \theta$. If AD $= m$ and DB $= n$; $\widehat{ACD} = x$ and $\widehat{DCB} = y$ then

$$(m+n)\cot\theta = n\cot A - m\cot B \quad (1)$$

$$\text{and } (m+n)\cot\theta = m\cot x - n\cot y \quad (2)$$

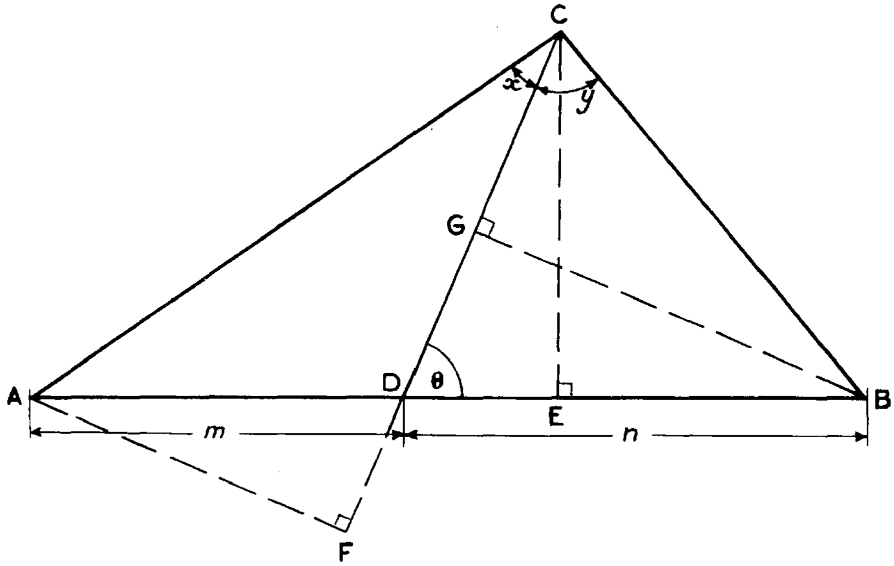

FIG. 3

(1) Construction : Draw CE perpendicular to AB meeting AB in E.

$$\cot A = \frac{AE}{CE}; \quad \cot B = \frac{EB}{CE}; \quad \cot \theta = \frac{DE}{CE}$$

$$AD = m = AE - DE \text{ and } DB = n = EB + DE$$

$$= CE (\cot A - \cot \theta) \text{ and } n = CE (\cot B + \cot \theta)$$

$$\frac{m}{n} = \frac{\cot A - \cot \theta}{\cot A + \cot \theta}.$$

i. e.

$$(m + n) \cot \theta = n \cot A - m \cot B \qquad (1)$$

(2) Construction : Draw AF perpendicular to CD; and BG perpendicular to CF, meeting them in F and G respectively. Then

$$\cot x = \frac{CF}{AF}; \quad \cot y = \frac{CG}{BG}; \quad \cot \theta = \frac{DF}{AF}; \quad \cot \theta = \frac{DG}{BG}.$$

Then

$$\frac{CF}{DF} = \frac{\cot x}{\cot \theta}; \qquad \frac{CG}{DG} = \frac{\cot y}{\cot \theta}$$

$$\frac{CF - DF}{DF} = \frac{\cot x - \cot \theta}{\cot \theta}; \qquad \frac{CG + DG}{DG} = \frac{\cot y + \cot \theta}{\cot \theta};$$

Then

$$DF \frac{(\cot x - \cot \theta)}{\cot \theta} = CD = DG \frac{(\cot y + \cot \theta)}{\cot \theta}$$

and

$$\frac{m}{n} = \frac{DF}{DG} = \frac{\cot y - \cot \theta}{\cot x - \cot \theta}.$$

$$\text{Then } (m + n) \cot \theta = m \cot x - n \cot y \qquad (2)$$

(B) *Tienstra Formula.* Refer to diagram (4).

By the intersection formula we have :

$$E_P\,(\cot 2 + \cot 3) = E_A \cot 3 + E_B \cot 2 - N_A + N_B$$
$$E_P\,(\cot 4 + \cot 5) = E_B \cot 5 + E_C \cot 4 - N_B + N_C$$
$$E_P\,(\cot 1 + \cot 6) = E_C \cot 1 + E_A \cot 6 - N_C + N_A$$

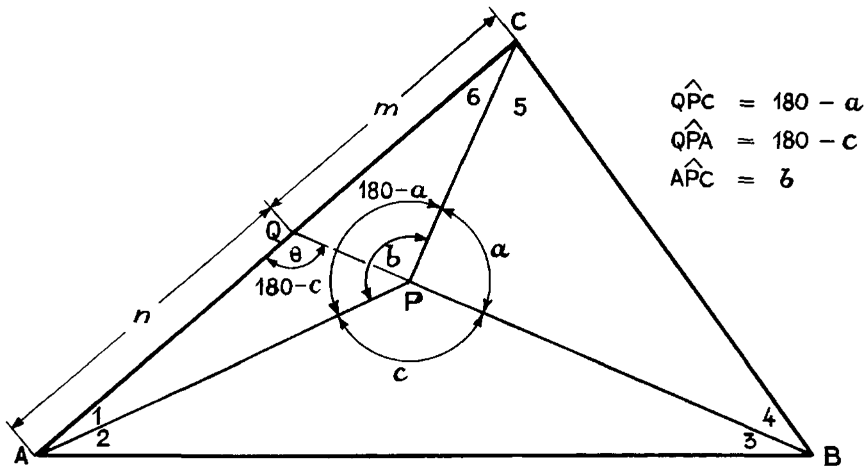

FIG. 4

Adding gives

$$E_P(\cot 1 + \cot 2 + \cot 3 + \cot 4 + \cot 5 + \cot 6) =$$
$$E_A(\cot 3 + \cot 6) + E_B(\cot 2 + \cot 5) + E_C(\cot 1 + \cot 4) \qquad (3)$$

From formulae (1) and (2) above we have :

$$(m + n) \cot \theta = m \cot 4 - n \cot 3$$
$$(m + n) \cot \theta = n \cot 6 - m \cot 1$$
$$(m + n) \cot \theta = - m \cot a + n \cot c$$
$$(m + n) \cot \theta = n \cot C - m \cot A$$

Hence

$$m\,(\cot 1 + \cot 4) = n\,(\cot 3 + \cot 6)$$
$$m\,(\cot a - \cot A) = n\,(\cot c - \cot C)$$

Therefore

$$\frac{\cot 1 + \cot 4}{\cot 3 + \cot 4} = \frac{n}{m} = \frac{\cot a - \cot A}{\cot c - \cot C}$$
$$= \frac{1/\cot c - \cot C}{1/\cot a - \cot A} \text{ or } \frac{1/\cot C - \cot c}{1/\cot A - \cot a}$$
$$= \frac{n_3}{n_1}$$

Similarily it can be shown that $\dfrac{\cot 1 + \cot 4}{\cot 2 + \cot 5} = \dfrac{n_3}{n_2}$.

Therefore :

$$\cot 1 + \cot 4 : \cot 3 + \cot 6 : \cot 2 + \cot 5 = n_1 : n_2 : n_3$$

$$\frac{\cot 1 + \cot 4}{\cot 1 + \cot 2 + \cot 3 + \cot 4 + \cot 5 + \cot 6} = \frac{n_1}{n_1 + n_2 + n_3}$$

Whence from equation (3) we obtain the required formula for E_P. Northings are identically proved.

Conclusion

In the case of a multiple ray resection, the method gives a quick trial point for further solution by least squares or graphical methods; also the mean of a cyclic solution of the rays in triplets gives a close approximation to the precise result.