IMPROVED METHOD OF CONSTRUCTION OF A HYPERBOLIC LATTICE CHART

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Paper received in March 1961

Introduction

The problem of the construction of hyperbolic lattice charts for use in hydrographic surveying is considered in this article, and an improved method known as *Hyperbolic Intersections on a Constant Grid Line* will be presented. The new method replaces the *Beta Increment Method* which was developed by the Hydrographic Office about fifteen years ago, shortly after the adoption by the Navy of hyperbolic systems. Both methods enable the computation of coordinates of points along hyperbolae, which subsequently enables the draftsman to plot these points and connect them thereby constructing a hyperbolic lattice chart. The computational procedure of the Beta method has not been entirely satisfactory, but up to the present time it has served in lieu of a better method. One of the major drawbacks to the Beta method is that it causes considerable drafting inefficiency. The draftsman is required to leaf through many pages of computations, in order to select a sufficient number of points to enable the construction of the hyperbolae. Only a fraction of the computed points is ever used; of those that are used, it is necessary to plot them with respect to an X and then a Y direction. A more satisfactory solution would be the use by the draftsman of the computed points in consecutive order, and the subsequent plotting of the hyperbolic intersections with respect to only one dimension. The new method achieves the desired results. Both methods are illustrated in this article, and the reader may judge for himself as to the quality of the improvement.

Electronic surveying systems

The positioning of vessels engaged in hydrographic surveying is largely accomplished by electronic surveying equipment. The major radio positioning systems fall into two categories, ranging and hyperbolic. The ranging systems use a minimum of two transmitting stations in known locations, and the signals from these stations are interpreted by the survey vessels as arcs of concentric circles having origins at the two transmitting stations.

In a hyperbolic system, three transmitting stations comprise a net. The centre and one end station generate a family of hyperbolae having as common foci the positions of the two stations involved. The centre and opposite end station generate another family of hyperbolae which is independent of the first.

In either the ranging or hyperbolic, intersecting curves will determine a position. Frequently the intersections will determine two positions; this presents no problem, since one of the positions can be rejected by inspection, as one point will lie on the opposite side of the net and usually will be many miles away.

Beta Increment Method

Following is a brief description of the computational and drafting procedure of the Beta Increment Method. This method employs a classical solution. The coordinates of the points along the hyperbolae are first expressed in terms of a local system, and then the coordinates are rotated and translated into the final frame of reference, the UTM grid.

In the construction of a pair of hyperbolic lattices, each of the two systems of hyperbolae is computed separately by identical methods. For simplicity, only one system of hyperbolae will be considered.

For most hydrographic work a plane coordinate projection is desirable, because of its simplicity, and it meets the needs for accuracy out to one hundred miles. For hydrographic work which utilizes electronic measurements in excess of one hundred miles, formulae which relate the electronic measurements to the spheroid are required. In the method illustrated here, a plane projection is employed. The projection is that of the Universal Transverse Mercator grid, and the coordinates of the transmitting stations as well as all of the points on the hyperbolae will be expressed on this grid.

In determining the coordinates of points on a hyperbolic lattice, the points are first expressed in terms of a local system with the mid-point of the system's baseline as the origin. The semi-major axis is equal to $+1$ at the master and -1 at the slave, and K is the scale factor equal to one-half the length of the baseline (see figure 1).

The parametric equations of the local system are expressed by :

$$X' = Ka \cosh B$$

$$Y' = Kb \sinh B$$

where :

$$a = 1 - L(\Delta a)$$

$$b = \pm \sqrt{1 - a^2}$$

$$-1 \leqq a \leqq +1$$

L = lane number

$$B \geqq 0$$

$$\Delta a = \frac{2}{\text{number of lanes in baseline}}$$

Observe that for every value of (*a*), two values of (*b*) result which are symmetric with respect to the baseline.

After expressing X′ and Y′ in the local system, the coordinates are rotated and translated onto the UTM coordinate system by the following expressions :

$$X = + X' \cos \psi + Y' \sin \psi + h$$
$$Y = - X' \sin \psi + Y' \cos \psi + k$$

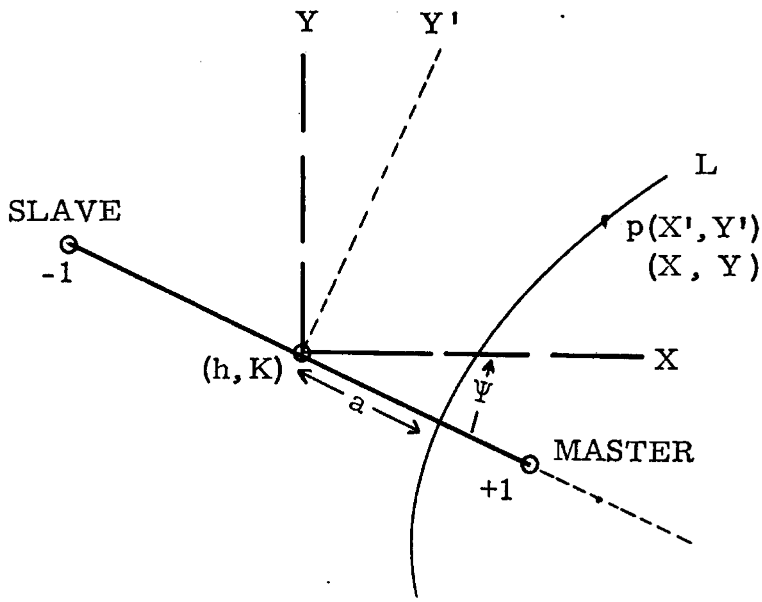

FIGURE 1

X′, Y′ = local coordinate axis and points
X, Y = UTM coordinate axis and points
h, *k* = UTM coordinates of mid-point of baseline
ψ = counter-clockwise angle of rotation
L = lane number of a hyperbola
a = distance from mid-point to L along baseline.

The computation of X and Y is usually performed by an electronic computer, and the results are presented to the draftsman in the form shown in figure 2. With each subsequent increment of B the computed point moves farther away from the baseline.

The draftsman receives a considerable quantity of material similar to that shown in figure 2, which is known as a *tab run*. Every lane or fractional lane which has been computed and which may be plotted is included in the tab run. The volume of paper required for these computations frequently weighs in excess of one hundred pounds. All of the points along one lane are plotted, in the sequence indicated in figure 2. Since the baseline normally lies along a sea coast and the area of interest is in the water, both sets of coordinates, that is the + b and — b, are seldom plotted, and of the remaining points about every other B value will be used. The result is that about 25 % of the coordinates will be used; however, the increment of B cannot be increased since sometimes the extra points are needed for sheets of large scale projections.

LANE 30	+ b		— b			+ b		— b	
B	Easting	Northing	Easting	Northing	B	Easting	Northing	Easting	Northing
0.0	337 654	4 586 083	337 654	4 596 083	0.1	337 490	4 585 900	337 414	4 586 137
0.2	336 923	4 585 586	336 770	4 586 061	0.3	335 945	4 585 137	335 713	4 585 855
0.4	334 546	4 584 548	334 234	4 585 517	0.5	332 715	4 583 814	332 319	4 585 044
.									
.									
.									
2.4	156 302	4 520 567	152 149	4 533 462	2.5	133 365	4 512 420	128 786	4 585 044
LANE 40									
0.0	337 660	4 586 085	337 660	4 586 085	0.1	337 502	4 585 884	337 414	4 527 808
0.2	336 940	4 585 551	336 764	4 586 099	0.3	335 968	4 585 083	335 701	4 585 912
0.4	334 577	4 584 475	334 217	4 585 594	0.5	332 752	4 583 721	332 295	4 585 141
.									
.									
.									
2.4	156 655	4 519 580	151 860	4 534 469	2.5	133 755	4 511 328	128 448	4 527 808

FIGURE 2. — An example of a tab run by the Beta Increment Method, where Beta is incremented from 0.0. to 2.5, in increments of 0.1, Beta value.

The points are plotted visually in a plane, on a grid sheet that has ten subdivisions for every two centimetres. A draftsman can plot to an accuracy of about one-quarter subdivision. For each point that is plotted, it is necessary to proceed in an X direction for the easting and then in a Y direction for the northing.

Objectives for improvement

1. To develop exclusively a specific area.
2. To reduce the large volume of computer paper, as this frequently imposes hardship in transporting and storing the tab runs.
3. To increase the percentage of the points plotted with respect to those computed. The 25 % usage represents a considerable waste, especially of expensive electronic computer time.
4. To make the plotting of coordinates simpler, and to improve the speed and accuracy at which this can be accomplished.

Hyperbolic intersections on constrant grid line

Following is an explanation of the new method.
The parameters employed are as follows :

$$R = \sqrt{(X_p - X_m)^2 + (Y_p - Y_m)^2} \quad (1)$$

$$D = \sqrt{(X_p - X_s)^2 + (Y_p - Y_s)^2} \quad (2)$$

See figure 3.

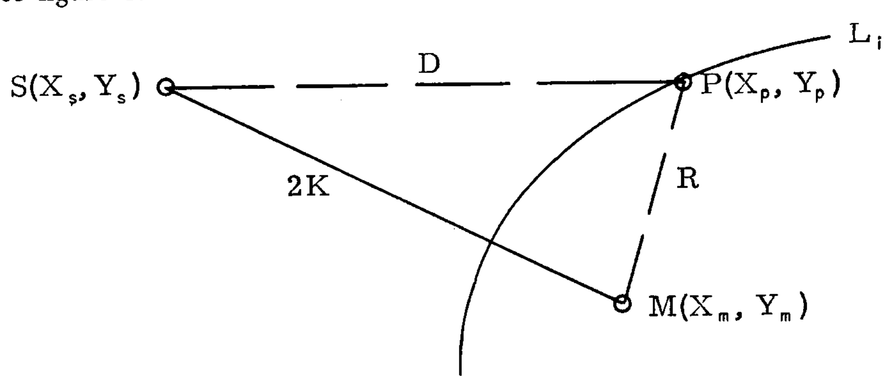

FIGURE 3

$P(X_p, Y_p)$ = coordinates of any point P on L_i
$M(X_m, Y_m)$ = coordinates of focus, also master station
$S(X_s, Y_s)$ = coordinates of focus, also slave station
2K = length of baseline
L_i = hyperbolic lane number
i = any number from the minimum to the maximum lane value of a system.

Definition of hyperbola : a hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is a constant.

For any lane L_i :

$$D - R = \text{a constant } k \quad (3)$$

This condition defines a hyperbola.

Along the baseline for any lane L_i :

$$D + R = 2K \tag{4}$$

and

$$R = L_i(GLW)$$

where a lane number (L_i) times one unit of grid lane width (GLW) along the baseline is equal to R on the baseline.

Subtracting 2R from (4) :

$$D - R = 2K - 2R$$

Substituting for R on the right hand side :

$$D - R = 2K - 2L_i(GLW)$$

Solving for L_i :

$$L_i = \frac{2K + R - D}{2(GLW)} \tag{5}$$

which is the general equation for the determination of lane value.

Lane increment (ΔL) will be such that :

$$1.25 \text{ centimetres} \leq \Delta L \leq 2.5 \text{ centimetres}$$

which will determine lane spacing.

Only the following multiples of ΔL are ever used :

.05	1	20
.1	2	50
.2	5	100
.5	10	

Hyperbolic lanes intersect a constant grid line in one of four ways (see figure 4).

In A and B, hyperbolic lanes are making grid intersections such that a constant northing would be selected upon which to compute intersections. In A the lanes are incrementing and in B they are decrementing. In C and D, a constant easting is selected upon which to compute intersections. Lanes are incrementing in C and decrementing in D.

The explanation which follows will be with respect to the case of 4A, where it is desired to compute the eastings of the lane crossings along a constant northing. The computation for the determination of the intersection points is not precise nor is it iterative, although iteration is employed whenever the rate of change of lane spacing becomes such that accuracy is jeopardized. The computational error is about one-half the magnitude of that produced by the draftsman.

The computation proceeds as follows : coordinates (X_p, Y_p) are selected, where :

$$X_p = X_a = 550\,000$$
$$Y_p = Y_a = 3\,200\,000$$

Parameters (1) and (2) are used for the determination of R and D. Employing equation (5), the precise lane value (L_i) is computed for P_a and is called L_a. The L_i value for P_b is determined by taking :

$$X_p = X_a + \Delta X = X_b$$
$$Y_p = Y_a = Y_b = Y_n$$

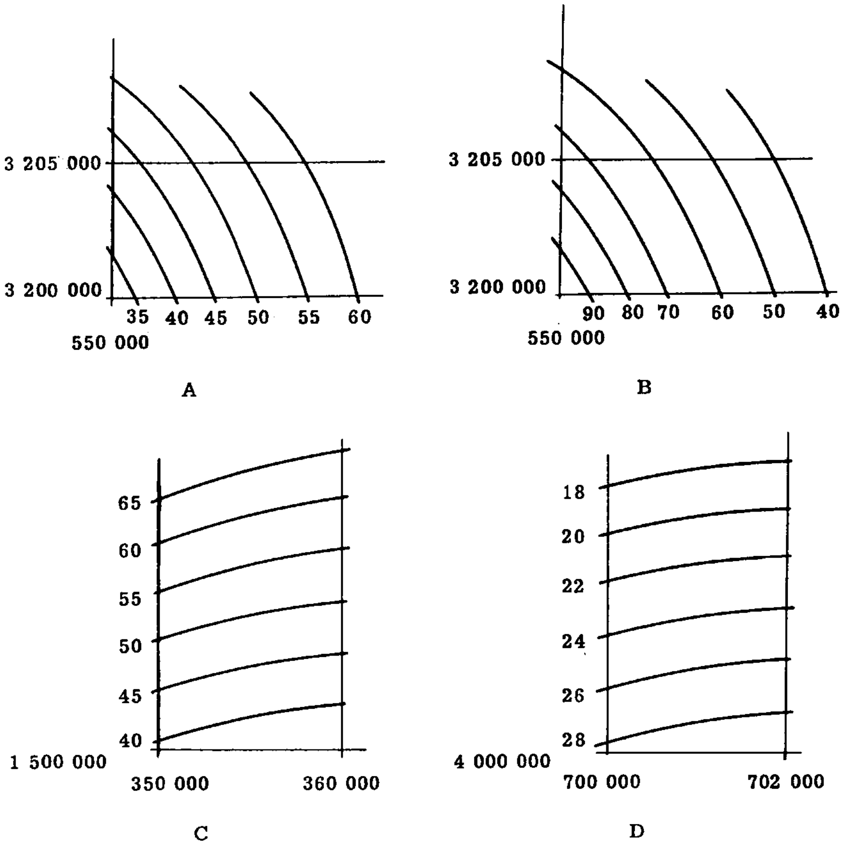

FIGURE 4

Parameters (1) and (2) and equation (5) yield for P_b :

$$L_i = L_b$$

Determine ΔL such that is the largest value less than $(L_b - L_a)$. The value $(L_b - L_a)$ is the lane difference for a known value of ΔX, where ΔX equals 2.5 centimetres on the grid sheet. Assume that 5 is equal to ΔL in the problem illustrated here.

The first value of L_i that it is desired to compute is an intercept which will be a multiple of ΔL which is just larger than L_a. In the problem illustrated, this will be lane 35 (L_{35}).

Take :

$$L_{35} - L_a$$

which is the lane fraction from Point P_a to the intersection point of L_{35}. Determine approximate easting for L_{35} :

$$\frac{(L_{35} - L_a)\,(\Delta X)}{(L_b - L_a)} + X_a = X_u\,.$$

The L_i value is determined by taking :

$$X_p = X_u$$

Parameters (1) and (2) and equation (5) determine L_u. The difference between L_{35} and L_u will be a small lane difference which will be translated into an easting correction to be applied to X_u, so that X_u approaches X_{35}.

Determine X_{35} :

$$\frac{(L_{35} - L_u)(\Delta X)}{(L_b - L_a)} + X_u = X_{35} .$$

where X_{35} is the easting of lane 35 along constant northing 3 200 000.

Take :

$$\frac{5(\Delta X)}{(L_b - L_a)} + X_{35} = X_u$$

where X_u is now the approximate easting for lane 40.

Take :

$$X_p = X_u$$

a value of L_u is determined.

Determine X_{40} :

$$\frac{(L_{40} - L_u)(\Delta X)}{(L_b - L_a)} + X_u = X_{40} .$$

After the first two lane intersections have been determined, each subsequent determination will be identical.

Take :

$$X_{40} - X_{35} = X_{n-1} - X_{n-2} = \Delta X \tag{6}$$

where ΔX is the difference between the two preceding eastings.

Take :

$$X_{40} + \Delta X = X_{n-1} + \Delta X = X_u \tag{7}$$

where ΔX when added to the preceding X is the approximate easting for lane $n(L_n)$ and will be identified as L_u.

Determine X_n :

$$\frac{(L_{45} - L_u)(\Delta X)}{5} + X_u = X_{45} = \frac{(L_n - L_u)(\Delta X)}{\Delta L} + X_u = X_n \tag{8}$$

where X_n is the easting of L_n.

When equation (8) has been completed, proceed to (6) to next intersection. Figure 5 lists a tab run based upon the method illustrated above.

With reference to figure 5, the draftsman will plot each point in the order in which it appears. Each intersection will be spaced at an optimum distance, and each intersection will be plotted, with the possible exception of an extra point at each end of a group. Additionally each intersection will be plotted with respect to only one dimension, that is the points will be plotted along a constant grid line.

Completion of objectives

1. In the development of a specific area, the new method delineates precisely the number of computations required. Formerly, it was not practical to develop exclusively a specific area.

2. By comparing figure 5 with figure 2, it will be observed how economical the new format is. The volume of paper required for a tab run by the new method will be about 10 % of the volume of the Beta Increment Method.

3 200 000 (constant northing)

Lane	Easting	Lane	Easting	Lane	Easting	Lane	Easting		Lane	Easting
30.00	549 769	35.00	551 130	40.00	552 450	45.00	553 727	....	65.00	558 455
70.00	559 554	75.00	560 625	80.00	561 669	85.00	562 687	....	105.00	566 538
110.00	567 452	115.00	568 347	120.00	569 226	125.00	570 089	....	145.00	573 400
.										
.										
.										
760.00	642 840	770.00	643 851	780.00	644 861	790.00	645 886	....	830.00	649 985

3 205 000

45.00	548 714	50.00	550 063	55.00	551 386	60.00	552 661	....	80.00	557 419
85.00	558 532	90.00	559 618	95.00	560 678	100.00	561 715	....	120.00	565 650
125.00	566 586	130.00	567 505	135.00	568 408	140.00	569 296	....	160.00	572 707
.										
.										
.										
770.00	644 192	780.00	645 241	790.00	646 307	800.00	647 374	....	840.00	649 990

FIGURE 5

Example of tab run by Hyperbolic Intersection on Constant Grid Line. (Observe that lane increment changes from 5 lanes to 10 lanes in order to maintain proper lane spacing).

3. Almost every intersection will be plotted by the Hyperbolic Intersection on Constant Grid Line versus about 25 % by the former. This represents a considerable saving in electronic computer time.

4. By enabling the draftsman to plot the intersection points along a line, as opposed to plotting points in a plane, the draftsman's speed is more than doubled. The accuracy of the plotted point should also be somewhat improved, inasmuch as only one dimension need be considered for the plotted point.

Conclusion

The Hyperbolic Intersection on Constant Grid Line Method was designed specifically to meet the needs of the Hydrographic Office. However, a number of users of hyperbolic systems in the past adopted the Beta Increment Method as standard computational procedure for the development of a hyperbolic lattice chart. The same users will probably be interested in the new method, for the reasons presented in this article. The application technique need not be confined to that of a hyperbolic lattice computation. A number of other problems under consideration at the Hydrographic Office can, and do, yield to similar treatment.

Bibliography

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