(ELECTRONIC) RIGOROUS LEAST SQUARE ADJUSTMENT OF DECCA SPEED RUNS

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1. — General remarks

1.1 — Speed trials do not strictly come within the category of surveying. They are, however, a precision application of radio position fixing methods and the mathematics involved in the computation of these trials are considered to be of interest to hydrographic surveyors.

1.2. — For a general description of procedures and methods used for ships' acceptance trials using radio position fixing systems, reference should be made to (1), which contains a list of references to more detailed articles.

This paper will deal with one of the aspects of the computation of the results of speed trials. The method developed is suitable for electronic computation.

1.3. — The ultimately required speed V through the water cannot, at sea, be measured directly with sufficient accuracy. All methods of speed determination — except those using drifting buoys — therefore determine the ground speed S. The effect of tidal stream, current and (not too adverse) meteorological conditions is eliminated by a proper combination of a number of runs S on opposite courses. The *method* of combination of runs will be described in section 6 of this paper. The *accuracy* with which the effect of tidal stream, etc. can be eliminated with this and other methods, will be discussed in a separate paper, to be published later.

1.4. — The minimum requirement for the determination of the ground speed S is two position fixes (as on the measured mile), one at the beginning and one at the end of each speed run. No conclusions as to absolute accuracy of a run can be drawn from these 2 fixes ; an error in one or both fixes will result in a systematic error in S that cannot be detected from the observations.

One of the big advantages of radio methods (including Decca) is that S is based on a large number of fixes that have only random, and no systematic, errors. In the Decca method, S for each run is based on 19 fixes of which 17 are consequently redundant. The use of proper methods of computation thus enables the reduction of the final error in S by a factor $\sqrt{17} \approx 4$, as compared with a run based on 2 fixes only, provided all fixes are of equal accuracy.

As Decca Navigator chains are used for the purpose of speed trials, extreme fixing accuracy cannot be obtained. The improvement factor of 4 however suffices for the limited accuracy of the Decca fixes. Operational figures show that S, as computed from a least square adjustment, can be determined with the same accuracy as on the measured mile ; in addition — and contrary to the measured mile — there is a guarantee against systematic errors in S.

The main economic advantages of the Decca method are :

a) In all European waters, Decca coverage is available free of cost.

b) Contrary to the measured mile, turning circles, stopways, etc. can be carried out.

c) Contrary to the measured mile, a sufficient number of deep water areas is available at reasonable distances from shipyards.

d) Trials can be carried out day and night (at night with reduced, but acceptable, accuracy), independent of visibility.

e) Trials can be made in areas of little shipping traffic.

f) An equal number of runs occupies less time than on the measured mile.

Keeping all these arguments, possibilities and advantages in mind, it is therefore evident that the analysis of the trials should not unnecessarily introduce computational errors. It is for this reason that a least square adjustment is the best possible method. The computations are complicated, but in the last few years this has no longer been an objection because electronic computation has been readily available and therefore — all things considered — at reasonable cost.

2. — Errors in Decca fixes

2.1. — Any least square adjustment of a number of observations is based on the assumption that the errors in the observations are of random character.

In the case of Decca therefore, it has first to be shown that the errors in the fixes are (sufficiently) distributed at random and that no systematic errors may influence the final result.

2.2. — As has been explained in reference (1), use is made only of *differences* between fixes. In this case, the only source of *systematic* errors is the propagation speed of the radio waves. This speed is known with an accuracy of 1 part in 10 000 or in most cases considerably better.

For the purpose of Decca acceptance trials it may therefore be concluded that systematic errors are non-existent. This conclusion is confirmed by the results of simultaneous speed determinations by Decca and on the Newbiggen measured mile. Of 6 of such combined determinations of the final speed V, the results agreed with each other within the limits of their standard errors (0.1 to 0.3 %).

2.3. — The sources of *random* errors are the following :

a) During the day, the radiated patterns — in which the combination receiver-decometers measures hyperbolic coordinates — show a short-periodic sway or instability of a small amplitude.

From many long series of observations at fixed stations (monitors) it can be shown that this daytime instability is of the order of ± 0.01 lane and usually even smaller over the periods of 10 to 15 minutes for one single run.

Because they are small, it is difficult to say whether they are of random or of short-periodic systematic character. Their effect can however — being of the same magnitude or smaller than the reading errors of 0.01 lane — be treated as random.

The "Decca day" covers the time when the sun is at least 10° to 15° above the horizon. The rest of a 24-hour period covers the " Decca night ". (This definition holds for short time instability limits of ± 0.01 lane.)

b) At night, the pattern instability is of a long periodic character (usually 1 to 2 hours) and of a much larger amplitude. It is also dependent on the location of the receiver with respect to the Decca transmitters.

No general figure as to the magnitude of the night instability can be given. In most cases however, it will be of the order of ± 0.02 to 0.05 lanes during the short period of a single trial run. It is for this reason that night trials offer only reduced accuracy.

In a few cases however (in Netherlands experience, 2 % of all night runs), the night-time instability is of a *systematic* character and in these cases will result in a systematic error in S and consequently also in the final V.

A method has been developed by which *speed* runs, thus systematically affected, can be *rejected*. The method will be described in a separate paper.

As there are no redundant fixes in a turning circle, stopway, etc., the rejection method cannot be applied for this type of trials and they should be carried out only during the day.

c) The combination receiver/decometers may be affected by random as well as systematic errors. It is for this reason that in all types of precision position fixing — as survey, acceptance trials, etc. — use has to be made of specially calibrated receivers. In these receivers provision has also been made against kicks from lane identification transmissions.

The remaining errors in these specially calibrated kick-free receivers are :

1) *systematic* errors less than 0.01 ; which moreover do not affect the results, because only *differences* between fixes are used ;

2) random errors smaller than ± 0.01 lane.

2.4. — Too large approximations in the computations should be avoided. Errors of this type can be kept negligibly small by using proper computational methods.

In the least square adjustment as given in this paper, the *computational* accuracy is 1 metre and therefore is completely negligible in comparison with the unavoidable errors in the Decca fixes themselves.

2.5. — Summarizing the above figures, it may therefore be concluded that :

a) *Systematic* errors are non-existent during the day and can be avoided (rejection method) during the night.

b) Other errors are of a sufficiently near *random* character to warrant the use of a least square adjustment of speed runs, provided a sufficient number of redundant fixes is available.

Daytime random errors can be expected to be of the order of ± 0.01 to 0.02 lane.

Night-time random errors may (although not always) be of a magnitude 3 times as large.

2.6. — The effect of the uncertainties mentioned in sub-section 2.5 on each individual fix (and consequently on S) is dependent on local lanewidth and the angle of cut of the hyperbolae.

No general figures can therefore be given, but the annexed example gives an idea of operational night-time accuracy in an area with lanewidths of the order of 1 000 and 1 800 metres and an angle of cut of 35°.

3. — Rigorous adjustment of S

3.1. — The first step is to convert the hyperbolic coordinates (decometer readings) into rectangular coordinates X, Y.

The conversion method used in the Netherlands for the computation of hyperbolic chart patterns, as well as for acceptance trials, has been described in a separate paper : "(Electronic) conversion of hyperbolic into rectangular coordinates".

3.2. — The most probable speed, to be derived from the 19 fixes in figure 1, is evidently given by some median straight line, for instance the dashed line in figure 1.

It is however, to a certain extent, anybody's guess how to draw this line in the most probable way and it is still more difficult to decide on the most probable beginning and end of the median line.

Any constructional method to decide on the most probable value of S and of the ground course of the run can therefore be no more than a guess, wherein incomplete use is made of the total available information, and of which the accuracy remains undetermined.

(For the purpose of illustration the spread of the 19 fixes with respect to the median line is, in all figures annexed to this paper, much exaggerated.)

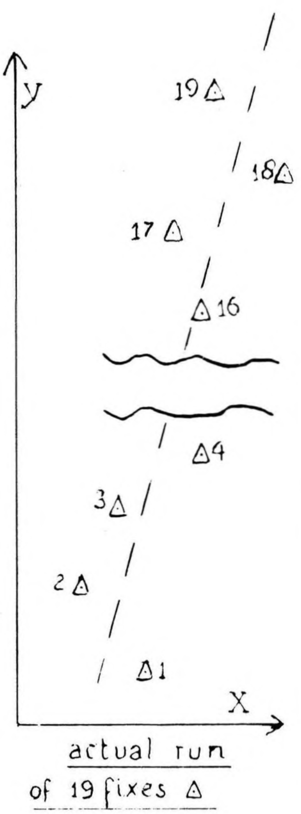


FIG. 1

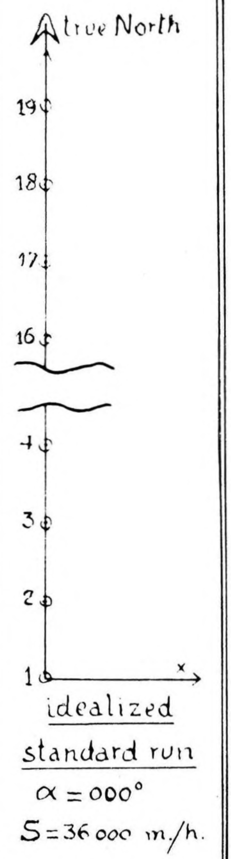


FIG. 2

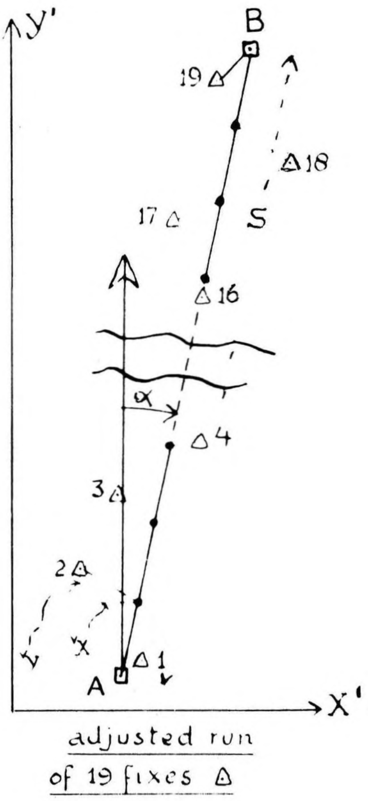


FIG. 3

3.3. — In all manner of technical problems, the question arises how to *compute* the most probable straight line through a number of points which show a random spread. The solution is a least square adjustment, and various methods are given in textbooks on higher mathematics and on geodesy.

The method as applied to speed trials is a variant of a similar geodetic problem. Its principles (not the full mathematical theory) will now be discussed.

3.4. — Figure 2 represents an idealized "standard" speed run at a ground speed of 36 000 m/h and at a true course of 000°. By 'idealized', it is understood that no errors whatsoever exist in this run. Consequently, all 19 fixes are in a straight line at equal distances of 300 metres for the usual interval of observations of 30 seconds.

There can be no objection to this idealization, because S is *by definition* proportional to time and lying on a straight line.

The reason that the "ideal" course has been chosen at 000° is that this simplifies the computations, because all values of all fixes then become zero.

A (mathematical) comparison of the 19 fixes in the actual run with those in the idealized one, provides the means of computing the most probable value of S, the ground course and the standard errors with which they can be determined.

3.5. — The principle of the least square method can be explained as follows.

Suppose the idealized run of figure 2 is taken off the paper, stretched or compressed uniformly and at the same time turned through an angle in such a way that it would fit on the actual run of figure 1 with the smallest possible discrepancies in the 19 fixes (*).

The best possible fit is illustrated in figure 3. The distance AB equals S, and the angle α through which the idealized run had to be turned is the ground course.

In fact this is what is done mathematically by the least square adjustment, which derives its name from the fact that the theory shows that the most probable value of S is the one that fulfils the condition that the sum of the squares of the remaining discrepancies v_X and v_Y (fig. 3) is *minimum*, i.e. as small as possible.

It should be noted that the procedure does not "disturb" the original fixes (see figure 3 in comparison with figure 1).

In order to obtain (mathematically) the most probable fit, the idealized run has to be uniformly stretched or compressed by a factor λ (lambda). The most probable value of S is obviously equal to 36 000 λ m/h. This factor follows from the least square adjustment.

An example of this rigorous least square adjustment, in a form suitable for computation on an electric desk calculator, appears at the end of the article (the values X, Y are the converted decometer readings ; see reference (2)).

In this form it is also suitable for electronic computation. As explained in reference (1), the programming for electronic computation is dependent on the type of computer, and is therefore not given here.

On the X-1 computer used for the Netherlands trials, the adjustment of 1 run — *including* the conversion from hyperbolic into rectangular coordinates X, Y — takes about 3 minutes for 1 run of *19 fixes*.

On an electric desk calculator — *excluding* the conversion of coordinates — it takes an experienced mathematician from 1 to 1 1/2 hours. The conversion would take him from 1/2 to 3/4 of an hour for *each fix*.

These examples may serve to illustrate that electronic computation is the only practical solution. Nowadays so many computers are available that this offers no practical problem ; moreover all manner of possible computational errors are avoided.

(*) The procedure can be very clearly illustrated by plotting the 19 fixes of the idealized run on a transparent rubber band.

4. — Accuracy of speed determination

4.1. — As usual in scientific, geodetic and hydrographic computations, accuracy figures are computed and given on the basis of standard errors, i.e. on a basis of 68 % probability.

4.2. — The example of a night run, appearing at the end of the article — which was first checked to be sure that it did not contain systematic errors likely to cause rejection — has been chosen at random from about two hundred speed runs.

The most probable speed of 14.587 knots has been determined with a standard error of ± 0.042 nautical miles (1 852 metres)/h or 0.29 % of S. The ground course of 312° 15′ was determined with a standard error of ± 12′ (*).

These figures are representative for Decca night runs. In only a very few was the standard error in S as high as ± 0.4 % and in quite a number of night runs it is not more than 0.2 %.

The standard errors of daytime runs vary between 0.1 and 0.3 % of S.

Consequently, speed errors are always considerably smaller than those that are unavoidable in the other parameters, such as shaft horse power and fuel consumption (see reference (1).

4.3. — The lanewidths in this example are 1 000 m and 1 800 m respectively, and the angle of cut of the hyperbolae 25°. Distances from speed trial area to Decca transmitters were 275, 240 and 260 kilometres.

Depth of water was 108 m (60 fathoms); draught was 11.4 m (38 ft).

4.4. — To the hydrographic surveyor, it is of interest that the *relative* accuracy (standard error σ_X and σ_Y) of a Decca fix, during the period of 9 1/2 minutes of the run, was ± 26 metres in X and ± 12 metres in Y. Due to the nature of the speed trial procedure, no figures as to absolute accuracy can be derived from these trials.

4.5. — In reference (1), it has been mentioned that the approximate adjustment (used until 1960) of speed runs results in very nearly the most probable speed, but — for *night* trials with a large spread of fixes — the standard error computed by this method is usually much too large. As an illustration, the run of the example has also computed by this now abandoned method. The results were :

Approximate method	Rigorous Method
S = 14.582 knots	S = 14.587 knots
σ_S = ± 0.60 % of S	σ_S = ± 0.29 % of S
α = 312° 19′	α = 312° 15′
σ_α = ± 21′	σ_α = ± 12′

From a practical point of view, the differences are not large and are even acceptable. Under unfavourable conditions however they might be

(*) There is no real need for this high accuracy in α, but it does not increase computer time to do the computations with that accuracy.

larger and moreover, with electronic computation, one may just as well use the rigorous method.

5. — Recommended compass course for Decca night runs

5.1. — One of the advantages of radio position fixing, and therefore also of Decca for speed trials, is that the runs can be made on any course that seems desirable from one point of view or another, for instance in the direction of the wind, waves, tidal streams, etc.

For *daytime* runs there are no restrictions from the point of view of accuracy, because the spread of the fixes (provided calibrated receivers are used) is always small (as is also illustrated by the daytime turning circles and stopways; see reference (1)).

5.2. — At *night* however, the pattern instability is much larger and so consequently is the spread of the 19 fixes.

The area of uncertainty (68 % probability) in the fix is represented by the error ellipse of figure 4. Its major and minor axes depend on the amount of pattern instability, width of lanes and angle of cut of position lines.

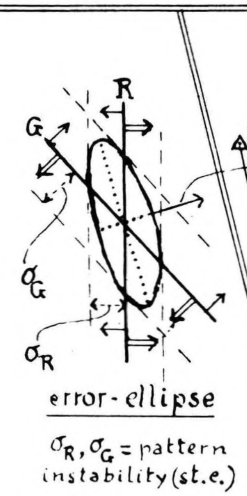


FIG. 4

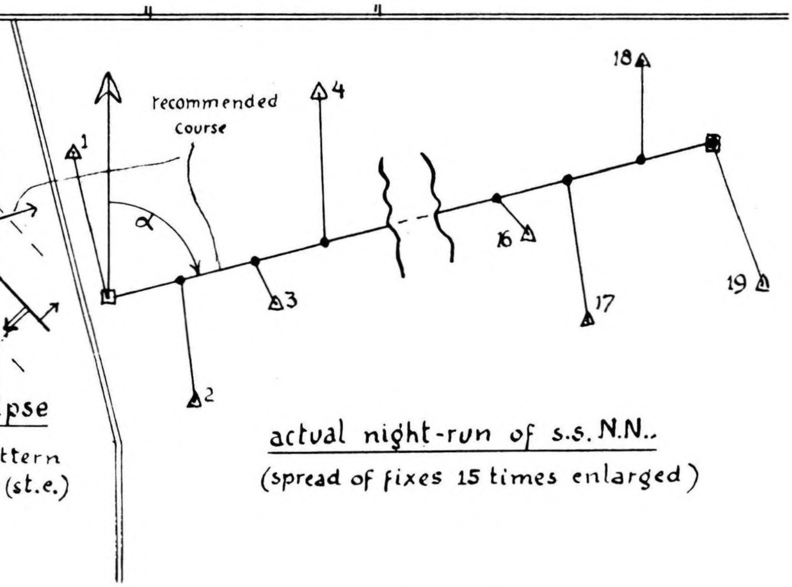


FIG. 5

In case the instability (skywave) is not excessively large and irregular, there is a strong correlation between 2 Decca patterns of one and the same chain; the reason is that both patterns have the master in common. As a consequence, the 2 patterns "move" in opposite directions (see figure 4) and the result is that the tendency of the "move" of the intersection of the 2 position lines in the direction of the long axis, is much larger than that in the direction of the small axis.

It is therefore preferable to make night-time speed runs roughly in the direction of the small axis of the error ellipse, because the spread of the fixes will then be mainly in a direction perpendicular to the course.

Fix errors perpendicular to the course will have a cosine effect only on the speed and will mainly affect the ground course. As will be shown in section 6, errors in ground course have a cosine effect only on speed and consequently, the effect of fairly large night-time pattern instability can be kept at a minimum by steering courses roughly in the direction of the small axis of the error ellipse. Figure 5 is an illustration of an actual night run chosen at random (scale of spread is 15 times as large as scale of run).

In a few cases however (in Netherlands experience, 2 % of hundreds of night runs), the skywave is so excessive and irregular that the spread of the fixes — even within the period of 9 1/2 minutes of a speed run — is systematically affected.

Runs thus affected can be discovered by the method mentioned in section 2.3 *b* ; such runs are then rejected and not used for the final computation of V (section 6).

5.3. — Since 1960, all Netherlands night runs are made on courses approximately in the direction of the minor axis of the local error ellipse.

6. — Elimination of drift

6.1. — By "drift" is understood the total effect of tidal stream, current, wind, etc.

In a paper to be published later, all these effects will be discussed separately as well as together.

This section deals only with the mathematical aspects of the elimination of the combined effect.

6.2. — The procedure as used for all Decca speed trials is as follows.

Two speed runs S_1 and S_2, as computed by the (rigorous) least square adjustment, are mathematically combined as illustrated in figure 6. V_1 is adopted as the provisional mean speed through the water, as derived from these 2 runs. The formula, derived from figure 6, is :

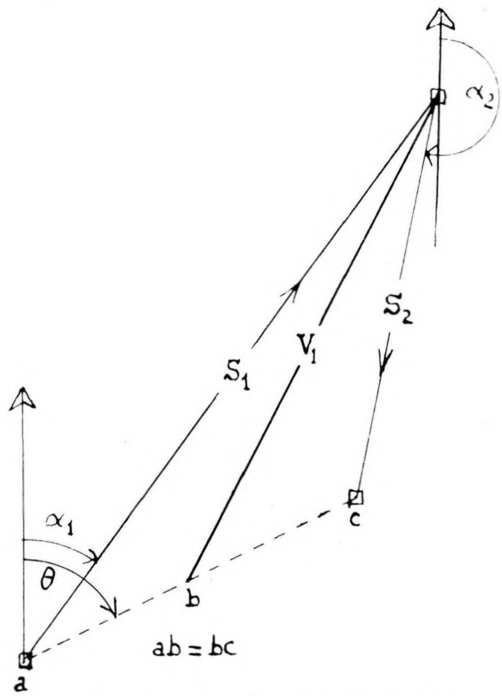
$$V_1 = \frac{1}{2}\sqrt{S_1^2 + S_2^2 - 2\, S_1\, S_2 \cos \Delta\alpha} \qquad (1)$$

where : $\Delta\alpha = \alpha_1 - \alpha_2$;

example :

$$\alpha_1 = 201^\circ\ 58'$$
$$\alpha_2 = 019^\circ\ 55'$$
$$\Delta\alpha = 182^\circ\ 03'$$
$$\cos \Delta\alpha = -\ 0.999360$$

Because of the lack of information as to the magnitude and direction of drift, it is (and has to be) assumed that the drift has remained constant during the period between the start of the first and the end of the second run ; in Decca trials, according to the procedure described in reference (1), this is approximately 30 minutes.



Approximate elimination of drift.

FIG. 6

Any other procedure might have certain advantages, but would have the considerable disadvantage of taking more time and consequently introducing larger uncertainties in the elimination of the drift ; this also applies to the procedure on the measured mile (see figures 1 and 1a in reference (1)).

This assumption, that has to be made in any type of speed trials, can therefore result in no more than a first approximation of V, and consequently a final speed V, derived from 2 speed runs only, cannot be expected to be accurate; moreover, no accuracy figure can be given, because no redundant data are available. The distance *ac* in figure 6 is twice the mean drift over a period of one hour (S_1 and S_2 are ground speeds per hour). The azimuth of the mean drift is equal to the angle θ.

The next step is to combine runs S_2 and S_3 in a similar way, resulting in a second approximation V_2. In this second approximation it is automatically assumed that the drift has been constant during the period from the start of the 2nd run till the end of the 3rd one, i.e. again during an overlapping period of 30 minutes.

For 4 runs, the third approximation V_3 is computed from a similar combination of runs S_3 and S_4. In case more than 4 runs have been made, more approximate speeds can be computed.

In the 3 approximate speeds thus computed, the drift effect has not yet been completely eliminated.

6.3. — The final speed V through the water is computed by applying the means of means method to V_1, V_2 and V_3 ; in a separate paper on the elimination of drift, it will be shown that the effect of tidal stream and current is then completely eliminated (to within 0.1 % of speed) in the final V.

Mathematically, the means of means (for 4 speed runs) equals :

$$V = \frac{V_1 + 2V_2 + V_3}{4} \tag{2}$$

6.4. — The standard errors in S_1 ... S_4 follow automatically from the least square adjustment ; see the example at the end of the article.

The formula for the computation of the standard error in the final V (not for V itself, which is computed by formula (2)), is derived as follows.

In the first place, it should be noted that the angle $\Delta\alpha$ in formula (1) is always very near 180° (cosine $\Delta\alpha \approx -1$). *For the purpose of computing a standard error,* the following approximations are more than sufficiently accurate :

$$S_I = \frac{S_1 + S_2}{2}$$

$$S_{II} = \frac{S_2 + S_3}{2}$$

$$S_{III} = \frac{S_3 + S_4}{2}$$

$$S_{IV} = \ldots\ldots \text{ etc.}$$

$$\frac{S_I + S_{II}}{2} = \frac{S_1 + 2S_2 + S_3}{4}$$

$$\frac{S_{II} + S_{III}}{2} = \frac{S_2 + 2S_3 + S_4}{4}$$

$$(V) = \frac{S_1 + 3S_2 + 3S_3 + S_4}{8} \tag{3}$$

Differentiating from (3) :

$$dV = \frac{1}{8}dS_1 + \frac{3}{8}dS_2 + \frac{3}{8}dS_3 + \frac{1}{8}dS_4$$

The 4 speeds S_1 ... S_4 are uncorrelated and the law of propagation of random errors may therefore be applied, resulting in :

$$\sigma_V^2 = \frac{1}{64}\sigma_{S_1}^2 + \frac{9}{64}\sigma_{S_2}^2 + \frac{9}{64}\sigma_{S_3}^2 + \frac{1}{64}\sigma_{S_4}^2$$

or :

$$\sigma_V^2 \approx \frac{1}{64}\sigma_{S_1}^2 + \frac{1}{7}\sigma_{S_2}^2 + \frac{1}{7}\sigma_{S_3}^2 + \frac{1}{64}\sigma_{S_4}^2 \tag{4}$$

6.5. — Formulae (1) and (2) are used for the computation of V_1 ... V_3 and of V, and formula (4) for the computation of σ_V.

This computation is included in the programme of the electronic computer, but could of course also be done quickly on an electric desk calculator.

References

[1] General information on the Decca method of ships acceptance trials. Published in the present volume (*).

[2] (Electronic) conversion of hyperbolic into rectangular coordinates. Published in the present volume.

(*) See also the references at the end of this article.

S.S. : N.N AREA : FARNDEEPS RUN n° : 13

LEAST SQUARE ADJUSTMENT

Date : 31 May 1961 time : 20h45 GMT (around sunset)

n°	stopwatch t	X	v_X	Y	v_Y	x	y	x'	y'	X'	Y'
1	02m27s	14 324	0	34 570	- 14	0	0	0	- 2 700	+ 1 500	- 1 349
2	57	14 174	- 17	34 727	- 19	0	300	0	- 2 400	+ 1 350	- 1 192
3	03 27	14 038	- 48	34 873	- 14	0	600	0	- 2 100	+ 1 214	- 1 046
4	57	13 844	- 20	35 016	- 3	0	900	0	- 1 800	+ 1 020	- 903
5	04 27	13 650	+ 7	35 159	+ 3	0	1 200	0	- 1 500	+ 826	- 760
6	57	13 483	+ 7	35 304	+ 9	0	1 500	0	- 1 200	+ 659	- 615
7	05 27	13 329	- 5	35 458	+ 7	0	1 800	0	- 900	+ 505	- 461
8	57	13 132	+ 25	35 601	+ 15	0	2 100	0	- 600	+ 308	- 318
9	06 27	12 949	+ 42	35 754	+ 14	0	2 400	0	- 300	+ 125	- 165
10	57	12 797	+ 27	35 909	+ 10	0	2 700	0	0	- 27	- 10
11	07 27	12 644	+ 13	36 064	+ 6	0	3 000	0	+ 300	- 180	+ 145
12	57	12 449	+ 42	36 205	+ 17	0	3 300	0	+ 600	- 375	+ 286
13	08 27	12 296	+ 28	36 360	+ 13	0	3 600	0	+ 900	- 528	+ 441
14	57	12 180	- 22	36 526	- 1	0	3 900	0	+ 1 200	- 644	+ 607
15	09 27	12 025	- 34	36 681	- 5	0	4 200	0	+ 1 500	- 799	+ 762
16	57	11 840	- 16	36 833	- 6	0	4 300	0	+ 1 800	- 984	+ 914
17	10 27	11 654	+ 4	36 985	- 6	0	4 800	0	+ 2 100	- 1 170	+ 1 066
18	57	11 490	+ 1	37 137	- 7	0	5 100	0	+ 2 400	- 1 334	+ 1 218
19	11 27	11 359	- 35	37 304	- 22	0	5 400	0	+ 2 700	- 1 465	+ 1 385
Z		12 824		35 919		0	2 700	0	0		
Σ			- 1		- 3					+ 1	+ 5
		$Z_X = \frac{[X]}{n}$		$Z_Y = \frac{[Y]}{n}$						$X' = X - Z_X$	$Y' = Y - Z_Y$

[y'y'] = + 51 300 000 = [1]

[y'Y'] = + 25 888 200 = [2]

[y'X'] = - 28 492 800 = [3]

$\underline{a} = \frac{[2]}{[1]} = \underline{+ 0.504\ 463}$ $\qquad \underline{a^2 = 0.25}$

$\underline{b} = \frac{[3]}{[1]} = \underline{- 0.555\ 415}$ $\qquad \underline{b^2 = 0.31}$

$\lambda^2 = a^2 + b^2 = 0.563\ 150$

$\lambda = 0.750\ 433$

$tg\ \alpha = \frac{b}{a} = - 1.100\ 610$

$\alpha = 312°15'$

note : [ ] = sum of squares

$v_X = b.y' - X'$

$[v_X v_X] = 12\ 089 \qquad \sigma_X^2 = \frac{[v_X v_X]}{n-1} = 671 \qquad \sigma_X = \pm 25.9$ m

$v_Y = a.y' - Y'$

$[v_Y v_Y] = 2\ 547 \qquad \sigma_Y^2 = \frac{[v_Y v_Y]}{n-1} = 141 \qquad \sigma_Y = \pm 11.9$ m

Observations with either v_X or v_Y larger than $2\frac{1}{2}$ times σ_X or σ_Y must be rejected from the run ; the run must then be readjusted.

$\sigma_a^2 = \frac{\sigma_Y^2}{[1]} = \frac{141}{51.3} \times 10^{-6} = 2.76 \times 10^{-6}$

$\sigma_b^2 = \frac{\sigma_X^2}{[1]} = \frac{671}{51.3} \times 10^{-6} = 13.09 \times 10^{-6}$

$\sigma_\lambda^2 = \frac{1}{\lambda^2}(a^2.\sigma_a^2 + b^2.\sigma_b^2) = 1.7757\ (0.25 \times 2.76 + 0.31 \times 13.09) = 8.43 \times 10^{-6}$

$\sigma_\lambda = 2.90 \times 10^{-3}$

$\sigma_\alpha^2 = \frac{1}{\lambda^4}(a^2.\sigma_b^2 + b^2.\sigma_a^2) = 3.1715\ (0.25 \times 13.09 + 0.31 \times 2.76) = 13.09 \times 10^{-6}$

$\sigma_\alpha = 3.62 \times 10^{-3}$ radians

n = 19 $\qquad \Delta t = t_{last} - t_{first} = 540$ seconds

$S = \frac{3600(y'_l - y'_f)}{\Delta t}.\lambda = 3600\ \frac{5400}{540}.\lambda = 27\ 016$ m./h.

$\sigma_S = \frac{S}{\lambda}.\sigma_\lambda = \frac{27\ 016}{0.7504}.2.9.10^{-3} = \pm 78$ m./h.

$\sigma_\alpha = 206\ 265''.\sigma_\alpha(rad.) = 725'' = \pm 12'.1$

knot = $\frac{m./h.}{1\ 852}$

S = 14.587 knots ; st.e. = ± 0.042 = ± 0.29 % of S

α = 312°15' ; st.e. = ± 12'.1