(ELECTRONIC) CONVERSION OF HYPERBOLIC INTO RECTANGULAR COORDINATES

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1. — Introduction

1.1. — Radionavigation — and (the more accurate) survey receivers — measure in the patterns radiated by a " chain " of radionavigation (survey) transmitters. The measured information is displayed in a suitable form on indicators.

The information required by the navigator (surveyor) is the coordinates of the location of his receiver.

1.2. — Most navigation and survey systems radiate hyperbolic patterns of Lines Of Position = L. O. P.s.; examples are Loran and Decca.

Some survey systems measure only distances and their L. O.P. s. are consequently circular; of this group Shoran (Hiran) is mentioned as an example.

Other survey systems, such as for example Raydist, Hi-Fix, etc., can operate as well on the hyperbolic as on the circular principle.

1.3. — The usual nautical and survey practice is to provide the user with charts, on which a *lattice* of the L. O. P. s. is overprinted together with a *grid* (X, Y rectangular coordinate systems) and/or a *graticule* (system of parallels and meridians).

Provided care is taken that the lattice L. O. P. s. correspond with the radiated L.O.P.s., the navigator or surveyor can plot his observed L.O.P.s. in the latticed chart and can scale off his position in the graticule or grid, thus obtaining the desired geographical or rectangular coordinates.

This is in fact a *graphical method of conversion* of hyperbolic (or circular) coordinates into geographical and/or Cartesian coordinates.

1.4. — The chart lattices have, of course, to be precomputed in such a way that the chart L. O. P. s. actually coincide with the radiated patterns.

Complete coincidence can, however, in practice never be achieved.

The reason is that the computation of the chart lattices requires knowledge of the propagation constants of the radiowaves which determine the shape and form of the *radiated* patterns. These propagation constants are, in most practical applications, only known approximately and moreover show local irregularities as will be explained in more detail in sections 2 and 3.

Any computation of chart patterns has therefore to be based on a more or less idealized mathematical model, in which use is made of the best possible estimates of the propagation constants.

1.5. — This paper will deal with one particular mathematical model of *hyperbolic* lattices, in which mathematical complications appear to have been reduced to a minimum.

This method has, since 1960, been used in the Netherlands for the computation of hyperbolic lattices (navigation and survey) and for the conversion of hyperbolic Decca coordinates into rectangular coordinates X, Y in acceptance trials of ships (see also references (1) and (2)).

The method is suitable for computation on an electric desk calculator as well as for electronic computation.

2. — Propagation of radiowaves

2.1. — Short radiowaves travel along nearly straight-line paths and consequently can cover short distances only at sea level; see figure 1.

It is for this reason that systems operating on long waves have to be used for radio position fixing at medium and long distances.

The direct (*) path of a radio transmission of a wave length of a few kilometres is illustrated in the left half of figure 1.

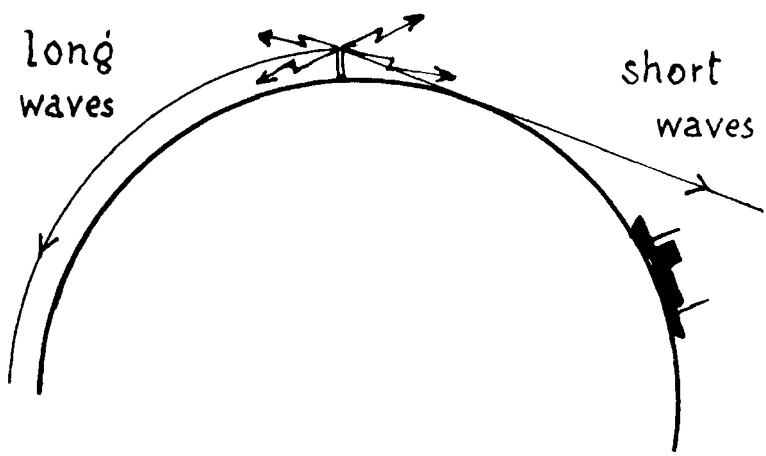


FIG. 1

(*) Indirect path (skywave) will not be discussed in this paper, because chart patterns are always computed for the direct or groundwave mode of propagation.

2.2. — Systems like Loran C and Decca transmit on a radio frequency of $\sim$ 100 Kc/s = a wavelength of about 3 kilometres. The *curvature* of the transmission path of waves of this order of length is approximately equal to the earth curvature; apart from energy losses with distance, the groundwave range is therefore very large.

The actual curvature is dependent on wavelength and on the dielectric constant of the atmosphere.

2.3. — It is generally assumed that the projection on earth of the curved transmission path coincides with the shortest distance on earth (ellipsoid) between transmitter and receiver; in other words it is assumed that no lateral refraction of the radio waves occurs. It has never been possible to prove this accurately from measurements, from which it could however be concluded that this projection is somewhere between the geodetic, the vertical section (both spheroidal lines) and the great circle (a spherical distance), joining transmitter and receiver.

Under the assumption of a radius of curvature of the (ground) path equal to that of the earth, the length of the transmission path would therefore — at sea level — be practically equal to that of its projection on earth.

2.4. — Directly or indirectly, the yardstick by which all radio position fixing systems measure is the propagation speed.

The propagation speed *in vacuo of all electromagnetic radiations* is constant and is internationally adopted at 299 792.5 km/sec; it has been possible to determine this constant *c* with a standard error (68 % probability) of 0.4 km/sec, i.e. with an accuracy of about 1 part in 750 000.

The speed with which radiowaves travel through the earth's atmosphere is dependent on density (height), atmospheric dielectric constant, wavelength and electrical conductivity of the terrain over which the waves travel.

2.5. — The anomalies in all these propagation parameters and their effect on terrestrial distances and L. O. P. s. will be discussed in detail in section 3.

3. — Uncertainties in propagation constants and their effect on distances and hyperbolic lines of position

3.1. — In this section, mean values of propagation constants will be given, together with the magnitude of their possible variations in actual practice.

All figures are for propagation of radio waves of the order of length of *3 kilometres* (Loran C and Decca) in the *lower regions of the atmosphere.*

3.2. — The radius of curvature of the transmission path is assumed to be equal to the earth's radius. The dielectric constant of lower atmosphere varies with meteorological conditions and as a result, the actual radius may differ by ± 1 % from the assumed one. The effect on the path length is as follows :

Pathlength	Relative error
100 km	± 1/4 800 000
200 km	± 1/1 200 000
500 km	± 1/ 192 000
1 000 km	± 1/ 48 000
1 500 km	± 1/ 21 000

3.3. — The propagation speed over *sea water* is V = 299 680 km/sec and is dependent on atmospheric conditions, which however are very unlikely to affect this speed by more than ± 1 part in 15 000; any error is, of course, proportional to distance.

Radiodistances, converted into terrestrial by means of V = 299 680, can therefore be in error to a total of ± 1 part in 15 000.

3.4. — The propagation speed over *fresh water* is of the order of 299 450 km/sec and variations therein may be of the order of ± 1 part in 10 000, making all distances uncertain by that amount.

3.5. — Over land, the propagation speed may — in extreme cases — vary between about 299 650 (salt water swamps) and about 298 800 (very " bad " bare rocks), mainly dependent on the electrical conductivity of the soil and sub-soil to a depth of about 50 metres. These extremes differ by as much as 1 part in 350.

3.6. — Due to the small lateral refraction (section 2.3.), the projection on earth of the curved radiopath will be somewhere between a geodesic and a great circle (on a sphere with " mean " radius for the area covered by the radio chain). Compared with (most of) the uncertainties mentioned above, the difference in terrestrial length of these 2 lines is of no practical importance whatsoever, as may be seen from the following figures (the actual differences are dependent on azimuth and on location on the terrestrial ellipsoid).

Geodesic	Great circle	Difference
100 km	100 km ± 0.0022 metre	1/460 000 000
500 km	500 km ± 0.14 metre	1/ 3 700 000
1 000 km	1 000 km ± 2.2 metres	1/ 460 000
1 500 km	1 500 km ± 11 metres	1/ 140 000

3.7. — Mixed land and sea paths can — for the purpose of conversion of radio into terrestrial distance — be split up, applying the most probable sea and (various) land speeds to the sections as scaled-off from an existing chart. In case ground conductivities are unknown (as is nearly always the case in underdeveloped countries), estimates of speeds have to be used and the uncertainties in V will still remain considerable.

3.8. — These figures about possible uncertainties in the propagation speed look very disappointing. Fortunately however, the situation is in actual practice not as bad as (most of) these figures suggest. The reasons are as follows :

a) Mean propagation speeds along the baselines Master-Slave can — in all radio position fixing systems where the terrestrial baselengths are accurately known — be determined by means of fairly simple measurements. The speeds thus determined give at least a fair idea as to the mean propagation constants in parts of the total area of coverage of the chain. As a consequence the estimates of the constants (mainly the speed) in the rest of the coverage will have considerably smaller errors than the extremes mentioned in the preceding sub-sections.

b) In *hyperbolic* systems, the hyperbolae are defined by the *difference* in distance from receiver to Master and Slave respectively. Although these 2 distances generally will be of unequal length and also usually will be (partly) over terrain of different conductivity, at least part of the errors in the adopted estimated speed will tend to cancel out and consequently will not have the full effect on the position of the hyperbolae.

3.9. — Because of the fact that the actual propagation constants are not uniform throughout the whole area of coverage of the chain, the radiated L. O. P. s. will not be of a pure mathematical hyperbolic (or circular) shape.

As the actual constants can only be estimated with a limited degree of accuracy, the computed chart patterns can never be more than a reasonable approximation of the radiated patterns. Even when the best possible estimates are used, there will remain discrepancies between the 2 patterns.

3.10. — The only way to determine the magnitude of these discrepancies is to compare chart patterns (computed L. O. P. s.) against radiated patterns (receiver-readings) at a number of points with known terrestrial coordinates, throughout the (used part of the) total chain coverage.

With reasonable estimates of the propagation constants, the differences thus determined will change only slowly from place to place and, from measurements on a limited number of these check-points, a plot can be made of lines of equal difference, showing the corrections that have to be applied to the receiver-readings before plotting them on the hyperbolic chart.

3.11. — These corrections for the correction chartlet can be kept as small as possible (and consequently the *number* of check-points) when all available information on propagation is used for the pattern computation. It would for instance be possible to split up all paths into sea, fresh water, swamps, " good " and " bad " conducting land, etc., and apply the most probable speeds. This would however very much complicate the formulae to be used; introducing too much detail would make the formulae system unmanageable for an electric desk calculator as well as for the programming of an electronic computer. Moreover, it would still be necessary to do the observations and computations for the correction chartlet.

The generally adopted compromise therefore is to compute spheroidal chart patterns (taking earth and path curvature into account), using one single mean propagation speed (the best possible mean estimate for the area) and to consider the remaining discrepancies as local — so-called " fixed " — corrections to be determined separately.

3.12. — The procedure mentioned in the last paragraph of sub-section 3.11. is also used for areas where no conductivity data and/or terrestrial triangulation are available. The correction chartlet is in this case *computed* by means of the best possible estimates of path split-up and assessed propagation speeds.

3.13. — It may be interesting to make mention of one of the few cases of propagation (largely over land) in which the electrical conductivity was known in detail. This enabled the computation of theoretical speeds by means of which the length of the 3 baselines of a Decca Navigation chain could be computed and compared against their lengths as known from a primary triangulation of an accuracy (standard error) of 1 part in about 150 000.

The accuracies of the " Decca " distances were as follows :

Red baseline ($\sim$ 162 km) : 1/54 000
Green baseline ($\sim$ 157 km) : 1/63 000
Purple baseline ($\sim$ 171 km) : 1/90 000

For further details, reference should be made to (3) and (4).

4. — Mathematical model

4.1. — A hyperbola is by definition a curved line, along the entire length of which the difference in distance to the 2 foci is constant. In a hyperbolic radio position fixing system, the transmitters are located at the foci.

In time-difference systems, as for instance Loran, the receiver measures the difference in arrival time of 2 pulses, one radiated by a Master and the other one by a Slave which is synchronized to the Master. With the known propagation speed of the radiowaves, this time difference is converted into the difference in distance, which defines the hyperbola.

In a phase comparison system, of which Decca is an example, the phase difference between synchronized continuous wave transmissions of Master and Slave is measured by the receiver; with the known propagation speed (more accurately : the phase velocity), this phase difference can again be converted into a difference in distance between receiver, Master and Slave respectively.

In both cases, the receiver therefore indirectly measures the differences $(l_M - l_R)$ and $(l_M - l_G)$, as is illustrated in figure 2.

4.2. — For long waves over long distances at or around sea level, all radio distances in figure 2 are very nearly equal to the corresponding distances on the terrestrial ellipsoid, as has been explained in sections 2 and 3.

Assuming constant propagation speed throughout the radiated patterns, such a mode of propagation would give rise to patterns of spheroidal (ellipsoidal) hyperbolae.

In reference 5, the Italian Professor S. BALLARIN developed the formulae for a mathematical model which is based on these assumptions. This method represents the best possible idealization of the actual radiated patterns.

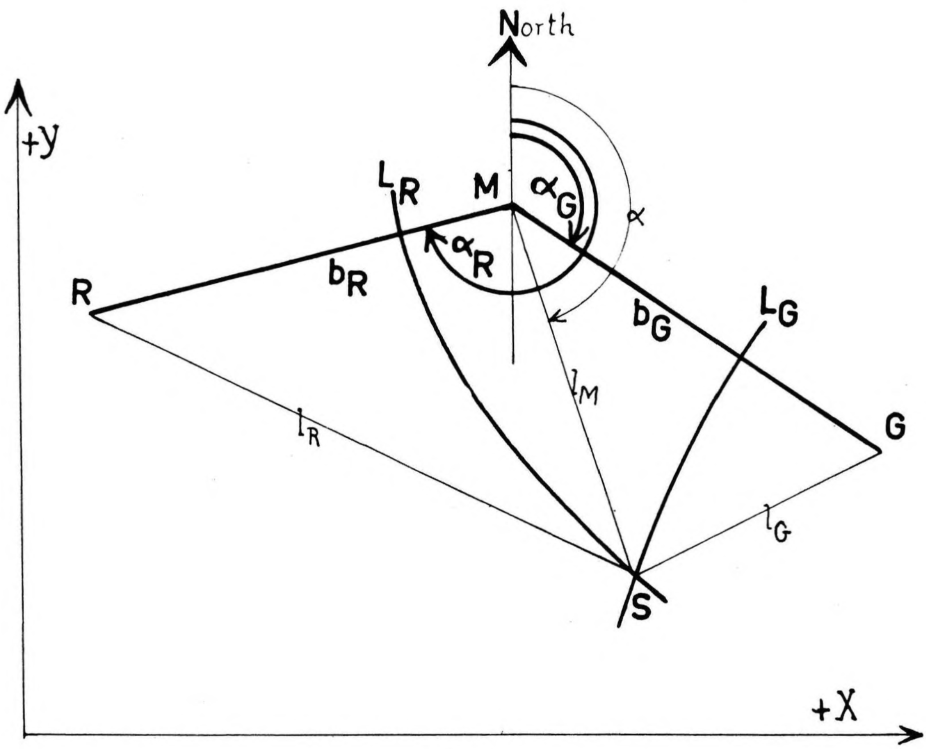


FIG. 2

The formulae would become much too complicated for practical use, in the case of various propagation speeds (if known) being introduced for various areas of the total coverage. Such deviations from idealized patterns must be cared for by local corrections; see sub-section 3.10.

Professor BALLARIN's method is no doubt mathematically the most elegant of the many others that are sometimes used. On an electric desk calculator the computations are quite lengthy (*). The BALLARIN formula system can in principle also be used for programming an electronic computer. Also, apart from being lengthy, difficulties arise in practice as to the algebraic signs of sines and tangents, upon which a computer cannot decide without being supplied with additional information; for further details see sub-section 5.10.

4.3. — Because of the complicated and lengthy BALLARIN computations, it was decided to change over to a still more simplified mathematical model, in which straight-line propagation — again at constant speed — is assumed along a flat earth.

The mathematical and computational advantages of this model are that the patterns now become plane hyperbolae, and that the much more

(*) Among other reasons because — in order to enable the use of " manageable " formulae — the fix coordinates are first computed on a sphere of " mean " radius; thereafter, these spheroidal coordinates are converted into the desired ellipsoidal latitude and longitude by means of a transformation from sphere to spheroid.

simple formulae of plane trigonometry can be used throughout the whole computational procedure.

4.4. — This simplified model differs from the best possible idealization as used in Professor BALLARIN's model, and chart patterns thus computed will not offer the best possible fit to the radiated patterns.

The difference between the two models is caused by the neglect of the curvature, the effect of which will evidently increase with increasing distances between receiver and transmitters. For any location in the coverage, the difference will however be of a fixed amount equal to the difference between the spheroidal and the plane hyperbolae for that particular location. To compensate for this difference which changes slowly from place to place, it is sufficient to compute both spheroidal and plane hyperbolae for a limited number of points in the total coverage and to apply their difference as a fixed local correction to the receiver-readings.

This system is quite acceptable, because even in the best possible idealization (spheroidal hyperbolae), corrections have anyhow to be applied to compensate for (the best possible estimate of) unequal propagation speeds over different areas of the total coverage of the chain.

The Dutch practice is to combine these speed corrections with those for the difference between spheroidal and plane hyperbolae into one combined correction chartlet.

4.5. — Provided a suitable chart projection is used, the difference between spheroidal and plane hyperbolae can be computed by means of formulae of plane trigonometry, as will be explained in section 6.

4.6. — *Summarizing,* the method used in the Netherlands has been adopted as a result of the following considerations :

a) Spheroidal hyperbolae must be considered as the best possible mathematical model of the radiated patterns; the computation is complicated and lengthy;

b) Speed corrections to computed chart patterns have to be applied in any case;

c) Chart patterns of plane hyperbolae can be computed with fairly simple formulae of plane trigonometry;

d) Patterns of plane hyperbolae need corrections that can be computed with simple formulae of plane trigonometry;

e) Speed corrections and those mentioned in d) can be combined in one simple correction chartlet.

5. — Mathematical model adopted and derivation of formulae

5.1. — The adopted mathematical model of plane hyperbolae is represented in figure 2.

Given : X_M, Y_M ; X_R, Y_R ; X_G, Y_G; (X_P, Y_P).
L_R ; L_G ; (L_P).

Wanted : X_S, Y_S.

NOTE 1. — The rectangular coordinates X, Y of Master and Slaves are assumed to be known. In case only geographical coordinates are given, they must be converted into X, Y; for most chart projections, conversion tables have been published.

NOTE 2. — L is the symbol for a Decca lanenumber, consecutively counted from 000 at the Master extensions of the baselines. Example : Green decometer reading = D 45.63; $L = 3 \times 30 + (45.63 - 30) = 105.63$.

5.2. — The formulae for plane hyperbolae as given in this section were developed by the author in 1956. The derivation is based on Professor BALLARIN's development, applied to plane triangles.

5.3. — From known X, Y coordinates, the length of the baselines and their azimuth is computed as follows :

$$\left.\begin{aligned} b_R &= \sqrt{(X_R - X_M)^2 + (Y_R - Y_M)^2} \\ b_G &= \sqrt{(X_G - X_M)^2 + (Y_G - Y_M)^2} \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \alpha_R &= \arctan \frac{X_R - X_M}{Y_R - Y_M} \\ \alpha_G &= \arctan \frac{X_G - X_M}{Y_G - Y_M} \end{aligned}\right\} \qquad (2)$$

5.4. — The observer's position S is determined by the intersection of the two (observed) lanenumbers L_R and L_G (fig. 2).

These lanenumbers equal :

$$\left.\begin{aligned} L_R &= \frac{b_R + l_M - l_R}{\Lambda_R} \\ L_G &= \frac{b_G + l_M - l_G}{\Lambda_G} \end{aligned}\right\} \qquad (3)$$

where :

Λ_R = comparison wavelength in Red pattern,

Λ_G = comparison wavelength in Green pattern.

Substituting :

$$\left.\begin{aligned} l_M - l_R &= -n_R \\ l_M - l_G &= -n_G \end{aligned}\right\} \text{ in (3),}$$

it follows that n_R and n_G can be computed from :

$$\left.\begin{aligned} n_R &= b_R - L_R \cdot \Lambda_R \\ n_G &= b_G - L_G \cdot \Lambda_G \end{aligned}\right\} \qquad (4)$$

In formulae (4), b_R, b_G, L_R, L_G, Λ_R and Λ_G are known quantities.

NOTE. — In the computation of Loran hyperbolae, n is equal to the difference in microseconds, converted into distance units (metres, yards, etc.).

5.5. — In triangle MRS (fig. 2) :

$$l_R^2 = l_M^2 + b_R^2 - 2 l_M\, b_R \cos(\alpha_R - \alpha)\,.$$

Substituting : $l_R = l_M - (l_M - l_R) = l_M + n_R$:

$$(l_M + n_R)^2 = l_M^2 + b_R^2 - 2\, l_M\, b_R \cos(\alpha_R - \alpha)$$
$$l_M^2 + n_R^2 + 2\, l_M\, n_R = l_M^2 + b_R - 2\, l_M\, b_R \cos(\alpha_R - \alpha)$$
$$2\, l_M\, n_R + 2\, l_M\, b_R \cos(\alpha_R - \alpha) = b_R^2 - n_R^2$$
$$2\, l_M \left(\frac{n_R + b_R \cos(\alpha_R - \alpha)}{b_R^2 - n_R^2} \right) = 1 \qquad (5)$$

Similarly, in triangle MGS, it follows that :

$$2\, l_M \left(\frac{n_G + b_G \cos(\alpha - \alpha_G)}{b_G^2 - n_G^2} \right) = 1 \qquad (6)$$

Note that $\cos(\alpha - \alpha_G) = \cos(\alpha_G - \alpha)$.

5.6. — From equations (5) and (6) it follows that :

$$\frac{n_R}{b_R^2 - n_R^2} + \frac{b_R}{b_R^2 - n_R^2} \cdot \cos(\alpha_R - \alpha) = \frac{n_G}{b_G^2 - n_G^2} + \frac{b_G}{b_G^2 - n_G^2} \cdot \cos(\alpha_G - \alpha)$$

Putting :

$$\left. \begin{array}{ll} \dfrac{n_R}{b_R^2 - n_R^2} = N_R & \dfrac{n_G}{b_G^2 - n_G^2} = N_G \\[2ex] \dfrac{b_G}{b_R^2 - n_R^2} = M_R & \dfrac{b_G}{b_G^2 - n_G^2} = M_G \end{array} \right\} \qquad (8)$$

equation (7) becomes :

$$N_R + M_R \cos(\alpha_R - \alpha) = N_G + M_G \cos(\alpha_G - \alpha) \qquad (9)$$

5.7.— In equation (9), α is the only unknown quantity and can be solved as follows :

$$M_R \cos\alpha_R \cos\alpha + M_R \sin\alpha_R \sin\alpha - M_G \cos\alpha_G \cos\alpha -$$
$$- M_G \sin\alpha_G \sin\alpha = N_G - N_R$$
$$(M_R \sin\alpha_R - M_G \sin\alpha_G) \sin\alpha + (M_R \cos\alpha_R - M_G \cos\alpha_G) \cos\alpha = N_G - N_R \qquad (10)$$

Putting :

$$\frac{M_R \cos\alpha_R - M_G \cos\alpha_G}{M_R \sin\alpha_R - M_G \sin\alpha_G} = \tan\varepsilon \; (*) \qquad (11)$$

and substituting (11) in (10) :

$$(M_R \cos\alpha_R - M_G \cos\alpha_G) \operatorname{cotan}\varepsilon \sin\alpha +$$
$$+ (M_R \cos\alpha_R - M_G \cos\alpha_G) \cos\alpha = N_G - N_R$$
$$\{M_R \cos\alpha_R - M_G \cos\alpha_G\} \left\{ \frac{\cos\varepsilon}{\sin\varepsilon} \cdot \sin\alpha + \cos\alpha \right\} \sin\varepsilon = (N_G - N_R) \sin\varepsilon$$
$$\{M_R \cos\alpha_R - M_G \cos\alpha_G\} \sin(\alpha + \varepsilon) = (N_G - N_R) \sin\varepsilon$$
$$\sin(\alpha + \varepsilon) = \frac{N_G - N_R}{M_R \cos\alpha_R - M_G \cos\alpha_G} \cdot \sin\varepsilon \qquad (12)$$

(*) Depending on the geometrical situation, the quotient in the left hand term of this equation may have any value between 0 and $\pm \infty$, and may therefore be made equal to the tangent of an auxiliary angle ε; ε is not an angle in figure 2.

From equation (12), $(\alpha + \varepsilon)$ can be computed and consequently :

$$\boxed{\alpha = (\alpha + \varepsilon) - \varepsilon} \tag{13}$$

5.8. — The next step is to compute l_M.

$$\frac{1}{2\,l_M} = N_R + M_R \cos(\alpha_R - \alpha) = N_G + M_G \cos(\alpha_G - \alpha)$$

$$\boxed{l_M = \frac{1}{2N_R + 2M_R \cos(\alpha_R - \alpha)} = \frac{1}{2N_G + 2M_G \cos(\alpha_G - \alpha)}} \tag{14}$$

5.9. — With the polar coordinates α and l_M from formulae (13) and (14), the rectangular coordinates X_S, Y_S follow from :

$$\boxed{\left.\begin{array}{l} X_S = l_M \sin\alpha \\ Y_S = l_M \cos\alpha \end{array}\right\}} \tag{15}$$

5.10. — The above formulae are suitable for computation on an electric desk calculator as well as for programming of an electronic computer (see references (6) and (7)). In both cases the computation is simple and fast. Just as in the BALLARIN computations, a difficulty arises with the algebraic signs of some of the goniometric functions, as is illustrated by the following example (actual computations are in 8 decimal places).

formula (11)	tan ε	+ 0. 158 38			
	ε	009°		189°	
	sin ε	+ 0. 15643		- 0. 15643	
formula (12)	sin $(\alpha + \varepsilon)$	+ 0. 25882		- 0. 25882	
	$\alpha + \varepsilon$	015°	165°	195°	345°
	ε	009°	009°	189°	189°
formula (13)	α	006°	156°	006°	156°

It remains therefore undecided whether α is 006° or 156°.

In nearly all normal geometrical configurations occurring in actual practice, this trouble can be overcome when the quadrant of α is known, which is always the case in the computation of chart patterns and also for the conversion of hyperbolic coordinates in a speed trial area.

The human computer would therefore have no difficulty. An electronic computer would get 2 angles, would " think " that some sort of an error had been made, would repeat that section of the computation (only to get the same 2 angles) and would go on indefinitely, without continuing the rest of the computation. In order to overcome this difficulty, information as to the quadrant of α is " built-into " the programme of the electronic computer.

There is however one geometrical configuration — figure 3 — in which information about the quadrant is insufficient (in actual practice, this seldom occurs).

In figure 3, both fixes S_1 and S_2 satisfy the same hyperbolae numbers and both azimuths α_1 and α_2 are in the 4th quadrant. In this case it is necessary to precompute α from estimated coordinates of S and to use the estimated α as "built-in " information.

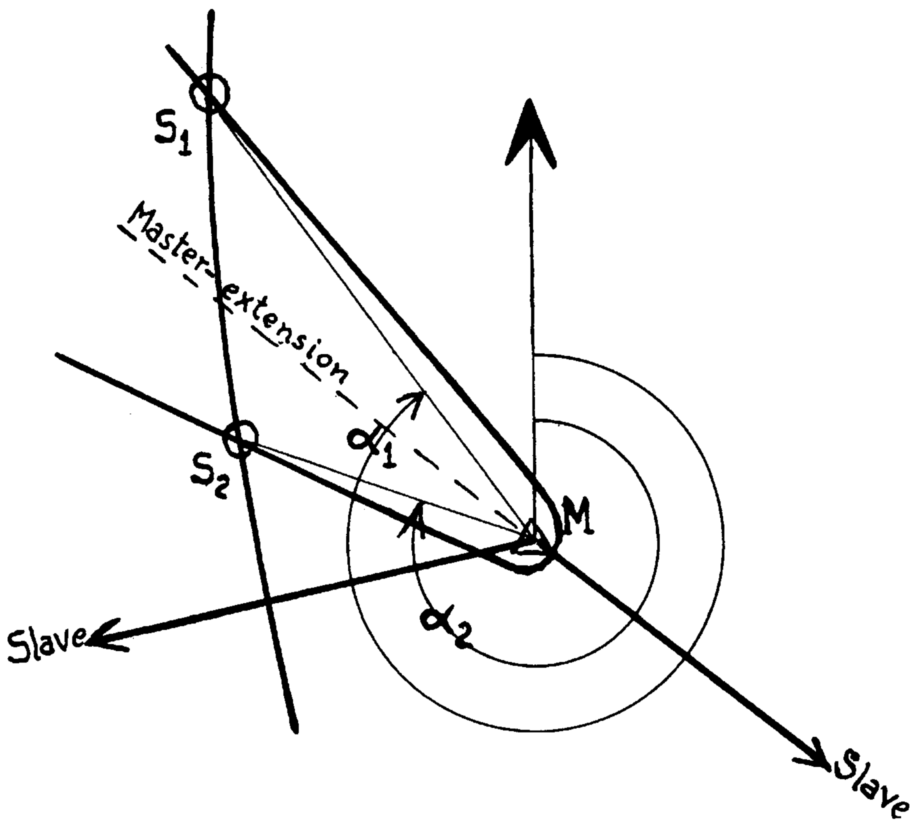


FIG. 3

NOTE. — Theoretically, S_1 and S_2 could be quite near to each other and in a limiting case, the estimated position S would have to be known very accurately in order to be able to decide on the proper value of α. In actual practice however, such extreme configurations will not occur or at least will not be used for position fixing, because hyperbolae very near a baseline extension are too far apart to be suitable for the purpose.

6. — Difference between spheroidal and plane hyperbolae

6.1. — As mentioned in sub-sections 4.4. and 4.6., the difference between plane and spheroidal hyperbolae has to be computed and applied as a correction to the receiver-readings before plotting on a chart of plane hyperbolae. For Decca patterns, this correction has to be computed in hundredths of lanes and for Loran patterns, in microseconds.

BALLARIN's inverse (*) method could be used to compute the spheroidal hyperbolae numbers for all points indicated in figure 4 for which it is decided to compute the difference; the plane hyperbolae numbers can be computed from the rectangular coordinates of these points and of the transmitters.

The BALLARIN computation would however be quite lengthy, the more so because the X, Y coordinates of all points would first have to be converted into φ and λ.

6.2. — A much faster method is to compute the spheroidal hyperbola numbers in the *same* rectangular system as has been used for the computation of the plane hyperbolae. It will be shown below that this method is indeed very simple and fast.

6.3. — The only universal coordinate system on the ellipsoidal earth is the *graticule* (φ and λ).

Rectangular *grids* (X, Y) are auxiliary coordinate systems which facilitate computations and sealing-off within the area of the chart.

The mathematical relation between graticule and grid is of course dependent on the chosen chart projection.

For the purpose discussed in this section, the most suitable chart projection is the one in which the chart distances (as defined by the X, Y system) have a mathematically simple relation to ellipsoidal distances. This enables chart distances (as used in the computation of plane hyperbolae) to be converted in a simple and fast way into ellipsoidal distances (as required for the computation of the spheroidal hyperbolae).

6.4. — One of the chart projections nearly ideally fulfilling this requirement is the *Transverse Mercator* projection and this projection has been chosen for (nearly) all Netherlands computations of hyperbolic chart patterns as well as for speed and manœuvring trials.

There are other suitable projections, but one of the reasons that the T. M. has been chosen is that very handy T. M. tables exist, published by the U. S. Army Map Service (A. M. S.).

In order to maintain consistency in all the computations, the geographical coordinates φ, λ of the transmitters, as used in the computation of the plane hyperbolae, naturally have to be converted into T. M. Easting and

(*) In the direct BALLARIN method, the geographical coordinates φ, λ of a fix are computed from given hyperbola numbers. In the inverse method, the latter are computed from given fix coordinates φ, λ. Both methods are described in reference (5).

Northing (*) (notations in this paper : X, Y) by means of these tables, before using the formulae of section 5.

NOTE. — The T. M. projection would become unsuitable in the case where the total range of the radio system extended over more than one T. M. zone, that is 6° difference in longitude and a latitude extension from 80° N to 80° S.

Radio survey systems and radio position fixing for speed trials, however, never cover such large ranges.

For very long range rădio navigation systems, chart patterns of plane hyperbolae would be unsuitable anyhow, because their difference with the radiated patterns would become too large.

6.5. — By *scale factor* k is to be understood the ratio between an *infinitely small distance* as derived from a measurement on the chart, and the corresponding distance on the ellipsoid.

The advantages of the T. M. projection are :

a) $k = 0.99960$ is constant throughout the length of the central meridian;

b) k can be computed for any Easting (X) and Northing (Y) in a very simple way by means of the A. M. S. tables;

c) k varies slowly with latitude, is otherwise constant along any particular Y-axis and may — for all practical purposes — be considered as a *constant* over the whole range of latitude covered by the radio system.

Summarizing, scale factors are — for all practical purposes — dependent only on Easting (X). As a consequence, one and the same scale factor applies to all points along the line 1-6 (fig. 4), another scale factor along the line 7-12, 0.99960 along the central meridian, etc.

6.6. — By *distance scale factor* is to be understood the ratio between a *long distance* as derived from measurements on the chart, and the corresponding distance on the ellipsoid.

Because of the property of the T. M. projection of small and slowly changing scale factors, this distance scale factor f can be computed by means of the following simple formula :

$$\frac{1}{f} = \frac{1}{6}\left(\frac{1}{k_A} + \frac{4}{k_C} + \frac{1}{k_B}\right) \tag{16}$$

where :

$$\left.\begin{array}{l} k_A = \text{scale factor along Y-axis A} \\ k_B = \text{scale factor along Y-axis B} \\ k_C = \text{scale factor along Y-axis C} \\ \text{and C = midway between A and B or} \\ X_C = \dfrac{X_A + X_B}{2} \end{array}\right\} \text{(see also figure } 4a\text{)}$$

(*) There are no theoretical objections to the use of any other chart projection. The only requirement is that one and the same projection be used for the computation of the plane and of the spheroidal hyperbolae.

$1/f$ is the factor by which *chart* distances have to be multiplied to convert them *into ellipsoidal* (spheroidal) distances.

6.7. — By way of example, it is assumed that it is required to compute the difference between plane and spheroidal hyperbola numbers at 36 points in the coverage of the (Decca) chain RMG (fig. 4). The procedure is as follows.

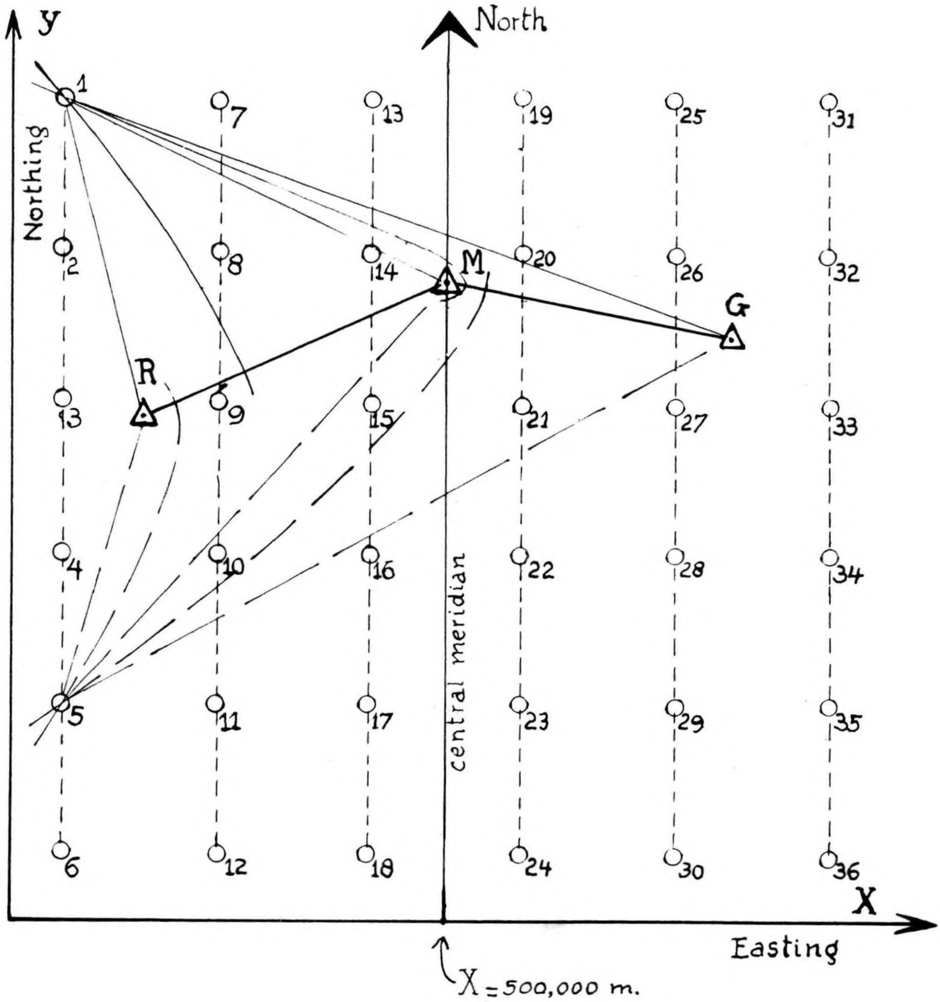


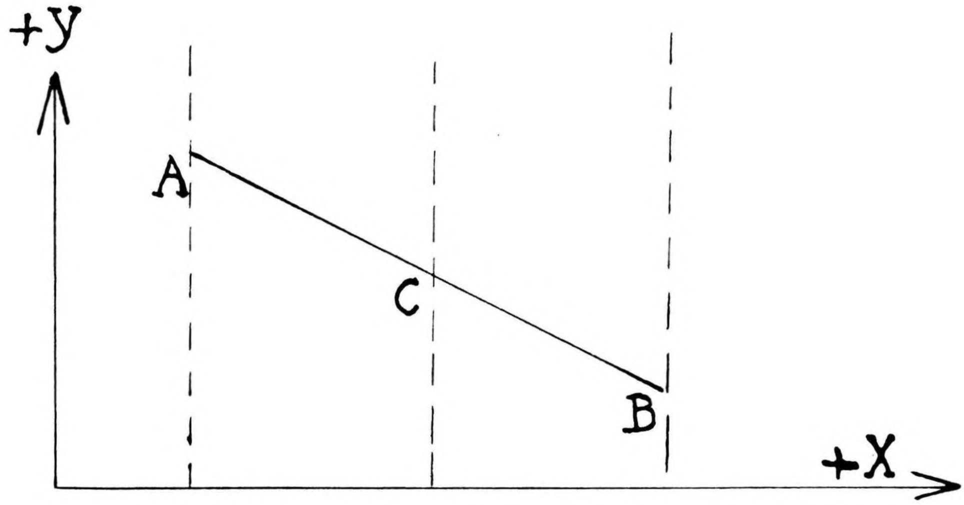
FIG. 4

6.7.1. — Chart distances along baselines are computed from :

$$\left.\begin{aligned} b_R &= \sqrt{(X_M - X_R)^2 + (Y_M - Y_R)^2} \\ b_G &= \sqrt{(X_G - X_M)^2 + (Y_G - Y_M)^2} \end{aligned}\right\} \qquad (17)$$

Chart distances between the 36 points and the transmitters are computed from :

$$\left.\begin{array}{l} 1R = l_{R1} = \sqrt{(X_R - X_1)^2 + (Y_R - Y_1)^2} \\ 1M = l_{M1} = \sqrt{(X_M - X_1)^2 + (Y_M - Y_1)^2} \\ 1G = l_{G_1} = \sqrt{(X_G - X_1)^2 + (Y_G - Y_1)^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 36G = l_{G_{36}} = \sqrt{(X_G - X_{36})^2 + (Y_G - Y_{36})^2} \end{array}\right\} \text{(see also sub-section 5.3.)} \qquad (18)$$



$$X_C = \frac{X_A + X_B}{2}$$

FIG. 4a

6.7.2. — *Plane hyperbola* numbers are computed from :

$$\left.\begin{array}{l} L_{R_1} = \dfrac{b_R + l_{M_1} - l_{R_1}}{\Lambda_R} \\ L_{G_1} = \dfrac{b_G + l_{M_1} - l_{G_1}}{\Lambda_G} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ L_{G_{36}} = \dfrac{b_G + l_{M_{36}} - l_{G_{36}}}{\Lambda_G} \end{array}\right\} \text{(see also sub-section 5.4.)} \qquad (19)$$

6.7.3. — Scale factors k are taken from the A. M. S. tables for :

a) the Y-axes through R, M and G.

b) the Y-axes 1-6, ... 31-36 (fig. 4).

c) the Y-axes midway between R and M, and M and G.

d) the Y-axes midway between 1 and R, 1 and M, 1 and G; 7 and R, ... ; 31 and R, 31 and M, 31 and G.

6.7.4. — Distance scale factors $1/f$ for *all* distances are computed by means of formula (16).

NOTE. — The advantage of the T. M. projection is that the distance scale factors are dependent on X only; example : factors for all lines 1M, 2M .... are equal.

6.7.5. — Spheroidal distances are computed from :

$$\left.\begin{aligned} \hat{b}_R &= b_R \cdot \frac{1}{f_{MR}} \\ \hat{b}_G &= b_G \cdot \frac{1}{f_{MG}} \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} \hat{l}_M &= l_M \cdot \frac{1}{f_{l_M}} \\ \hat{l}_R &= l_R \cdot \frac{1}{f_{l_R}} \\ \hat{l}_G &= l_G \cdot \frac{1}{f_{l_G}} \end{aligned}\right\} \quad \text{(for all 36 points)} \tag{21}$$

6.7.6. — Spheroidal hyperbola numbers follow from :

$$\left.\begin{aligned} \hat{L}_R &= \frac{\hat{b}_R + \hat{l}_M - \hat{l}_R}{\Lambda_R} \\ \hat{L}_G &= \frac{\hat{b}_G + \hat{l}_M - \hat{l}_G}{\Lambda_G} \end{aligned}\right\} \quad \text{(for all 36 points)} \tag{22}$$

6.7.7. — The difference $(\hat{L} - L)$ is the correction to be applied to the Decca decometer readings before plotting them in chart patterns of *plane* hyperbolae.

For a position thus plotted, the coordinates X, Y (and/or φ, λ) scaled-off from the chart, are the same as those that would be scaled-off from a plot of uncorrected readings on a chart, latticed with spheroidal hyperbolae.

6.8. — These corrections change only slowly from place to place and 36 " check " points are usually enough to enable the drawing of correction *curves* on a correction chartlet, of which figure 5 is an example.

As already remarked at the end of sub-section 4.4., these corrections can be combined with those for different propagation speeds over land and over sea, resulting in one single correction chartlet.

6.9. — This method of computation of corrections may look fairly complicated and lengthy. Thanks to the favourable properties of the T. M. projection however, it is in fact very simple and fast on an electric desk calculator as well as on an electronic computer.

6.10. — The method of splitting-up propagation paths and of computation of speed corrections has been described in section 14 of reference

(8). It is also a simple and fast one suitable for hand, electric or electronic computation.

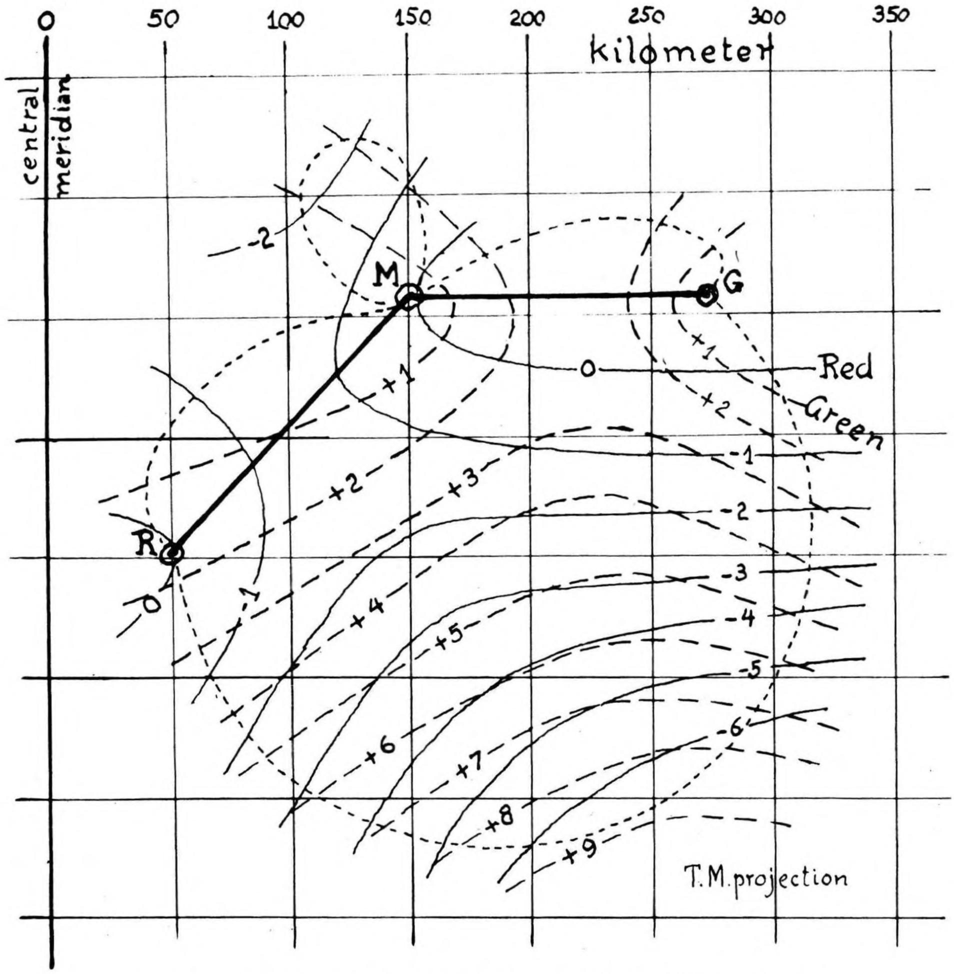


FIG. 5. — Differences between plane and spheroidal hyperbolae.
Corrections in 0.01 of Decca lanes to be applied to decometer readings.
The short-dash curves limit the areas of sufficient angle of cut of hyperbolae.

7. — Application to Decca acceptance trials of ships

7.1. — The operational procedure has been described in reference (1).

7.2. — All computations are carried out electronically on the X-1 computer of the Netherlands Ship Model Testing Basin at Wageningen. The mathematical model of plane hyperbolae, as described in this paper, is used throughout. For the programming, reference should be made to (6).

7.3. — Because of limitations as to depth of water (in W. Europe), the number of suitable areas, although large enough, is limited. For these areas, the corrections (L — L) (sub-section 6.7.7.) have been precomputed.

As soon as the films of the photographic registrations are developed, these corrections are applied to the decometer readings (spheroidal lane-numbers in the *radiated* patterns) as taken from the 35 mm films and converted in the consecutive numbering system (see note 2 of sub-section 5.1.). The readings thus corrected are sent to the computation centre, together with the name of the ship, of the speed trial area and of the Decca chain used.

7.4. — For all areas in use, the computation centre has the following information :

a) the X, Y coordinates of the Decca transmitters (T. M. projection with the central meridian chosen through the Master);

b) the propagation speed to be used;

c) the quadrant of the angle α (fig. 2).

7.5. — The first step in the electronic computation is to convert all decometer readings — run by run of course — into X, Y; computational accuracy is 0.1 metre (see also sub-sections 8.4., 8.5. and 8.6.).

In the case of *turning circle* or *stopway* runs, a stop is built-into the programme and the computation is finished with this conversion.

The coordinates X, Y are used for a graphical plot on a scale of 1/5 000 or 1/2 500. Examples are given in reference (1).

In the case of *speed* runs, 17 redundant observations are available for each run. In this case the electronic computation automatically continues with the least square adjustment, the computation of ground course and standard errors, and the computation of the final speed V from the means of means and its standard error, as has been described in reference (2).

7.6. — The computation of a speed run of 19 fixes, including the conversion of the 19 sets of hyperbolic coordinates into X, Y, the least square adjustment complete with standard errors, and the final means of means, takes only 3 minutes.

Because the method is so fast and because an arrangement could be made that avoids computer waiting time, the computational results of an acceptance trial of (sometimes) as many as 50 runs (*) become available with a delay of not more than one day. It usually takes two days to make the plots of turning circles and stopways, to prepare and type out the final report to the shipowner, shipyard, etc.

7.7. — The very big advantages of electronic computation of acceptance trials are that the method is so fast, that no human computational errors are made and that — once formulae have been developed and the programme has been made up — no specialists (apart from the already available computer operator) are needed for the computation of the results. A computation and adjustment of comparable accuracy would — on an electric desk calculator — take weeks for an experienced mathematician.

(*) In 50 runs there would be something of the order of $50 \times 20 = 1\,000$ fixes from 2 000 hyperbolic L.O.P.s. to be converted into X, Y, and (for speed runs) to be adjusted.

8. — Computational accuracy requirements

8.1. — A general requirement of all radio navigation and survey position fixing systems is that the basic system accuracy should be sufficient for the envisaged application.

8.2. — Because of the general practice of graphical conversion of hyperbolic or circular coordinates (instrument readings) into chart coordinates by means of latticed charts in combination with correction chartlets or lists of " fixed " corrections, it is in the first place a requirement that the basic system accuracy (in the areas of maximum basic system accuracy) should not fall below the accuracy of (careful) plotting of a L. O. P. at the largest chart scale that is to be used.

This is in fact a matter of proper choice of system, or — for a given system — proper choice of chart scale(s).

8.3. — Maximum basic system accuracy and plotting being given, it is evidently a requirement that the method of establishing and charting the lattices should be at least of comparable accuracy. Otherwise, the radio system would not be used to its full capabilities.

8.4. — The basic maximum accuracy (standard error in one L. O. P.) of *survey* systems may be estimated as follows :

a) $\sim$ 4 metres in $\sim$ 100 Kc/s phase comparison systems of the Decca type.

b) $\sim$ 0.7 metre in $\sim$ 2 Mc/s phase comparison systems like Raydist, Loran, Hi-Fix, Minifix, Rana, Toran, etc.

c) Not better than that in other systems (except the tellurometer and geodimeter which however are not *position* fixing systems).

These figures are maximum accuracies (standard error) under favourable conditions and those of the order of 1 metre often cannot be realized in actual practice because of propagation anomalies, effects of reflections (also in 2 Mc/s systems), etc.

8.5. — The plotting accuracy of charting and drawing any lattices and of plotting in the latticed charts for *survey* work, is generally adopted as 0.2 millimetre = 0.0002 metre ($\sim$ 0.01 inch) at the scale of the chart.

8.6. — In order to avoid unnecessary loss of accuracy, chart lattices of *survey* systems should therefore be plotted with an accuracy of 0.2 millimetre and be computed with an accuracy corresponding to the maximum basic accuracy.

For the most accurate systems and under the most favourable circumstances, the latter is of the order of 0.5 metre.

8.7. — For reasons of uniformity in pattern computations, it is desirable to develop and use one single universal computational system for all radio position fixing systems (provided, of course, that such a uniform system does not introduce unnecessary mathematical complications in cases where ultimate accuracy is not a requirement).

In these days of electronic computers, a universal system is of particular importance, because one single programme can then be used for all (in this case, hyperbolic) lattice computations. Apart from being so fast, one of the advantages of an electronic computer is that excessive (perhaps for some purposes) computational accuracy has a negligible effect on computer time.

The method developed in this paper appears to meet these requirements for electronic computation as well as for computations on an electric desk calculator, where an increase in decimal places also has few consequences. (Desk calculators may be the only available means for a number of survey projects). The *computational* accuracy is in both cases 0.1 metre (*) thereby avoiding possible accumulation of rounding-off errors to a total amount in excess of the maximum basic system accuracy (**).

8.8. — *Navigational* systems also require the computation and charting of lattices. Navigation systems are however not intended for, and are sometimes incapable of, achieving survey accuracy. Their chart patterns are plotted on much smaller scales and — for navigational use — the chart scale is often the limiting factor, determining the required computational accuracy.

Some of these navigation systems are however — under favourable and carefully controlled circumstances — capable of much higher basic system accuracy than that for which they were originally intended and there are many cases in which they are used for precision position fixing of fairly high accuracy; the application of a navigation system to ships' acceptance trials is one example, and there are many others.

It is especially with these (frequent) precision applications in mind that the chart patterns of (some) navigation systems could just as well be computed according to the same system as that for survey patterns. It has already been explained that no extra time or other trouble is involved.

8.9. — Some other methods of determining chart patterns will be discussed in section 9.

9. — Other systems

9.1. — The opinion is sometimes expressed that there is a general tendency towards making many problems look more complicated than they really are, or, in other words, that much ado is made about little.

With respect to the matter dealt in this paper, it may appear that much simpler methods would probably do just as well.

The answer to the question whether or not other and/or simpler methods are acceptable evidently depends on the accuracy of the methods in relation to the basic system accuracy and ultimate accuracy requirements of the user.

(*) 8 decimal places have to be used in goniometric functions.
(**) In Decca computations usually rounded-off to the nearest metre.

9.2. — For a number of other methods of computation of *spherical* or *spheroidal* hyperbolae, reference should be made to (5), (11), (12), (13), (14), (15), (16), (17), (18), (19) and (20). It has already been remarked that Professeur BALLARIN's method (5) is, mathematically, the most elegant one; in addition, it has the advantage that it leaves the user free as to the choice of the chart projection and consequently is universal (an advantage which only some of the other methods also have).

However, because of the fact that the formulae are developed for spherical or spheroidal hyperbolae, all these systems are more complicated than one making use of plane hyperbolae.

9.3. — The method developed by SADLER (21) makes use of precomputed intersections with X and/or Y axes of a standard pattern of 400 *plane* hyperbolae on a standard baseline of a length of 40 000 arbitrary units. The X-axis in this coordinate system coincides with the baseline and the Y-axis with its bisector.

In actual practice, the length of the baseline as well as the number of radiated (zero phase) hyperbolae is known. As this radiated pattern is conformal with the standard 400 hyperbola pattern, the intersections of the hyperbolae actually radiated with the X and/or Y-axes can be obtained by simple interpolation in " Hyperbolic Grid Tables " that have been computed by the British Hydrographic Office. Dependent on length of baseline, the interpolated intersections can be obtained with an accuracy of 1 metre or better.

The ultimate requirement for latticing *nautical* charts is to plot *spheroidal* hyperbolae on a chart in Mercator projection. The X, Y coordinates obtained from the interpolation in the table therefore have to be corrected for the difference between plane and spheroidal hyperbolae. For the method of deriving corrections Δ X and Δ Y, developed by Mr. SADLER for the Mercator projection, reference should be made to (21).

An inconvenience to this method is that the X, Y coordinate systems have, for each baseline, a different orientation and do not coincide with the normal chart grid. Separate auxiliary grids have therefore first to be plotted on the nautical chart to be latticed. Another solution — for one application applied in the Netherlands — is mathematical transformation of the separate X, Y systems to the normal rectangular chart grid.

9.4. — Another method of computation of plane hyperbolae was described in reference (8). It required a mathematical transformation of the separate X, Y systems for each baseline to the main chart grid.

The difference between the plane and spheroidal hyperbolae was incorporated in a specially developed system of parallels and meridians;correction chartlets for this difference were consequently not required.

9.5. — In 1954, Mr. KUIPERS of Royal Shell made a design for an automatic plotter of plane hyperbolae in the X, Y system of the T. M. projection. With this instrument it was intended to engrave plane hyperbolae very accurately (a few microns) on a glass plate. The plotting scale had of course to be very small, but the accuracy of engraving was felt to be high enough to permit photographic enlargement to the desired chart scale.

Mainly because of extremely high mechanical requirements (a few microns) in the moving parts of the instrument, the idea has never been realized in practice.

9.6. — In reference (22), the use of standard sheets of 400 plane hyperbolae at a scale of 1/400 000 drawn on dimensionally stable material, was suggested. These standard sheets were plotted by means of the Hyperbolic Grid Tables of the British Hydrographic Office and the idea is therefore an application of Mr. SADLER's method. These standard sheets — of which 16 would cover all practical needs anywhere in the world — could then be photographically enlarged to the desired chart scale and printed in the correct azimuthal orientation of the baseline(s).

The number of hyperbolae actually radiated will, of course, always differ from the chart number of 400 and it was therefore suggested to multiply the *receiver indications* of each pattern by a factor compensating for this difference; such a factor would be a constant for each pattern in any particular chain. A separate correction chartlet was needed to compensate for the difference between plane and spheroidal hyperbolae.

The method has never been used in practice.

9.7. — A pure constructional method has been described in (24). In this method plane hyperbolae are drawn through the intersection of circles constructed with the transmitters as centres; this construction is made at the scale of the final chart to be latticed. As the radius of these circles may become as large as 7 metres and as their angle of intersection becomes very small near the baseline extensions, the accuracy with which the hyperbolae can be drawn is limited and even quite poor near the baseline extensions. In the larger part of the coverage area, the accuracy falls considerably below the basic accuracy of a survey system.

As far as known, the use of this method has been discontinued.

9.8. — Another constructional method, based on the same principle of drawing plane hyperbolae through the intersection of concentric circles, has been developed by Mr. WERNINK of the Netherlands Hydrographic Office. In this case the circles are engraved on dimensionally stable material at a scale of about 1/250 000 by means of a slightly modified coordinatograph (*).

By keeping a constant room temperature and humidity, it is possible to engrave these circles with an accuracy of 0.05 millimetre and to determine their intersections with an accuracy of 0.07 mm.

The intersections are pricked through and this sheet is photographically enlarged to the desired chart scale. Afterwards the hyperbolae are drawn with an accuracy of 0.2 mm by a draftsman.

This constructional method is much more accurate than the Swedish

(*) The actual scale is dependent on the length of the baselines and the dimensions of the coordinatograph.

one described in (24). Its weak point is again near the baseline intersections, where the angle of cut of the circles becomes very small.

In the Netherlands, this method is used for the construction of *provisional* hyperbolic charts of *survey* chains. They are based on provisional — usually estimated — propagation speeds and are usually prepared even before the chain is operational.

Such provisional charts are of course not very accurate, but their advantage is that the survey vessel can start sounding work as soon as the chain is operational. Sounding tracks are then provisionally plotted and only the definite plotting — and not the survey work itself — has to wait until the chain constants have been computed from lanecounts, opposite readings, etc. (see reference (8)); this usually takes 2 weeks and thereafter the definite lattices are (electronically) computed according to the method developed in this paper.

10. — Closing remarks

10.1. — One may differ in opinion as to the question whether hyperbolic chart lattices should be based on spheroidal or plane hyperbolae.

For long range systems, the chart lattices should be spheroidal in any case, because the difference with plane hyperbolae would become too large.

For medium range systems, such as Decca, it is general practice to lattice nautical charts with spheroidal hyperbolae. In my personal opinion, plane hyperbolae would do just as well and in this case the difference with spheroidal hyperbolae could, together with the so-called " fixed " corrections, be given in the Decca Data Sheets.

In the Netherlands practice of *survey* chains, we gradually came to the conclusion that the simplest way is to lattice the survey charts with plane hyperbolae and to use " combined " correction chartlets.

10.2. — Another point about which opinions differ is whether or not " fixed " corrections of *nautical* charts should be incorporated in the chart lattices. In my opinion, they should not be included, because these corrections may not only change with seasons, but also because they usually only become known in the course of time and are not sufficiently known at the time when the latticed chart has to be produced. Another reason is that the synchronization of Master and Slaves is sometimes — for one reason or another — changed. It is impossible to change the latticed charts, whereas Data Sheets or correction chartlets can be easily corrected.

10.3. — Having decided in favour of plane hyperbolae, the advantage of the method given in this paper appears to be that it is universal for all hyperbolic systems and that mathematical complications have been reduced to a minimum.

Even for computation by means of an electric desk calculator — as may for instance be necessary on board a survey vessel — the method appears to be much simpler and faster than any other computational method of comparable accuracy.

10.4. — The formula system of plane hyperbolae, developed in section 5, is independent of the chart projection. The Transverse Mercator projection is introduced only because it facilitates the computation of the difference between plane and spheroidal hyperbolae. Any other chart projection could be used just as well; the only consequence would be a slightly more complicated computation (for a limited number of " check " points only) of the difference between plane and spheroidal hyperbolae.

10.5. — For *operational* accuracy figures of the Decca system itself (under unfavourable night conditions) and for an example of a least square adjustment of a Decca speed trial run, reference should be made to (2).

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