CONNECTION OF ELEVATIONS
OF THE MIDDLE DALMATIAN ISLANDS
WITH ELEVATIONS OF THE CONTINENT

by Professor Dr. Ing. Nikola Čubranić

The Institute of Higher Geodesy of the Faculty of Geodesy of the University of Zagreb, in collaboration with the Hydrographic Institute at Split, undertook to include the Middle Dalmatian Islands of Drvenik, Šolta, Brač and Hvar in a homogeneous system of elevations.

For the determination of the elevations on the islands, that is to say, for the determination of the zero sea level, portable tide gauges were used. The determination of the zero sea level by means of such tide gauges depends, however, on the duration of observations, as the mean sea level obtained in observations over a period of a year may not be sufficiently accurate. The mean yearly levels of the Adriatic Sea differ, according to data supplied by Yugoslav tide gauges, by over 10 cm. Similar differences were shown by the tide gauge in Trieste (for the period 1949-1951) when the difference amounted to 13 cm [1].

Besides, if such a mean sea level could be determined as precisely as possible, even by continuous observations, it would not comply with the generally accepted mean, on the basis of which elevations above sea level on the continent were computed. The elevations above sea level in Yugoslavia are based on the mean obtained by observations over a period of one year with the tide gauge in Trieste in the year 1875. From the results of observations carried out in later years, the Austrian astronomer STERNECK found in 1905 that the mean level of the Adriatic was 8.99 cm higher than the accepted level of 1875. The results of observations obtained at the tide gauge at Bakar show the same. In order to arrive at a new and better mean sea level, a series of fixed tide gauges were installed along the Adriatic coast where observations have been carried out for years.

From the above, it appears clear that by using portable tide gauges the islands cannot be connected with sufficient accuracy to the continent with regard to their elevations. As geodetic levelling as a measuring method for connecting islands cannot be taken into consideration, only hydrostatic and trigonometric levelling can be used. Hydrostatic levelling is in all probability more reliable and accurate, but being unusually expensive it is difficult to carry out in practice. The method applied hitherto in carrying out hydrostatic levelling consisted of laying a lead tube along the sea bottom from coast to coast, the tube being filled with water so that

the level of the water in the tube will be at the same height at both ends. At first glance, this appears very simple, reliable and accurate, but it demands higher expenditure and a larger organization. Also, its accuracy is not as absolute as appears at first glance, because the accuracy of the levels at both ends of the tube depends on the friction in the tube and even more on the formation of air bubbles, their number and size. Since today there are available various types of synthetic materials, hydrostatic levelling may be carried out by the use of plastic transparent tubes of a greater diameter than the lead tubes. Using such plastic tubes, friction would be reduced, the filling of the tube would be supervised, and the air bubbles eventually left behind would not affect so much the level of the water at the ends of the tube in case the tube did not sink, but remained floating on the surface of the sea. This method could be carried out in places where sea traffic is not dense (author's idea).

Because the funds available were insufficient and there was a lack of the equipment needed for hydrostatic levelling, we decided to apply trigonometric levelling, considering on the basis of our investigations that this method might also provide the necessary accuracy or at least the accuracy prescribed for precision levelling of the second order if, during the observations, precautions were taken to eliminate those factors which, in regular practice, greatly reduce the accuracy of the method; this is particularly important because of refraction.

Differences in elevation in trigonometric levelling are usually cumputed by the formula :

$$H_b - H_a = \Delta H_a = S \cot Z_a + i_a - l_b + \left(\frac{1 - k_a}{2R} \right) S^2 + \lambda \qquad (1)$$

where S represents the distance between the points A and B on the ellipsoid, Z_a the zenith angle at point A, i_a the height of the instrument at point A; l_b the height of the signal at point B, while the next term represents the influence of the curvature of the earth and of refraction; λ is the correction factor which depends on the difference in the elevation and the mean of the altitude above sea level of the line of sight :

$$\lambda = \Delta H \frac{H_m}{2R}$$

and is obtained from these elements by using pre-prepared tables.

In a similar way, we may obtain the difference in elevation in a reversed sense, i.e. $H_a - H_b = \Delta H_b$, if we insert the corresponding elements into formula (1).

The accuracy in determining ΔH depends on the accuracy of the measured elements of formula (1), respectively :

$$m_h^2 = \left(\frac{m_2''}{\rho''} \right)^2 S^2 + m_k^2 \left(\frac{S^2}{R} \right)^2 + m_i^2 + m_l^2 + (\cot Z)^2 m_s^2 \qquad (2)$$

By careful work, the heights of the instrument i and of the signals i and l may be measured to an accuracy of one millimetre, and we can neglect the terms m_i^2 and m_l^2 of formula (2). We may obtain the values from the coordinates of the trigonometric points by reduction to the ellipsoid.

Presuming that $m_s = \pm 10$ cm, we can reduce the influence of this error in the difference in elevation *ad libitum* by selecting the convenient points so that the points A and B are on approximately the same elevation above sea level. If the altitude angle is not wider than 25″ (in measurements discussed here, this was the widest altitude angle), the value of the last term in formula (2) amounts to 0.7 mm, i.e., a value which can also be neglected.

According to this, the accuracy in the determination of differences in elevations will be influenced by the accuracy of the measured angle Z and the accuracy of the coefficient of refraction, i.e., m_z and m_k.

For the measurement of the zenith angles we used two Wild T/3 theodolites, posted simultaneously at opposite points. The task was to measure with each theodolite, reciprocal observations being taken, direct and reverse. The measurements were made every half-hour, beginning exactly on the hour and the half-hour. Each reading was made in both the positions of the telescope with the spirit level on the vertical circle centred and with repeated coincidence. Knowing the accuracy of these theodolites, the author considers he is justified in assuming that the mean error in measuring the altitude angle (oscillations caused by refraction were not taken into account) measured in both positions of the telescope cannot exceed one second. Every half-hour, during the entire day, approximately 31 measurements were carried out from each of the two stations. If we take the lowest number of accomplished measurements as being 25, then the mean error in the zenith angle will be reduced to $\dfrac{1''}{\sqrt{25}} = 0.2''$. According to this, the influence of the first term in formula (2) could amount in the measurements described here, with the longest distance S = 7 km, to a maximum of 7 millimetres. The arithmetic mean in measurements carried out from both the stations will reduce these values to 0.14″ or to 5 millimetres.

The coefficient of refraction, particularly on unfavourable terrain where the line of sight is close to the ground for quite a long distance, may be very variable [2]. Therefore, the influence of refraction reduces considerably the accuracy of this method for the determination of elevations. Geodetic practice, as well as earlier works of the author, have shown that the angle of refraction, even in unfavourable conditions, if measurements were carried out simultaneously from two stations, was almost the same at the same moment at each station, i.e., that the coefficients k_a and k_b are at the same moment practically equal. Applying this supposition, the influence of the penultimate term in formula (1) will be cancelled out, regardless of which value we have taken for k (usually 0.13) in the arithmetical mean formed as :

$$\Delta H_{a-b} = \frac{\Delta H_a + (-\Delta H_b)}{2}$$

This is also shown by the formula by which we compute directly the difference in altitude on the basis of both zenith angles :

$$H_{a-b} = S \tan \frac{Z_a - Z_b}{2} + \frac{i_a - i_b}{2} + \frac{l_a - l_b}{2} + \lambda \ldots \ldots \quad (3)$$

i. e., even if Z_a and Z_b are influenced by refraction, the value $\dfrac{Z_a - Z_b}{2}$ would be free from refraction, for we suppose that $\dfrac{r_a - r_b}{2} = 0$, where r_a and r_b represent the angles of refraction.

The supposition that at the actual moment of observation $r_a = r_b$ or $k_a = k_b$ would be more correct if the slope of the ground at both stations were almost equal (because here the line of sight runs nearest to the ground) and of equal composition and temperature (exposure to the sun). This cannot be achieved when selecting the sites and therefore the coefficients k_a and k_b will not be entirely equal and in the term $\dfrac{Z_a - Z_b}{2}$ the influence of refraction will not be eliminated completely. This difference in refraction will influence directly the accuracy in the determination of the difference in elevation carried out by the trigonometric method.

We tried first to reduce the influence of the remaining refraction through the selection of stations, i.e., such stations were selected which ensured the shortest possible line of sight, elevated as high as possible above sea level. Further, we tried to select stations so that that the ground at both stations along the line of sight should be as steep as possible, and so that both stations should be at an approximately equal elevation above sea level. Naturally, it is not always possible to meet all these conditions. We also tried to reduce the influence of the remaining refraction on the result by observations which were carried out every half-hour throughout the 24 hours.

Connections between the islands of Drvenik, Šolta, Brač and Hvar and their connection to the continent are shown on the attached map. Besides the selected primary stations necessary for measurements across the sea, bench marks were established 5 to 10 metres from the stations. These bench marks were connected on the islands with regard to elevation by precision levelling and the lines of sight running across the sea by trigonometric levelling. The theodolites were usually placed on the triangulation points. The altitude i of the instrument was obtained from the adjacent bench mark (for verification purposes the height was measured with a tape). For sighting we used partly a level rod equipped with a board, divided horizontally into two parts, a black and a white field, and partly, particularly for longer distances, a heliotrope and at night a searchlight. The heliotrope and the searchlight were mounted alternately on the same tripod. The level rod, as well as the heliotrope and the searchlight, were usually placed above the bench mark. The altitude at the level rod was read directly and that at the heliotrope and the searchlight was found by levelling, and for verification purposes by measuring with a tape from the bench mark. The changes in the length S caused by these eccentricities were taken into account. Measurements were planned so that at each station 2 to 3 observers were on duty, alternating in turn. The work started at about 16 hours and lasted, with a break of 2 to 5 hours, until noon the next day. During the following 24 hours the team rested or prepared for the work in the next period.

DATA ON PRECISION LEVELLING

Between 1950 and 1959 traverses on the continent were levelled and based on some bench marks of the Austrian levelling. In this study we shall rely only upon levelling data because the elevations of the bench marks include errors of the old Austrian levelling which at that time was not as accurate as it is today. Besides, the elevation of some bench marks could easily have changed considerably during the 50 years that have elapsed.

From the files of the Geodetic Administration in Zagreb we obtained the original data on levelling (arithmetical means of measurements in both directions) carried out between 1950 and 1959 :

From	To	Length in km	Difference in altitude in metres	Order of livelling
3 436	MCCC	45.4	× 86.0773	I
MCCC	MCXLIV	50.0	209.1358	II
MCXLI	MCL	6.6	36.8857	I
MCL	22 452	20.8	× 807.8232	II
22 452	47	33.9	56.8541	II
Σ 3 436	47	156.7	96.7761	

For connecting the islands with regard to elevations, supplementary levelling on the continent and levelling on the islands were carried out in 1960 and 1961 and the following results were obtained :

From	To	Length in km	Difference in altitude in metres
3 436	14 730	11.95	× 85.9082
14 730	59	5.97	105.0337
1/573	7/573	3.88	× 77.5754
1/571	25/571	19.31	42.3006
53/574	70/574	62.69	5.9002
1/572	41/572	34.06	321.6794
41/572	61/572	17.82	× 700.2362
41/572	73/572	12.78	× 617.0494
73/572	75/572	1.09	4.9284
103/569	47/	1.00	33.8104

As regards accuracy, this levelling was carried out as second order levelling and the instrument used was a Zeiss Ni 2 without a plan parallel plate. From the differences in the data obtained from levelling in opposite

directions, the accuracy expressed by the mean error has been obtained as [3] :

$$m = \sqrt{0.98^2\,S + 0.32^2\,S^2}$$

This estimate refers only to the levelling on the island of Brač, but we can take it for all the levelling work carried out in 1960 and 1961. We must stress here that the work was carried out partly on rocky and hardly accessible ground and partly on rocky foot-paths, and that on short distances relatively great differences in altitude were encountered. For these reasons and owing to the fact that the mean error was estimated only on the basis of measurements in both directions and not the basis of closed polygons, the author thinks that the actual mean error will be something greater than that arrived at in the above formula.

We were particularly interested in the transfer of elevations across the sea, i.e. in measurements accomplished by trigonometric levelling, and whether these results can be accepted with respect to accuracy and be merged, together with the described geometric levelling, into a whole, i.e. into a homogeneous system of elevations. Therefore we shall also discuss here the results obtained by trigonometric levelling, their accuracy, as well as the accuracy of such connections.

The above described trigonometric levelling was carried out in the early summer of 1960. These measurements were repeated in the early summer of 1961 in the same way, but with other observers. In the table below we show the results of these measurements, i.e., the arithmetical mean of a corresponding number of measurements made in both directions, the actual number of measurements and their accuracy, i.e., the mean error computed on the basis of discrepancies between single results and the arithmetical mean.

Survey of measurements

Carried out in :

1960					
From - to	Number of measurements in both directions	S in km	Difference in altitude (arithmetic mean in metres)	Mean error in mm	Mean error per km in mm
59 - 1/573	29	2.7	— 2.5836	± 4.2	± 1.6
7/573 - 25/571	22	4.6	+ 29.1369	± 9.0	± 2.0
1/571 - 1/572	31	2.3	+ 7.8619	± 4.1	± 1.7
61/572 - 22 452	24	7.0	— 42.7191	± 13.0	± 1.8
73/572 - 53/574	29	5.0	+ 39.3475	± 10.0	± 2.0
75/572 - 53/574	26	5.0	+ 34.4211	± 11.4	± 2.3
70/574 - 103/569	33	4.8	+ 18.3131	± 8.0	± 1.7
Mean	28				± 1.9

1961					
59 - 1/573	35	2.7	+ 2.6162	± 4.2	± 1.6
7/573 - 25/571	33	4.6	+ 29,1400	± 7.9	± 1.7
1/571 - 1/572	34	2.3	+ 7,8844	± 3.9	± 1.7
61/572 - 22 452	32	7.0	— 42,7305	± 9.5	± 1.4
73/572 - 53/574	35	5.0	+ 39.3009	± 10,5	± 2.1
75/572 - 53/574	31	5.0	+ 34.4554	± 6.6	± 1.5
70/574 - 103/569	35	4.8	+ 18,3190	± 14,2	± 3.0
Mean	34				± 1.9

On both occasions equal accuracy in the determination of differences in elevation was achieved, i.e., 1.9 millimetres per km. Il we use the arithmetical means of the results obtained in 1960 and 1961, the accuracy of these means should be $\pm 1.9/\sqrt{2} = \pm 1.4$ mm per kilometre. But from the differences in the results obtained in 1960 and 1961 which in sequence amount to (in cm) : — 3.26; + 0.31; + 2.25; — 1.14; — 4.66; + 3.43; + 0.59, accuracy will be much worse. If, on the basis of these differences, we compute the mean error per kilometre for each year according to the formula

$$m_o = \sqrt{\frac{[p\,d^2]}{2\,n}}$$

where $p = \dfrac{1}{S^2}$, we shall obtain $m_o = \pm 0.492$ cm/km, and the mean errors in the results (the mean of 1960 and 1961) would read :

$$m_o = \pm 0.35 \text{ cm/km}$$

Apart from the connection between Brač and Hvar, all other connections accomplished across the sea were carried out at altitudes of 60 to 110 metres above sea level. As the nearest portion of the ground suitable for connecting Brač and Hvar is low and gently sloping (see map) we had to carry out for the purpose of greater accuracy a double connection, i.e., connecting point 73 (altitude 17 metres) and point 75 (altitude 22 metres) to point 53 (alittude 56 metres). It appeared that the most unfavourable connection was between point 73 and point 53 because, as mentioned above, between these two points the ground is gently sloping and besides, the line of sight runs for some 80 metres, 3 to 5 metres above trees, commencing from point 73, which may probably result in greater changes of refraction. As the difference between points 73 and 75 was obtained by precision levelling, we can assume it free from errors and we can investigate the accuracy of the results of trigonometric levelling from point 73 to point 53 and from point 75 to point 53.

Measurements

From	to	... 1960	1961	Mean	Adjusted
73	75	4.9284	4.9284	4.9284	4.9284
75	53	34.4211	34.4554	34.4377	34.4168
53	73	× 60.6525	× 60.6991	× 60.6758	× 60.6548
Σ		= 0.0020	= 0.0829	= 0.0419	= 0.0000

We see that the measurements of 1960 agree to 2 millimetres, but in 1961 they show a discrepancy of 8.3 centimetres. We cannot, however, say that the measurements carried out in 1961 were less accurate than those of 1960 and we have therefore taken into account both measurements as shown here.

We are now going to show the closing of the altitude polygons of the second order precision levelling, supplemented by data of the trigonometric levelling of 1960 and 1961, as well as the mean of the measurements of 1960 plus 1961.

Polygon I

From	To	Km	1960	1961	Mean 1960+1961	Method of levelling and designation
22 452	14 730	144.8	45.9862	× 45.9862	× 45.9862	precision lev. h_{10}
14 730	59	6.0	105.0337	105.0337	105.0337	precision lev. h_1
59	1/573	2.7	× 7.4164	× 7.3838	× 7.4001	trigonom. lev. h_2
1/573	7/573	3.9	× 77.5754	× 77.5754	× 77.5754	precision lev. h_3
7/573	25/571	4.6	29.1369	29.1400	29.1385	trigonom. lev. h_4
25/571	1/571	19.3	× 57.6994	× 57.6994	× 57.6994	precision lev. h_5
1/571	1/572	2.3	7.8619	7.8844	7.8732	trigonom. lev. h_6
1/572	41/572	34.1	321.6794	321.6794	321.6794	precision lev. h_7
41/572	61/572	17.8	× 700.2362	× 700.2362	× 700.2362	precision lev. h_8
61/572	22 452	7.0	× 57.2809	× 57.2695	× 57.2752	trigonom. lev. h_9
Σ		242.5	× 999.9064	× 999.8880	× 999.8973	
Discrepancy			$w = -93.6$ mm	$w = -112.0$ mm	$w = -102.7$ mm	

Discrepancy allowed in second order levelling for $L = 242.5$ km amounts to :

$$\Delta = 8 \cdot 1.5 \sqrt{L} = 186 \ \text{mm}$$

Polygon II

From	To	Km	1960	1961	Mean 1960+1961	Method of levelling and designation
2 452	61	7.0	42.7191	42.7305	42.7248	trigonom. lev. h_9
61	41	17.8	299.7638	299.7638	299.7638	precision lev. h_8
41	73	12.8	× 617.0494	× 617.0494	× 617.0494	precision lev. h_{10}
73	53	5.0	39.3475	39.3009	39.3452	trigonom. lev. h_{11}
53	70	62.7	5.9002	5.9002	5.9002	precision lev. h_{12}
70	103	4.8	18.3131	18.3190	18.3160	trigonom. lev. h_{13}
103	47	1.0	33.8104	33.8104	33.8104	precision lev. h_{14}
47	22 452	33.9	× 43.1459	× 43.1459	× 43.1459	precision lev. h_{15}
Σ		145.0	0.0494	0.0201	0.0557	
Discrepancy			$w = + 49.4$ mm	$w = + 20.1$ mm	$w = + 55.7$ mm	

Discrepancy allowed in second order levelling for $L = 145$ km amounts to :

$$\Delta = 8 \cdot 1.5 \sqrt{L} = 144 \ \text{mm}$$

Polygon III includes both the above-mentioned polygons and the corresponding discrepancies will read : $w = -91.9$ mm; $w = -47.0$ mm, and the discrepancy allowed in second order levelling for $L = 337.9$ km amounts to :

$$\Delta = 8 \cdot 1.5 \sqrt{L} = 220 \ \text{mm}$$

From the discrepancies in these polygons, we see that the results obtained by trigonometric levelling, which enable here the connection and thereby the closing of the polygons with respect to elevations, are acceptable because the polygons close far below the allowed discrepancies. Even data showing single differences in elevation obtained by trigonometric levelling will meet, by their accuracy, the demands of precise second order levelling. We examined the previous accuracies of trigonometric levelling on the basis of differences of 1960 and 1961 and arrived at the value of 3.5 mm per kilometre. Let us take as an example the most unfavourable case, i.e., the difference in elevation obtained by trigonometric levelling

for a distance of 7 kilometres. The mean error for this distance would amount to $3.5 \times 7 = 24.5$ millimetres. According to the formula on allowable discrepancies in second order levelling we arrive at :

$$8 \cdot 1.5 \sqrt{7} = 32 \text{ millimetres}$$

From this analysis we can conclude that the accomplished trigonometric levelling meets entirely the accuracy requirements for precise second order levelling.

For the adjustment of these two polygons we shall take the mean of the results of 1960 plus the results of 1961. Corrections in measurements obtained by this adjustment will further improve the accuracy achieved in the accomplished connections, particularly in those made by trigonometric levelling. We also know the altitudes of the bench marks on the continent which are : No. 14730, altitude 3.3771 m; No. 22452, altitude 57.4363 m and No. 47, altitude 114.2794 metres.

If we link polygon I with bench marks Nos. 14730 and 22452 we shall obtain the difference : $w = -148.1$ millimetres and in linking polygon II with bench marks Nos. 22452 and 47, the resulting difference will be : $w = +66.7$ millimetres. Thus we see that the difference in polygon II shows an increase of about 50 %. This condition is explained by the fact that levelling on the continent was not adjusted independently but was connected to several old bench marks dating from the past century, when levelling was not as accurate as it is at present. Additionally, the elevation of these bench marks could be expected to change in the course of the years. In order to carry out a new adjustment of the levelling on the continent, data were collected by new measurements. Since this work had not been completed, the author carried out an adjustment of the existing bench marks to serve for the practical uses of the moment. We do not quote these data here, as previously explained, because they fail to show the accuracy accomplished by trigonometric levelling.

As we cannot rely upon the elevations of the existing bench marks for estimating accuracy, we tried to adjust independently the two closed polygons.

The question arises as to which weights have to be assigned to single differences in altitude. Differences in altitude measured with more accuracy, i.e. of higher weight, have to be subjected to a smaller correction in adjustment. The weights are in inverse proportion to the square of the mean error. If the weights are designated by p and the mean error in the difference in altitude by m, then p is equal to $\dfrac{1}{m^2}$ or $\dfrac{1}{p} = m^2$, respectively. Corrections in measurements will depend upon the selection of weights and will be more accurate if the assigned weight ratio is carefully evaluated.

For connections made across the sea by trigonometric levelling we found that the mean error was 3.5 millimetres, i.e. the differences in altitude h_2; h_4; h_6; h_9; h_{11} and h_{13} were computed according to the formula $\dfrac{1}{p} = m^2 = (3.5 . S)^2$, where S represents the length of the trigonometric crossing.

The connecting of the geometric levelling on the islands we shall compute according to the above mentioned formula $\dfrac{1}{p} = m^2 = 0.96^2\, S +$ $+\ 0.32^2\, S^2$ (for differences in altitude h_1; h_3; h_5; h_7; h_8; h_{10}; h_{12} and h_{14}).

In a similar way, we could also estimate (from the difference in measurements in both directions) the accuracy of the precision levelling on the continent. But for this purpose we can rely on experience. The levelling on the continent is certainly more accurate than that carried out on the islands, because it was accomplished on a gravel road and the author thinks that no major mistake would be made if for the selection of the weight ratios for the difference in altitude of points 47 - 22452 - 14730, i.e., for h_{15} and h_{16}, we set the term :

$$\frac{1}{p} = m^2 = 0.80^2\, S + 0.14^2\, S^2 = 0.64\, S + 0.02\, S^2.$$

According to the distances S to which single differences in altitude refer, we obtained (all values for $\dfrac{1}{p}$ were divided by 100) :

$$\frac{1}{p_1} = 0.09\ ;\quad \frac{1}{p_2} = 0.90\ ;\quad \frac{1}{p_3} = 0.06\ ;\quad \frac{1}{p_4} = 2.59\ ;\quad \frac{1}{p_5} = 0.56;\quad \frac{1}{p_6} = 0.66\ ;$$

$$\frac{1}{p_7} = 1.50\ ;\quad \frac{1}{p_8} = 0.50\ ;\quad \frac{1}{p_9} = 6.00\ ;\quad \frac{1}{p_{10}} = 0.29\ ;\quad \frac{1}{p_{11}} = 3.06\ :\quad \frac{1}{p_{12}} = 4.64\ ;$$

$$\frac{1}{p_{13}} = 2.82\ :\quad \frac{1}{p_{14}} = 0.01\ ;\quad \frac{1}{p_{15}} = 0.44\ ;\quad \frac{1}{p_{16}} = 4.46\ .$$

The condition equations for the two closed polygons read:

$$-\, v_{16} + v_1 - v_2 - v_3 + v_4 - v_5 + v_6 + v_7 - v_8 - v_9 - 102.7 = 0\ ;$$
$$v_9 + v_8 - v_{10} + v_{11} + v_{12} + v_{13} + v_{14} - v_{15} + 55.7 = 0\ ;$$

v_i designates the corrections of the corresponding differences in altitude. In solving these equations by the method of the least squares we shall obtain corrections v_i in millimetres on condition that $[pv^2]$ is the minimum:

$$v_{16} = -\, 24.5;\ v_1 = +\, 0.5;\ v_2 = -\, 5.0;\ v_3 = -\, 0.3;\ v_4 = +\, 14.3;$$
$$v_5 = -\, 3.1;\ v_6 = +\, 3.6;\ v_7 = +\, 8.3;\ v_8 = -\, 3.3;\ v_9 = -\, 39.8;\ v_{10} = +\, 0.3;$$
$$v_{11} = -\, 3.4;\ v_{12} = -\, 5.2;\ v_{13} = -\, 3.2;\ v_{14} = 0.0;\ v_{15} = +\, 0.5.$$

The allocations and the extent of these corrections show that the weight ratios of the trigonometric levelling and of the weights of geometric levelling were quite accurately established, but slightly to the detriment of the trigonometric levelling, because the difference in altitude h_9, for instance, determined by trigonometric levelling, experienced a relatively considerable correction of 39.8 millimetres, i.e. 40 % of the discrepancy with polygon I, or over 70 % of the discrepancy with polygon II.

The corrected differences in altitude will read:

$h_{16} = \times\ 46.0107$ (i.e.— 53.9893 metres); $h_1 = 105.0342$; $h_2 = \times\ 7.4051$; $h_3 = \times\ 77.5757$; $h_4 = 29.1528$; $h_5 = \times 57.7025$; $h_6 = 7.8768$; $h_7 = 321.6877$; $h_8 = \times\ 700.2395$; $h_9 = \times\ 57.3150$; $([h] = 0.0000)$;

.

and
$h_9 = 42.6850$; $h_8 = 299.7605$; $h_{10} = \times 617.0491$; $h_{11} = 39.3418$; $h_{12} = 5.8950$;
$h_{13} = 18.3128$; $h_{14} = 33.8104$; $h_{15} = \times 43.1454$ ($[h] = 0.0000$).

If the altitude above sea level of a bench is known, e.g. that of the bench mark 22452, which is 57.4363 metres, we can easily obtain the altitude above sea level of all points in both polygons.

The accuracy achieved in all measurements is also characterized by the values of the correction v_i. Here we are particularly interested in the accuracy of measured connections across the sea, accomplished by trigono-metric levelling, and therefore we shall give here particularly these corrections : $v_2 = - 5.0$ mm / 2.7 km; $v_4 = + 14.3$ mm / 4.6 km; $v_6 = + 3.6$ mm / 2.3 km; $v_9 = - 39.8$ mm / 7 km; $v_{11} = - 3.4$ mm / 5 km; and $v_{13} = - 3.2$ mm / 4.8 km. On the basis of these data the mean value of corrections in trigonometric levelling will amount to 2.3 millimetres per kilometre.

As the conditions on the basis of which the above quoted corrections were obtained are purely mathematical, i.e. on closing the polygons, the accuracy of the corrected differences must apparently be higher.

The above-mentioned corrections of the trigonometric levelling with respect to accuracy must be ascribed mainly to refraction, i.e. in the arithmetical means $\dfrac{Z_a - Z_b}{2}$, computed on the basis of 62 single results on an average, refraction could not be eliminated entirely. Ascribing the v inaccuracies of definite values of $\dfrac{Z_a - Z_b}{2}$, we can compute for each v the corresponding error in seconds. If we differentiate the formula (3) according to the above term we shall obtain :

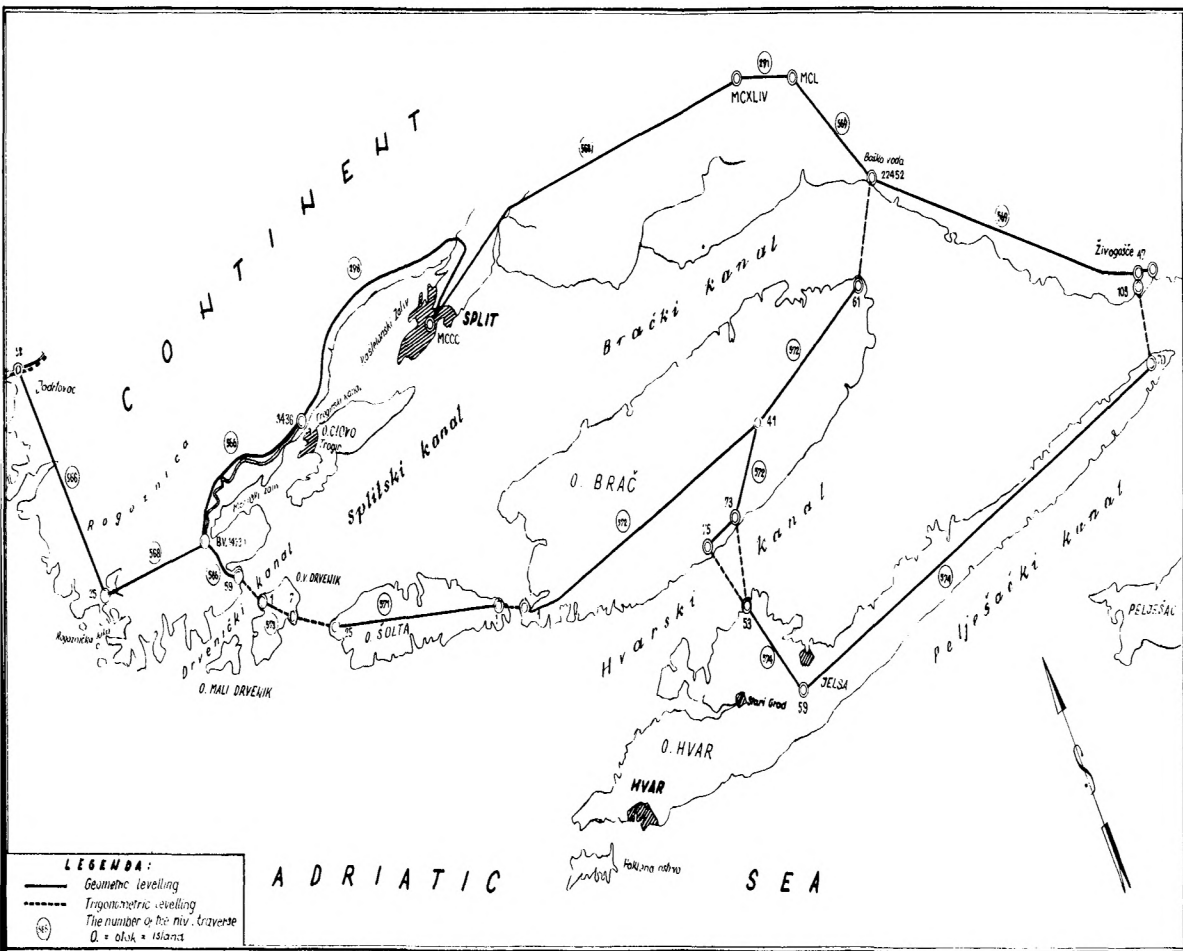$$\frac{S}{\rho''} \delta \left(\frac{Z_a - Z_b}{2} \right)'' = \delta h = v .$$

If we consider $\delta \left(\dfrac{Z_a - Z_b}{2} \right)'$ as an error in the difference of zenith angles and if we call them m, we shall obtain :

$$\delta \left(\frac{Z_a - Z_b}{2} \right)'' = m'' = \frac{v \rho''}{S} .$$

Inserting the corresponding v and S for individual differences in altitude we shall obtain : $m_2 = 0.37''$; $m_4 = 0.62''$; $m_6 = 0.31''$; $m_9 = 1.16''$; $m_{11} = 0.13''$ and $m_{13} = 0.13''$.

Earlier we established the accuracy of the measured values (arithmetical means) $\dfrac{Z_a - Z_b}{2}$ by 0.14'' for a constant refraction. We can therefore say that in the measured differences in altitude h_{11} and h_{13} the influence of refraction is cancelled out.

The highest difference in the mean value of the refraction angles r_a and r_b in the definite values of the zenith angles Z_a and Z_b emerged with the determinaiton of the difference in altitude h_9, which, with respect to

LEGENDA:
——— Geometric levelling
- - - - Trigonometric levelling
(145) The number of the niv. traverse
O. = otok = island

the accuracy in measuring (0.14″), we can ascribe entirely to refraction. Neither was the mean value $\dfrac{r_a - r_b}{2}$ equal to zero in the 56 measurements accomplished, as we assumed in the explanation accompanying formula (3), but it developed the value 1.16″.

In this paper we have shown only the results of the measurements and their further development. We did not show individual measurements because it would make this paper too long. But we still wish to supplement it by the results of the detailed analysis of single measurements.

Single measurements (carried out every half-hour) differ from the arithmetical mean by as much as ± 10″ and on some occasions up to ± 15″. Z_a and Z_b usually change in the same way nearly equally at the same time.

The differences of the single values of $\dfrac{Z_a - Z_b}{2}$ from the arithmetical mean amount to ± 5″. The smallest variations in single measurements of Z appeared at points 103, 61 and 1/572. At these points the ground sloped abruptly in the direction of the line of sight and therefore the line of sight passed far above the ground. Greatest variations in single Z appeared at points 73 and 75 and later at point 24452. At these points the ground sloped gently in the direction of the line of sight, i.e. the distance between the lines of sight and the ground increases only slowly. Besides, points 73 and 75 have a small altitude above sea level : 17 metres and 22 metres, respectively.

The analysis of the data supplies a guide for the selection of points for connections which have to be made by trigonometric levelling.

Analysing further the measured data we tried to find out the time of the day when the variations of the zenith angles Z are smallest. It may be expected that the zenith angles are largest at noon (smallest refraction) and smallest early in the morning (largest refraction) so that at noon the variations in refraction would be smallest and the results of this period of measuring may supply the most accurate results. But from the data of single measurements we could not establish any fast rule in this respect.

REFERENCES

[1] Ferruccio MOSETTI : Sulla tendenza secolare del livello medio marino a Trieste. Publ. n. 119ᵇ¹ˢ dell' Osservatorio Geofisico Sperimentale, Trieste, 1961.

[2] Nikola ČUBRANIĆ : Razmatranje veličine i utjecaja refrakcije kod trigonometrijskog mjerenja visina. — Geodetski List, 1962, br. 7-9, str. 233-244. (A study of the extent and influence of refraction in trigonometric measuring of altitudes. — Geodetic Review, 1962, No. 7-9, pages 233-244.

[3] S. KLAK : Upotreba nivelira Zeiss N/2 bez plan ploče u preciznom nivelmanu. (The use of the Zeiss N/2 levelling instrument without the plan parallel plate in precision levelling).