HARMONIC ANALYSIS
OF THE TIDE BY THE SEMI-GRAPHIC METHOD

by Vice-Admiral A. DOS SANTOS FRANCO

1. — Introduction

In 1959, the British Admiralty introduced a method of harmonic analysis of the tide using the hourly heights registered over 30 consecutive days.

On a sheet of squared paper (fig. 1.1) on which the vertical lines represent the days and the horizontal lines the hours, one plots the hourly heights at intervals of 2 squares (1 cm on the original of the diagram), in decimetres or in feet, in such a manner that the position of the decimal point corresponds exactly to the reading recorded. Then the points of equal reading are joined by contours. In this way one obtains the impression of a continuous topographical surface. The contours may be traced at intervals of 20 cm or 1 foot (or closer if needed), as the case may be.

Fig. 1.1 also includes one interesting detail : we see that the heights are plotted for the hours — 1 and — 2 of the 2nd, 3rd, 4th, etc., days, and for the hours 24 and 25 every day. It is evident that the heights recorded at the hours — 1 and — 2 of a given day are the repetition of the heights relating to the hours 23 and 22 of the previous day. This is to facilitate the interpolation of contours in the upper and lower parts of fig. 1.1 (*).

With the help of a geometrical explanation together with very simple arithmetical calculations, the *Admiralty Tidal Handbook N° 1* (H.D. 505) shows the possibility of making the analysis using particular sections of the surface, the construction of which we have just described, and which we shall call " the tidal-surface ". The principal advantage of the process rests in the minimizing during the tracing of the contours, of the effect of accidental errors which manifest themselves by irregularities in contours, irregularities which may be reduced by eye.

Having confidence in the efficiency of the method we have decided to study it in minute detail, using a more general mathematical procedure, based on the equation of the " tidal-surface ". In addition, we will generalise the method in order to be able to obtain all the constituents furnished by the method of the *Tidal Institute.*

(*) Tide of Aratú harbour (Brázil), 1948.

2. — Study of the "tidal-surface"

The height of the tide above the zero mark of the tide-gauge may be expressed by :

$$y = S_0 + \sum_c fH \cos (V_0 + u + qt + \rho d - g) \qquad (2a)$$

In this expression we shall define only q and ρ, because the other symbols are well known. It is evident that q is the hourly phase variation and ρ its daily variation (t being the standard time counted from 0 to 24 and d the number of days elapsed since the start).

To facilitate the writing of the equation let us make :

$$fH = R \qquad (2b)$$

$$V_0 + u - g = -r \qquad (2c)$$

thus we shall have :

$$y = S_0 + \sum_c R \cos (qt + \rho d - r) \qquad (2d)$$

If we consider t and d as *independent variables*, y becomes a continuous function which represents a surface referred to 3 axes of rectangular co-ordinates Ot, Od, Oy. This is the " tidal-surface " we have defined above. It must be pointed out that this surface is not the surface of the tide, but an artificial surface representing the phenomenon with the help of 3 parameters. Only the values of y obtained on this surface in the vertical planes parallel to Ot and for which we have $d - 1 = 0, 1, 2, 3$ ... are the true heights of the tide. It is evident that the intersection of the surface (2d) with a network of vertical planes parallel to Od and at intervals of one hour, and a network of vertical planes parallel to Ot and at intervals of one day, will give the observation points of fig. 1.1.

The method of harmonic analysis of the tide which we are going to describe is based on particular sections of the " tidal-surface " which are not sections parallel to Od and Ot, which we are generally accustomed to using.

We shall establish the mathematical expression for such sections of the " tidal-surface " and demonstrate that their use is legitimate for making the harmonic analysis of the *tide.*

Simplifying, let us suppose that the tide is made up of a single constituent the height of which above the mean level is expressed by :

$$y' = R \cos (qt + \rho d - r) \qquad (2f)$$

Allowing that this expression is the equation of a surface referred to 3 axes of rectangular co-ordinates Ot, Od, Oy, the graphic representation of this surface is that which we see in fig. 2.1 (for which the chosen values of q and ρ are those of M_2).

Let us suppose that this surface is cut by a plane perpendicular to tOd for which the expression is

$$at + bd = c \qquad (2g)$$

The common points to the plane (2g) and to the surface (2f) will satisfy simultaneously (2f) and (2g). Consequently, their locus will be obtained by substituting in (2f) one of the variables taken from (2g). From

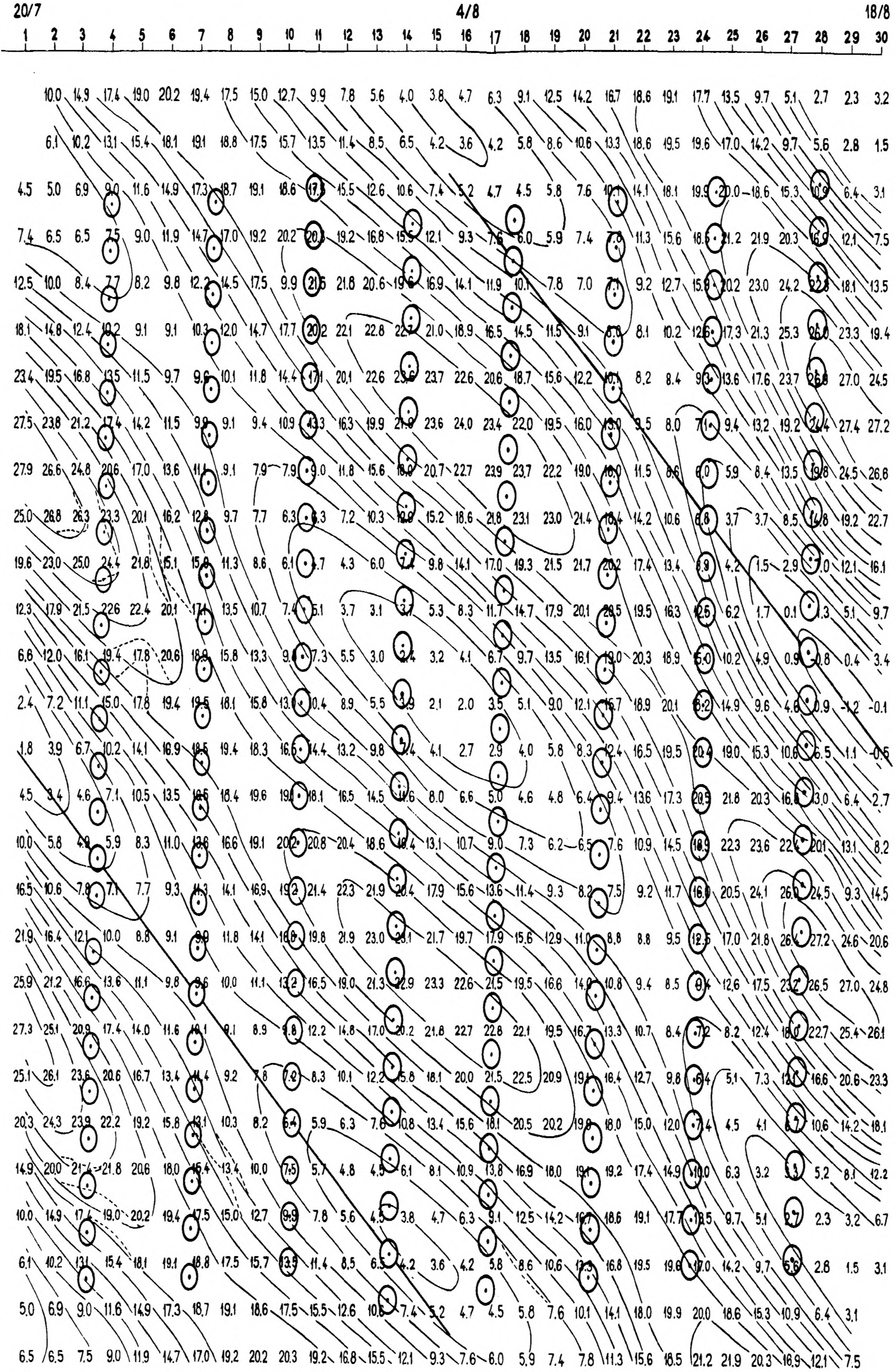
20/7
4/8
18/8
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

Fig. 1.1

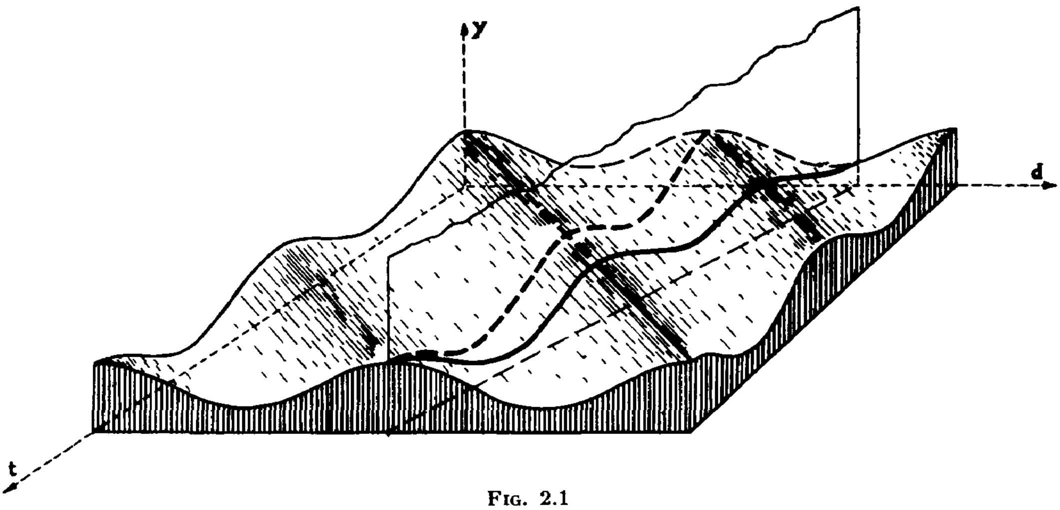

FIG. 2.1

this expression we derive :

$$d = c/b - at/b \tag{2h}$$

or

$$t = c/a - bd/a \tag{2i}$$

which used in (2*f*) will give :

$$y' = \mathrm{R}\ \cos\ [(q - \rho\, a/b)t + \rho\, c/b - r] \tag{2j}$$

or

$$y' = \mathrm{R}\ \cos\ [(\rho - q\ b/a)d + q\ c/a - r] \tag{2k}$$

(2*j*) is the equation of a curve situated in the plane (2*g*) expressed as a function of t, and (2*k*) is the equation of the same curve expressed as a function of d. As soon as we have fixed the values of a, b and c, which define the position of the secant plane, (2*j*) and (2*k*) are the harmonic expressions from which we may determine the values of R and $-r$ with the aid of a certain number of values *y' measured at the intersection of the " tidal-surface " with the secant plane*. These values of R and $-r$ will enable us to obtain, using the formulae (2*b*) and (2*c*), the harmonic constants of the *constituent of the tide*.

It is evident that the same will apply for the total " tidal-surface " which is the representation of the sum of the constituents. It is therefore legitimate to say that one may make the harmonic analysis is of the *tide* with the aid of any sections of the " *tidal-surface* ".

We note that $a/b = -tg\,\alpha$ defines the inclination of the secant plane in relation to yOt. The expressions (2*j*) and (2*k*) show that the hourly phase variations depend on this inclination, which may be observed in fig. 2.1.

Moreover for

$$q - \rho\ a/b = 0$$

or

$$\rho - q\ b/a = 0$$

for which we have

$$b/a = \rho/q \tag{2l}$$

the phase variation is zero. Therefore in all the planes having an inclination defined by (2*l*), the height y' of the constituent will be constant. All the sections thus defined will be straight lines as we can see in fig. 2.1.

To obtain the expressions of the curves determined by the planes (2*g*) on the total " tidal-surface ", it is evident that it suffices to replace in (2*d*) the values of t or d given respectively by (2*k*) and (2*i*). Proceeding in this way we have :

$$y = S_0 + \sum_c R \cos \left[(q - \rho\, a/b)t + \rho\, c/b - r\right] \tag{2m}$$

and

$$y = S_0 + \sum_c R \cos \left[(\rho - q\, b/a)d + q\, c/a - r\right] \tag{2n}$$

If we count t from zero hour and d from the first day of observation, the expressions (2*h*) and (2*j*) show that c/b is the co-ordinate on Od of the intersection of (2*g*) with the plane yOd and that c/a is the co-ordinate on Ot of the intersection of the plane (2*i*) with yOt. Thus we may write :

$$c/b = d'$$

$$c/a = t'$$

The expressions (2*m*) and (2*n*) are then transformed :

$$y = S_0 + \sum_c R \cos \left[(q - \rho\, a/b)t + \rho\, d' - r\right] \tag{2o}$$

and

$$y = S_0 + \sum_c R \cos \left[(\rho - q\, b/a)d + qt' - r\right] \tag{2p}$$

3. — Choice of the Sections

On examining the expression (2*o*) we notice that it is possible to modify the hourly phase variation by working on the basis of the relationship a/b. As we know, the separation of the various groups of constituents of the same species becomes appreciably simplified if the coefficient of t in the expression (2*o*) is a multiple of 15°, as is the case for the solar constituents (S_1, S_2 ...). Now, if we designate by n the number which defines the species of the constituents (0 for the long period series, 1 for the diurnal series, 2 for the semi-diurnal series, etc.), we can write the expression :

$$q - \rho\, a/b = 15°n$$

but as

$$\rho = 24\, q - 360°n$$

we obtain

$$a/b = 1/24 \tag{3a}$$

Thus we see that for the inclination $-\tan^{-1} 1/24$ of the secant plane, the hourly phase variation is completely independent of the values of q and ρ. This very interesting property constitutes the true basis of the semi-graphic method. In fig. 1.1, we see the traces of 8 secant planes inclined by $-\tan^{-1} 1/24$. These traces are defined by the centres of small circumferences which form the " columns " slightly oblique in relation to Ot. The values of y corresponding to the centres of these circumferences are those used for the analysis. The values, numbering 24 per "column ", enable one to separate the species by means of an adequate combination of these values. These combinations will lead to the determination of the linear functions of the unknown values of $R \cos r$ and $R \sin r$ for all the constituents of the species. Since, by reason of the short period of analysis not more than 6 constituents of each species will be considered, the systems formed in dealing with the 8 columns will furnish, as we shall see, a number of equations larger than the number of unknowns which we will deal with by the method of least squares. Thus, this number of columns which may be varied at will, was chosen after having accepted it as reasonable for the reduction of accidental errors, already reduced during the tracing of the contours.

In order to be able to study the distribution of observation points in the columns, it is necessary previously to study the distribution of these columns, which necessitates some preliminary explanations.

To simplify the calculation of the coefficients needed for the analysis, and for the analysis itself, the author of the Admiralty method chose as the central hour of the period to be analysed 0 hour on the 16th day (point N in fig. 3.1). Furthermore, instead of considering the phase variation of the constituents along the axis Od, he preferred to consider it on the constant phase planes of the constituents M_1, M_2, M_3 ... From this second choice there results considerable simplification in the formation of the necessary systems for the separation of constituents of the same species.

Furthermore, we will understand why the choice of these planes, the traces of which are represented in fig. 3.1 by AM and NH, leads to an appreciable simplification in the calculation of the invariable coefficients needed for the analysis.

According to the expression (2l), the inclination of the planes on which the phase of a constituent of speeds q and ρ is constant, is given by $-\cot^{-1}(\rho/q)$. Now, as any constituent M_n has a speed n times greater than M_1, the relationship ρ/q does not change and the expression (2l) for the whole group M will be :

$$b/a = 12°.191/14°.492 = -0.841$$

It is in this way that the straight lines AM and NH in fig. 3.1 are inclined by $\cot^{-1} 0.841$ (49°54′) in relation to Ot. Let us now see the phase variation of the constituents on these lines. For that, it will suffice to replace in (2p) b/a by $-0{,}841$, which gives us :

$$y = S_0 + \sum_c R \cos\ [(\rho + 0.841q)d + qt' - r]$$

If one takes as the origin 0 hour on the 16th day, or at 24 hours on the 15th day, M and N will both be points of origin for which $t' = 0$ and the above expression will be transformed as follows :

$$y = S_0 + \sum_c R \cos\ [(\rho + 0.841q)(d - 16) - r] \tag{3b}$$

If we designate by $\bar{t}$ the hour which corresponds to any point on AM or NH and if we imagine that a vertical plane passes through this point making an angle with. Ot equal to $-\tan^{-1}(1/24)$, the phases of the constituents from this point and in the above plane, will vary by $15°n \times (t - \bar{t})$. Thus we may write for any point situated on a line parallel to the " columns " the expression of y for any instant t :

$$y = S_0 + \sum_c R \cos\ [15°n(t - \bar{t}) + (\rho + 0.841q)(d - 16) - r] \tag{3c}$$

TABLE 3-I

Constituents	$\rho + 0.841\,q$	Days per 360° cycle	degres per $26^j,4$	A	B	C	D	E	F	G	H
				$d - 16 = 1,65 + 3,3k$							
				$k = -4$	$k = -3$	$k = -2$	$k = -1$	$k = 0$	$k = 1$	$k = 2$	$k = 3$
				—11,55	—8,25	—4,95	—1,65	1,65	4,95	8,25	11,55
				$(\rho + 0,841q)\ (1,65 + 3,3k)$							
Q_1	—27.16	13.25	—717°0	313°6	224°0	134°4	44°8	—44°8	—134°4	—224°0	—313°6
O_1 MO_3	—13.64	26.40	—360.0	157.5	112.5	67.5	22.5	—22.5	— 67.5	—112.5	—157.5
M_1 M_3	0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
K_1 MK_3	13.64	26.40	360.0	—157.5	—112.5	— 67.5	—22.5	22.5	67.5	112.5	157.5
J_1	27.16	13.25	717.0	—313.6	—224.0	—134.4	—44.8	44.8	134.4	224.0	313.6
OO_1	40.91	8.80	1080.0	—112.5	—337.5	—202.5	—67.5	67.5	202.5	337.5	112.5
μ_2	—25.24	14.26	—666.3	291.5	208.2	124.9	41.6	—41.6	—124.9	—208.2	—291.5
N_2 MN_4 $2MN_6$	—13.52	26.62	—356.9	156.1	111.5	66.9	22.3	—22.3	— 66.9	—111.5	—156.1
M_2 M_4 M_6 So	0		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
L_2 Mm	13.52	26.62	356.9	—156.1	—111.5	— 66.9	—22.3	22.3	66.9	111.5	156.1
S_2 MS_4 $2MS_6$ MS_f	25.24	24.26	666.3	—291.5	—208.2	—124.9	—41.6	41.6	124.9	208.2	291.5
$2SM_2$ S_4 $2SM_6$	50.47	7.13	1332.4	—222.9	— 56.4	—249.8	—83.3	83.3	249.8	56.4	222.9
SN_4 MSN_6	11.70	30.77	308.9	—135.4	— 96.7	— 58.0	—19.3	19.3	58.0	96.7	135.4

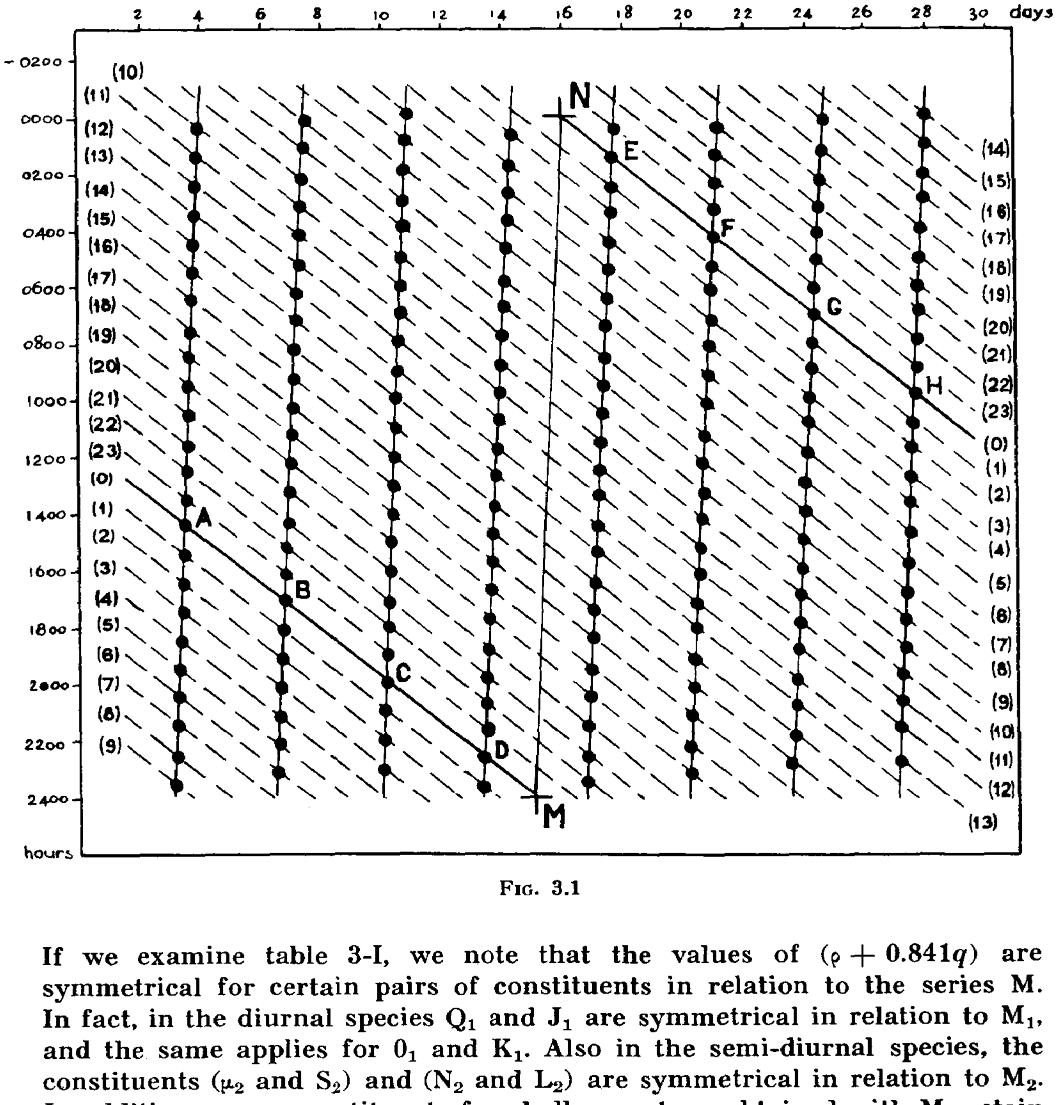

FIG. 3.1

If we examine table 3-I, we note that the values of $(\rho + 0.841q)$ are symmetrical for certain pairs of constituents in relation to the series M. In fact, in the diurnal species Q_1 and J_1 are symmetrical in relation to M_1, and the same applies for O_1 and K_1. Also in the semi-diurnal species, the constituents (μ_2 and S_2) and (N_2 and L_2) are symmetrical in relation to M_2. In addition, many constituents for shallow waters, obtained with M_2 retain the values of certain semi-diurnal constituents. Thus, in the formation of the systems of condition equations for the analysis there are numerous repetitions and, due to this fact, the calculation of the coefficients becomes considerably reduced.

We may now place the " columns " in the " tidal-surface ". For this purpose let us first see how many days are necessary for the analysis. To make the analysis it is advisable that all the constituents should have achieved about one complete cycle of 360° on AM and NH. The number of days required for each constituent to make this cycle will be given by the relationship $360°/(\rho + 0.841q)$.

The constituents O_1 and K_1 make the cycle in exactly 26.40 days, a figure which should be chosen as reasonably meeting the requirements of the analysis. Indeed, as table 3-I shows, the constituents N_2 and L_2 make the cycle of 360° in 26.62 days, SN_4 and MSN_6 in 30.77 days and the others complete much more than 360° in the same time.

However, in the British Admiralty publication the number of days chosen is equal to 23.1. Although this figure does not correspond to any special cycle, the method, nevertheless, gives good results. For purposes of comparison, we have retained 23.1 days in our study.

Under these conditions, if we choose 8 symmetrical sections arranged two by two in relation to NM in order to be able to separate a maximum of 6 constituents of each species, the points of intersection of the " columns " with AM and NH should have as a *difference in co-ordinate d* $23.1/7 = 3.3$ days.

Under these conditions, N being the central point of the observations (0^h,16), the point E (fig. 3.1) will have as co-ordinate $16 + 1.65$, the point $F = 16 + 1.65 + 3.3$, point $G = 16 + 1.65 + 2\ (3.3)$, etc. In short the co-ordinates of the points A, B, C ... H on Od will be given by the expression :

$$d = 16 + 1.65 + 3.3\ k$$

i.e.

$$d - 16 = 1.65 + 3.3\ k \qquad (3d)$$

in which :

$k = 0, 1, 2, 3$ for the right hand columns

and

$k = -1, -2, -3, -4$ for the left hand columns.

As AM and NH are inclined in relation to Od by $tan^{-1}\ 0.841$ the co-ordinates $\bar{t}$ of these points will be :

$$\bar{t} = 0.841\ (1.65 + 3.3\ k) \qquad (3e)$$

Consequently, if from M or N we draw the lines AM and NH with the inclination $tan^{-1}\ 0.841$, the points A, B, C ... H may be plotted on these lines by means of ($3d$) or of ($3e$). In table 3-I we see that the values of $d - 16$ have been calculated for the 8 columns as well as the phase variations from 0^h on the 16th day (or 24 hours on the 15th day) up to points A, B, C ... H of all the constituents isolated in the analysis. It is clear that these phase variations are given by the expression :

$$(\rho + 0.841q)\ (1.65 + 3.3\ k)$$

The numerous repetitions of values which we see in the table show how much the calculation will be simplified.

To place the " columns " in fig. 1.1 it is more convenient to establish the formula which gives the co-ordinates d of the intersection of the " columns " A, B, ... with the axes Od.

Let NP (fig. 3.2) be the constant phase line of M, and QP one of these columns. In the figure $\alpha' = tan^{-1}\ (1/24)$

and

$$NR = 1.65 + 3.3\ k$$

now,

$$RQ = \bar{t}\ \ \tan\alpha' = \frac{\bar{t}}{24}$$

and according to (3*e*) we shall have

$$RQ = 0.841\ (1.65 + 3.3\,k)\ /\ 24$$
$$= 0.035\ (1.65 + 3.3\,k)$$

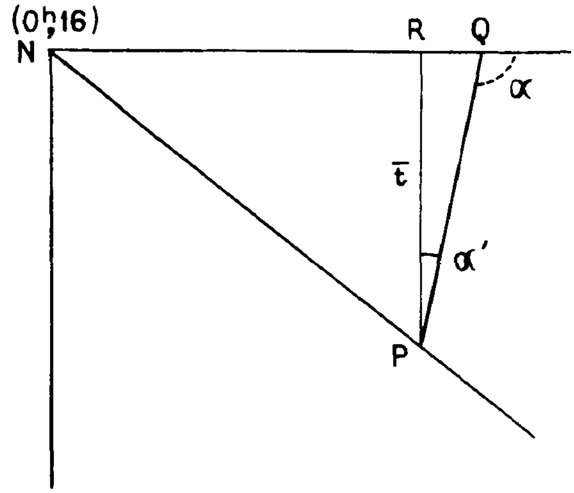

FIG. 3.2

Therefore :

$$NQ = NR + RQ = 1.035\ (1.65 + 3.3\,k)$$

i.e.

$$NQ = 1.71 + 3.41\,k \qquad (3f)$$

Consequently, on marking on the axis O*d*, from 0 hrs. on the 16th day the values of NQ calculated by means of (3*f*), and on the straight line $t = 24$ h parallel to O*d* the same values from the point (24 h. 15) and by joining these points 2 to 2, we shall obtain the 8 " columns " at the desired inclination.

The points A, B, C ... H, correspond to the time origin $\bar{t}$ of each " column ". Although in the original method the author judged it necessary to work with 16 values of y only on each column, the process of elimination will be much more accurate if we work, as we shall now, with 24 points on each " column ".

In fig. 1.1, we see the straight lines starting from points (0 h. 16) and (24 h. 15) which are the locus of the instants $\bar{t}$ from which one must take the ordinates on each column to transcribe them on the form 3-A. It is in this way that column A of the form 3-A is the result of the transcription of column A in fig. 1.1, carried out in the following manner : we take as the first ordinate that one which is in the circle situated at the intersection of the " column " A of fig. 1.1 with the straight line which passes through M (24 h. 15). The following ordinates are those which are situated on the circles of column A which follow one another in the same manner as the hours t increasing up to the last point situated in the lower part of column A. From there one passes to the upper part of the same column and continues the transcription of the values of y descending as far as the last point before the initial hour. One transcribes the other columns in the same manner, in columns B, C ... H of form 3-A.

FORM 3-A

$t - \bar{t}$	A	B	C	D	E	F	G	H
(0)	56	97	65	48	80	105	60	0
(1)	80	106	75	80	125	138	78	20
(2)	121	121	98	125	170	168	110	75
(3)	170	142	134	172	212	192	148	140
(4)	210	165	174	210	232	203	181	201
(5)	230	183	200	234	230	202	202	240
(6)	230	188	214	225	205	178	204	268
(7)	210	175	202	200	160	140	190	250
(8)	168	152	168	145	100	100	160	205
(9)	120	128	130	90	55	76	122	142
(10)	85	105	90	48	30	70	95	88
(11)	76	96	63	28	36	80	75	41
(12)	81	98	55	35	65	105	75	29
(13)	112	110	60	60	104	130	88	50
(14)	150	128	85	102	152	160	120	90
(15)	190	150	118	155	195	190	150	160
(16)	225	172	150	198	224	195	180	218
(17)	245	190	185	225	224	190	199	255
(18)	244	194	204	230	202	168	195	268
(19)	210	181	200	205	160	133	180	244
(20)	168	160	180	160	110	100	144	195
(21)	120	130	148	110	68	78	112	140
(22)	55	110	108	63	45	70	78	70
(23)	55	98	72	40	50	82	59	18

4. — Separation of the groups of constituents of the same species

If in the expression (3*c*) we make

$$(\rho + 0.841q)\ (d - 16) - r = - r' \qquad (4a)$$

we obtain :

$$y = S_0 + \sum_c R \cos r' \cos 15°n\,(t - \bar{t})$$
$$+ \sum_c R \sin r' \sin 15°n\,(t - \bar{t}) \qquad (4b)$$

The values of $15n\ (t - \bar{t})$ and the values corresponding to cosines and sines may now be put in the table. In this way we obtain table 4-I. We may easily see that a sum of the products of $\cos 15°n\ (t - \bar{t}) \times R \cos r'$ and of $\sin 15°n\ (t - \bar{t}) \times R \sin r'$ in each horizontal line is the mathematical expression of y for the point considered in the " column ". Afterwards, it is possible to choose a suitable combination of the terms y in such a way as to isolate a group of the same species. An example will serve to make the process clear. Let us take the column $\cos 30° \tau$ in table 4-I. If we assign to all the negative values the multiplier -1, to all the positive values of the multiplier 1, and to all the 0 values the multiplier 0, we have the set of multipliers of column (2) of table 4-II. Let us take a strip of paper and write on it these multipliers in such a way that each corresponds to a horizontal line of table 4-I. We may make these multipliers correspond to

TABLE 4-I

$t-\bar{t}=\tau$	L.P.				Diurnal		Semi-diurnal			Third-diurnal			Quarter-diurnal			Sixth-diurnal		
	Coefficients of $S_0 + \Sigma R_0 \cos r'_0$				Coefficients of			Coefficients of			Coefficients of			Coefficients of			Coefficients of	
					$\Sigma R_1^c \cos r'_1$	$\Sigma R_1^c \sin r'_1$		$\Sigma R_2^c \cos r'_2$	$\Sigma R_2^c \sin r'_2$		$\Sigma R_3^c \cos r'_3$	$\Sigma R_3^c \sin r'_3$		$\Sigma R_4^c \cos r'_4$	$\Sigma R_4^c \sin r'_4$		$\Sigma R_6^c \cos r'_6$	$\Sigma R_6^c \sin r'_6$
	$0°\tau$	C	S	$15°\tau$	cos $15°\tau$	sin $15°\tau$	$30°\tau$	cos $30°\tau$	sin $30°\tau$	$45°\tau$	cos $45°\tau$	sin $45°\tau$	$60°\tau$	cos $60°\tau$	sin $60°\tau$	$90°\tau$	cos $90°\tau$	sin $90°\tau$
0	0°	1	0	10°	1.000	0.000	0°	1.000	0.000	0°	1.000	0.000	0°	1.000	0.000	0°	1.000	0.000
1	0	1	0	15	0.966	0.259	30	0.866	0.500	45	0.707	0.707	60	0.500	0.866	90	0.000	1.000
2	0	1	0	30	0.866	0.500	60	0.500	0.866	90	0.000	1.000	120	—0.500	0.866	180	—1.000	0.000
3	0	1	0	45	0.707	0.707	90	0.000	1.000	135	—0.707	0.707	180	—1.000	0.000	270	0.000	—1.000
4	0	1	0	60	0.500	0.866	120	—0.500	0.866	180	—1.000	0.000	240	—0.500	—0.866	0	1.000	0.000
5	0	1	0	75	0.259	0.966	150	—0.866	0.500	225	—0.707	—0.707	300	0.500	—0.866	90	0.000	1.000
6	0	1	0	90	0.000	1.000	180	—1.000	0.000	270	0.000	—1.000	0	1.000	0.000	180	—1.000	0.000
7	0	1	0	105	—0.259	0.966	210	—0.866	—0.500	315	0.707	—0.707	120	0.500	0.866	270	0.000	—1.000
8	0	1	0	120	—0.500	0.866	240	—0.500	—0.866	0	1.000	0.000	180	—0.500	0.866	0	1.000	0.000
9	0	1	0	135	—0.707	0.707	270	0.000	—1.000	45	0.707	0.707	240	—1.000	0.000	90	0.000	1.000
10	0	1	0	150	—0.866	0.500	300	0.500	—0.866	90	0.000	1.000	300	—0.500	—0.866	270	—1.000	0.000
11	0	1	0	165	—0.966	0.259	330	0.866	—0.500	135	0.707	0.707	0	0.500	—0.866	0	0.000	—1.000
12 to 23				Repeat with opposite sign.			Repeat with same sign.			Repeat with opposite sign.			Repeat with same sign.					

each column of $\sum\limits_{c} cos\ r'$ and $\sum\limits_{c} sin\ r'$ of the table and add the products.

We see that by this means, all the sums of sines are 0 and all the sums of cosines are 0, except that of the columns $R_2\ cos\ r'_2$ and $R_6\ cos\ r'_6$. With the multipliers of column (*b*) of table 4-II, drawn up in the same way but considering the signs of the values of column sin 30° τ, the sum of all the products will be 0 except for $R_2\ sin\ r'_2$, and $R_6\ sin\ r'_6$.

TABLE 4-II

1	a	2	b	3	c	4	d	6	f
1	0	1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1	0	1
1	1	1	1	0	1	—1	1	—1	0
1	1	0	1	—1	1	—1	0	0	—1
1	1	—1	1	—1	0	—1	—1	1	0
1	1	—1	1	—1	—1	1	—1	0	1
0	1	—1	0	0	—1	1	0	—1	0
—1	1	—1	—1	1	—1	1	1	0	—1
—1	1	—1	—1	1	0	—1	1	1	0
—1	1	0	—1	1	1	—1	0	0	1
—1	1	1	—1	0	1	—1	—1	—1	0
—1	1	1	—1	—1	1	1	—1	0	—1
—1	0	1	0	—1	0	1	0	1	0
—1	—1	1	1	—1	—1	1	1	0	1
—1	—1	1	1	0	—1	—1	1	—1	0
—1	—1	0	1	1	—1	—1	0	0	—1
—1	—1	—1	1	1	0	—1	—1	1	0
—1	—1	—1	1	1	1	1	—1	0	1
0	—1	—1	0	0	1	1	0	—1	0
1	—1	—1	—1	—1	1	1	1	0	—1
1	—1	—1	—1	—1	0	—1	1	1	0
1	—1	0	—1	—1	—1	—1	0	0	1
1	—1	1	—1	0	—1	—1	—1	—1	0
1	—1	1	—1	1	—1	1	—1	0	—1

This procedure is a general one and each of the columns of multipliers appearing in table 4-II has been prepared in this way to separate a group of constituents of the same species. The interference of the sixth-diurnal species on the semi-diurnal and of the third-diurnal on the diurnal can be very easily eliminated as will be seen below. The symbol which is found at the head of each column of this table represents the group to be separated. The symbol 1, which is the subscript of the diurnal group, is used to indicate the separation of the cosines of the diurnal group, and the symbol *a*, first letter of the alphabet, for the separation of the sines of the diurnal group; 2 for the cosines of the semi-diurnal group, and *b* for the sines of the semi-diurnal group, etc.

The sum of the values of ($cos\ 15°n \sum\limits_{c} R\ cos\ r'$) and ($sin\ 15°n \sum\limits_{c} R\ sin\ r'$) in each horizontal row of (0) to (23) of table 4-I is the expression of *y* which corresponds to the points of intersection of the lines (0) to (23) with the " columns " A, B, C, ... H of fig. 3.1. Thus, if we combine the 24 values of *y* of each " column " A, B, C, etc., using the multipliers of table 4-II, we shall obtain the numerical values, which are recorded in form 4-A, for

TABLE 4-III

X	$S_0 + \sum_c R_0 \cos r'_0$	$\sum_c R_1 \cos r'_1$	$\sum_c R_1 \sin r'_1$	$\sum_c R_2 \cos r'_2$	$\sum_c R_2 \sin r'_2$	$\sum_c R_3 \cos r'_3$	$\sum_c R_3 \sin r'_3$	$\sum_c R_4 \cos r'_4$	$\sum_c R_4 \sin r'_4$	$\sum_c R_6 \cos r'_6$	$\sum_c R_6 \sin r'_6$
0	24.000	0	0	0	0	0	0	0	0	0	0
1	0	15.192	0	0	0	—4.828	0	0	0	0	0
a	0	0	15.192	0	0	0	4.828	0	0	0	0
2	0	0	0	14.928	0	0	0	0	0	—4.000	0
b	0	0	0	0	14.928	0	0	0	0	0	4.000
3	0	0	0	0	0	14.484	0	0	0	0	0
c	0	0	0	0	0	0	14.484	0	0	0	0
4	0	0	0	0	0	0	0	16.000	0	0	0
d	0	0	0	0	0	0	0	0	13.856	0	0
6	0	0	0	0	0	0	0	0	0	12.000	0
f	0	0	0	0	0	0	0	0	0	0	12.000

the functions which we shall call X to which we shall apply as subscripts the symbols which are found at the head of the columns in table 4-II.

The table 4-III gives the coefficients $\sum\limits_{c} R \cos r'$ and $\sum\limits_{c} R \sin r'$ in these functions. This table was constructed by using the multipliers of table 4-II and the values of cosines and sines of the angles of table 4-I. Now let us write explicitly the functions of table 4-III :

$$X_0 = 24 \sum_c R_0 \cos r'_0 \qquad (4c)$$

$$\left\{\begin{aligned} X_1 &= 15.192 \sum_c R_1 \cos r'_1 - 4.828 \sum_c R_3 \cos r'_3 && (4d) \\ X_a &= 15.192 \sum_c R_1 \sin r'_1 + 4.828 \sum_c R_3 \sin r'_3 && (4e) \end{aligned}\right.$$

$$\left\{\begin{aligned} X_2 &= 14.928 \sum_c R_2 \cos r'_2 - 4.000 \sum_c R_6 \cos r'_6 && (4f) \\ X_b &= 14.928 \sum_c R_2 \sin r'_2 + 4.000 \sum_c R_6 \sin r'_6 && (4g) \end{aligned}\right.$$

$$\left\{\begin{aligned} X_3 &= 14.484 \sum_c R_3 \cos r'_3 && (4h) \\ X_c &= 14.484 \sum_c R_3 \sin r'_3 && (4i) \end{aligned}\right.$$

$$\left\{\begin{aligned} X_4 &= 16.000 \sum_c R_4 \cos r'_4 && (4j) \\ 1.15\, X_d &= 16.000 \sum_c R_4 \sin r'_4 = U_d && (4k) \end{aligned}\right.$$

$$\left\{\begin{aligned} X_6 &= 12.000 \sum_c R_6 \cos r'_6 && (4l) \\ X_f &= 12.000 \sum_c R_6 \sin r'_6 && (4m) \end{aligned}\right.$$

The expressions (4*d*), (4*e*), (4*j*) and (4*k*) give :

$$15.192 \sum_c R_1 \cos r'_1 = X_1 + \frac{X_3}{3} = U_1 \qquad (4h)$$

$$15.192 \sum_c R_1 \sin r'_1 = X_a - \frac{X_c}{3} = U_a \qquad (4o)$$

and the expressions (4*f*), (4*g*), (4*l*) and (4*m*) give us :

$$14.928 \sum_c R_2 \cos r'_2 = X_2 + \frac{X_6}{3} = U_2 \qquad (4p)$$

$$14.928 \sum_c R_2 \sin r'_2 = X_b - \frac{X_f}{3} = U_b \qquad (4q)$$

The form 4-A shows us the numerical values of X and U as given by the above formulas.

5. — Separation of the constituents

Now we shall be able to proceed with the separation of the constituents of each species. We have obtained functions X and U the numerical values of which are equal to the contributions of all the constituents of a same group. Furthermore, table 4-III shows that all the functions X with numerical subscripts have only contributions in *R cos r'* and that the

FORM 4-A

	A	B	C	D	E	F	G	H
X_0	3611	3379	3178	3188	3234	3253	3205	3407
X_1	— 187	— 11	48	161	137	35	— 102	— 185
$0.333X_3$	— 9	1	— 16	— 11	— 1	— 2	— 2	— 4
U_1	— 196	— 10	32	150	136	53	— 104	— 189
X_a	— 74	— 62	38	9	21	51	60	— 38
$-0.333X_c$	0	3	9	— 5	— 2	5	0	4
U_a	— 74	— 59	47	4	19	56	60	— 34
X_2	—1269	—691	—1106	—1403	— 990	— 501	— 997	—1863
$0,333X_6$	8	— 1	— 2	1	2	— 2	— 1	— 4
U_2	—1261	—692	—1108	—1402	— 988	— 503	— 998	—1867
X_b	466	132	— 82	472	1054	839	241	56
$-0.333X_f$	1	— 2	— 3	0	2	1	0	2
U_b	467	130	— 85	472	1056	840	241	58
X_3	— 28	3	— 49	— 33	— 4	— 7	— 7	— 12
X_c	— 1	— 9	— 28	14	5	— 15	— 1	— 13
X_4	47	53	12	32	48	49	5	— 41
X_d	38	14	26	31	10	— 23	1	— 2
$U_f = 1.15X_d$	44	16	30	36	12	— 26	1	— 2
X_6	23	— 2	— 7	3	7	— 6	— 2	— 11
X_f	— 4	5	9	— 1	— 7	— 3	— 1	— 6

functions X with literal subscripts have only contributions in $R \sin r'$. The same is true for the functions U as we can see in the expressions (4*k*) and (4*h*) to (4*q*). Therefore, if we designate by l the sum of all the terms of the functions with numerical subscripts and by l' the sum of all the terms of the functions with literal subscripts, we may write :

$$l = C \sum_c R \cos r' \tag{5a}$$

$$l' = - C \sum_c R \sin r' \tag{5b}$$

where C represents the numerical coefficients for X and U. If in (4*a*) we replace $(d - 16)$ by the second part of (3*d*) and if we transfer to (5*a*) and (5*b*) the value of r' which results from this, we shall obtain after development :

$$l = C \sum_c [R \cos r \cos (\rho + 0.841\, q)\ (1.65 + 3.3\, k) + R \sin r \sin (\rho + 0.841\, q)\ (1.65 + 3.3\, k)] \tag{5c}$$

$$l' = - C \sum_c [R \cos r \sin (\rho + 0.841\, q)\ (1.65 + 3.3\, k) - R \sin r \cos (\rho + 0.841\, q)\ (1.65 + 3.3\, k)] \tag{5d}$$

TABLE 5-I

Constituents				$\cos(\rho + 0.841q)(1.65 + 3.3k)$							
				A	B	C	D	E	F	G	H
Q_1				0.690	—0.719	—0.700	0.710	0.710	—0.700	—0.719	0.690
O_1	MO_3			—0.924	—0.383	0.383	0.924	0.924	0.383	—0.383	—0.924
M_1	M_3			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
K_1	MK_3			—0.924	—0.383	0.383	0.924	0.924	0.383	—0.383	—0.924
J_1				0.690	—0.719	—0.700	0.710	0.710	—0.700	—0.719	0.690
OO_1				—0.383	0.924	—0.924	0.383	0.383	—0.924	0.924	—0.383
μ_2				0.366	—0.881	—0.572	0.748	0.748	—0.572	—0.881	0.366
N_2	MN_4	$2MN_6$		—0.914	—0.366	0.392	0.925	0.925	0.392	—0.366	—0.914
M_2	M_4	M_6	So	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
L_2			Mm	—0.914	—0.366	0.392	0.925	0.925	0.392	—0.366	—0.914
S_2	MS_4	$2MS_6$	MS_f	0.366	—0.881	—0.572	0.748	0.748	—0.572	—0.881	0.366
$2SM_2$	S_4	$2SM_6$		—0.733	0.553	—0.345	0.117	0.117	—0.345	0.553	—0.733
	SN_4	MSN_6		—0.712	—0.117	0.530	0.944	0.944	0.530	—0.117	—0.712
Constituents				$\sin(\rho + 0.841q)(1.65 + 3.3k)$							
				A	B	C	D	E	F	G	H
Q_1				—0.724	—0.695	0.714	0.705	—0.705	—0.714	0.695	0.724
O_1	MO_3			0.383	0.924	0.924	0.383	—0.383	—0.924	—0.924	—0.383
M_1	M_3			0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
K_1	MK_3			—0.383	—0.924	—0.924	—0.383	0.383	0.924	0.924	0.383
J_1				0.724	0.695	—0.714	—0.705	0.705	0.714	—0.695	—0.724
OO_1				—0.924	0.383	0.383	—0.924	0.924	—0.383	—0.383	0.924
μ_2				—0.930	—0.437	0.820	0.664	—0.664	—0.820	0.437	0.930
N_2	MN_4	$2MN_6$		0.405	0.930	0.920	0.379	—0.379	—0.920	—0.930	—0.405
M_2	M_4	M_6	So	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
L_2			Mm	—0.405	—0.930	—0.920	—0.379	0.379	0.920	0.930	0.405
S_2	MS_4	$2MS_6$	MS_f	0.930	0.437	—0.820	—0.664	0.664	0.820	—0.437	—0.930
$2SM_2$	S_4	$2SM_6$		0.682	—0.833	0.938	—0.993	0.993	—0.938	0.833	—0.682
	SN_4	MSN_6		—0.702	—0.993	—0.848	—0.331	0.331	0.848	0.993	0.702

Table 3-I gives the values of $(\rho + 0.841\ q)\ (1.65 + 3.3\ k)$ for the " columns " A to H, of the " tidal-surface ". We may see that the angles of columns A and H have the same numerical values, but are of opposite signs. The same applies to columns B and G, C and F, D and E. Thus, the cosines of these angles will be equal in value and in signs, and the sines will be equal and of opposite sign. This is to be seen in table 5-II which gives the values of the sines and cosines of the angles $(\rho + 0.841\ q)\ (1.65 + 3.3\ k)$ for each column A, B, etc., and each constituent Q_1, O_1, etc., μ_2, N_2, etc.

Consequently, we may add the l functions for A and H, B and G, etc., and the results will be freed from the terms in $R\ sin\ r$. The same applies to the differences between the values of l' for A and H, B and G, etc., which will be freed from the term in $R\ sin\ r$. In form 5-A, we see the practical values of l and l' calculated as just described. We can then write the following equations

FORM 5-A

		A + H	B + G	C + F	D + E		A—H	B—G	C—F	D—E
X_0	R cos r	7018	6584	6431	6422	R sin r	204	174	— 75	— 46
R cos r	U_1	— 385	— 114	65	286	U_a	— 40	— 119	— 9	— 15
	U_2	—3128	—1690	—1611	—2390	U_b	409	— 111	— 925	— 584
	X_3	— 40	— 4	— 56	— 37	X_c	12	— 8	— 13	9
	X_4	6	58	61	80	U_d	46	15	56	24
	X_6	12	— 4	— 13	10	X_f	2	6	12	6
		A + B	B + G	C + F	D + E		—(A—H)	—(B—G)	—(C—F)	—(D—E)
R sin r	U_a	— 108	1	103	23	U_1	— 606	— 94	1	— 14
	U_b	525	371	755	1528	U_2	7	— 306	605	414
	X_c	— 14	— 10	— 43	19	X_3	16	— 10	42	29
	U_d	— 42	17	4	48	X_4	— 88	— 48	37	16
	X_f	— 10	4	6	— 8	X_6	— 34	0	1	4

where $c(A)$, $c(B)$, etc., represent the cosines of table 5-I for columns A, B, etc., and $s(A)$, $s(B)$, etc., the sines :

$$\left.\begin{array}{l}
l\,(A) + l\,(H) = C \sum\limits_c \{[c(A) + c(H)]\ R\cos r\} \\
l\,(B) + l\,(G) = C \sum\limits_c \{[c(B) + c(G)]\ R\cos r\} \\
l\,(C) + l\,(F) = C \sum\limits_c \{[c(C) + c(F)]\ R\cos r\} \\
l\,(D) + l\,(E) = C \sum\limits_c \{[c(D) + c(E)]\ R\cos r\} \\
l'(A) - l'(H) = -C \sum\limits_c \{[s(A) - s(H)]\ R\cos r\} \\
l'(B) - l'(G) = -C \sum\limits_c \{[s(B) - s(G)]\ R\cos r\} \\
l'(C) - l'(F) = -C \sum\limits_c \{[s(C) - s(F)]\ R\cos r\} \\
l'(D) - l'(E) = -C \sum\limits_c \{[s(D) - s(E)]\ R\cos r\}
\end{array}\right\} \qquad (5f)$$

In the same way, if we total the symmetrical values of l', and the differences in symmetrical values of l, the results will be freed from $R\ cos\ r$.

Thus we obtain :

$$\left.\begin{aligned}
l'(A) + l'(H) &= C\sum_c \{[c(A) + c(H)]\ R\ \sin r\} \\
l'(B) + l'(G) &= C\sum_c \{[c(B) + c(G)]\ R\ \sin r\} \\
l'(C) + l'(F) &= C\sum_c \{[c(C) + c(F)]\ R\ \sin r\} \\
l'(D) + l'(E) &= C\sum_c \{[c(D) + c(E)]\ R\ \sin r\} \\
l\ (A) - l\ (H) &= C\sum_c \{[s(A) - s(H)]\ R\ \sin r\} \\
l\ (B) - l\ (G) &= C\sum_c \{[s(B) - s(G)]\ R\ \sin r\} \\
l\ (C) - l\ (F) &= C\sum_c \{[s(C) - s(F)]\ R\ \sin r\} \\
l\ (D) - l\ (E) &= C\sum_c \{[s(D) - s(E)]\ R\ \sin r\}
\end{aligned}\right\} \quad (5g)$$

We obtain in this way 8 equations in *R cos r* and in *R sin r*. As we have not considered more than 6 constituents in each group, we have for each group a number of systems of linear equations larger than the number of unknowns, which we may resolve by the method of least squares.

We may now make a very important observation : the expressions (5*f*) and (5*g*) show that the system (5*f*) only differs from (5*g*) by the known terms (first left-hand terms) if we invert the signs of the 4 last equations of (5*g*). We shall therefore have only one matrix for the two systems of each group. Table 5-I again shows us that $c(A) = c(H)$, $c(B) = c(G)$, etc., and $s(A) = -s(H)$, $s(B) = s(G)$, etc., which permits us to double the coefficients C of (5*f*) to (5*i*) and organize the matrices by copying the values of cosine and sine of (A) to (D) of table 5-I for each consittuent. This is what has been done to construct table 5-II.

Table 5-II shows us the systems in which the repetition of the coefficients is remarkable. We see that there are only long period constituents which have different matrices for the unknown *R cos r* and *R sin r*. We also see that the diurnal constituents and third-diurnal constituents may be determined as a function of a same matrix, and likewise for the fourth-diurnal and sixth-diurnal constituents, the matrix of which is slightly different from that which corresponds to semi-diurnal constituents. To take account of the difference between the general coefficients, it suffices to divide the final values of R by a coefficient equal to their ratio. In the case of third-diurnal constituents, this coefficient will be 0.953 and in the case of the fourth-diurnal constituents it will be 1.333.

In form 6-D, we see that these coefficients are designated by *c*.

The solution of these very numerous equations by the method of least squares leads to inverse matrices, i.e. to the multipliers of the known terms which we see in table 5-III. On transcribing these multipliers on a sheet of paper, in such a way that the columns shall have the same width as the columns of form 5-A, and in placing the sheet of paper in such a way that the values corresponding to M_2 for example are coincident with the row U_2, the sum of the products will be *R cos r* for this component. The same multipliers juxtaposed to row X_b will give *R sin r*. This is a general rule for obtaining all the values of *R cos r* and *R sin r* which appear in form 5-B. These values are not definitive for the diurnal and semi-

TABLE 5-II

General coefficient : 48,000

R cos r		So	Mm	MS$_f$	Mm	MS$_f$	R sin r	
X_0	A+H	1	—0.914	0.366	0.405	—0.930	A—H	X_0
	B+G	1	—0.366	—0.881	0.930	—0.437	B—G	
	C+F	1	0.392	—0.572	0.920	0.820	C—F	
	D+E	1	0.925	0.748	0.379	0.664	D—E	

General coefficient : 30,384

		Q_1	O_1	M_1	K_1	J_1	OO_1		
U_1	A+H	0.690	—0.924	1.000	—0.924	0.690	—0.383	A+H	U_a
	B+G	—0.719	—0.383	1.000	—0.383	—0.719	0.924	B+G	
	C+F	—0.700	0.383	1.000	0.383	—0.700	—0.924	C+F	
	D+E	0.710	0.924	1.000	0.924	0.710	0.383	D+E	
U_a	A—H	0.724	—0.383	0	0.383	—0.724	0.924	—(A—H)	U_1
	B—G	0.695	—0.924	0	0.924	—0.695	—0.383	—(B—G)	
	C—F	—0.714	—0.924	0	0.924	0.714	—0.383	—(C—F)	
	D—E	—0.705	—0.383	0	0.383	0.705	0.924	—(D—E)	
			MO_3	M_3	MK_3				

General coefficient : 28,968

General coefficient : 29,856

		μ_2	N_2	M_2	L_2	S_2	$2SM_2$		
U_2	A+H	0.366	—0.914	1.000	—0.914	0.366	—0.731	A+H	U_b
	B+G	—0.881	—0.366	1.000	—0.366	—0.881	0.553	B+G	
	C+F	—0.572	0.392	1.000	0.392	—0.572	—0.345	C+F	
	D+E	0.748	0.925	1.000	0.925	0.748	0.121	D+E	
U_b	A—H	0.930	—0.405	0	0.405	—0.930	—0.682	—(A—H)	U_2
	B—G	0.437	—0.930	0	0.930	—0.437	0.833	—(B—G)	
	C—F	0.820	—0.920	0	0.920	0.820	—0.938	—(C—F)	
	D—E	—0.664	—0.379	0	0.379	0.664	0.993	—(D—E)	

General coefficient : 32,000

			MN_4	M_4	SN_4	MS_4	S_4		
X_4	A+H		—0.914	1.000	—0.712	—0.366	—0.731	A+H	U_d
	B+G		0.366	1.000	—0.117	—0.881	0.553	B+G	
X_6	C+F		0.392	1.000	0.530	—0.572	—0.345	C+F	X_f
	D+E		0.925	1.000	0.944	0.748	0.121	D+E	
	A—H		—0.405	0	0.702	—0.930	—0.682	—(A—H)	
U_d	B—G		—0.930	0	0.993	—0.437	—0.833	—(B—G)	X_4
X_f	C—F		—0.920	0	0.848	0.820	—0.938	—(C—F)	X_6
	D—E		—0.379	0	0.332	0.664	0.993	—(D—E)	
			$2MN_6$	M_6	MSN_6	$2MS_6$	$2SM_6$		

General coefficient : 24,000

TABLE 5-III

		A+H	B+G	C+F	D+E		± (A—H)	± (B—G)	± (C—F)	± (D—E)	
So	10^5 R cos r	414	513	576	580	10^5 R sin r					
M*m*		—1165	— 189	571	784		522	1017	872	324	
MS_f		834	— 889	— 722	778		— 947	— 524	698	604	
Σ_0		83	— 565	425	2142		— 425	493	1570	928	
10^5 R cos r or 10^5 R sin r	Q_1	586	— 590	— 577	581		590	573	— 578	— 577	
	O_1	— 768	— 307	323	752		— 318	— 763	— 757	— 312	MO_3
	M_1	828	818	817	828		0	0	0	0	M_3
	K_1	— 768	— 307	323	752		318	763	757	312	MK_3
	J_1	583	— 581	— 586	585		— 586	— 574	577	581	
	OO_1	— 312	757	— 763	318		— 752	— 322	— 308	767	
	Σ_1	149	— 210	— 463	3816		756	— 323	— 309	771	
	μ_2	633	— 617	— 690	674		638	597	— 763	— 294	
	N_2	— 911	— 222	537	596		— 331	— 944	— 557	— 397	
	M_2	945	778	663	962		— 76	108	— 124	117	
	L_2	— 991	— 220	536	596		507	694	844	126	
	S_2	646	— 650	— 653	657		— 843	— 305	427	612	
	$2MS_2$	— 167	444	— 499	222		— 565	802	— 920	873	
	Σ_2	235	— 487	— 106	3707		— 670	952	—1093	1037	
	MN_4	— 956	— 525	212	856		— 570	—1058	— 960	— 337	$2MN_6$
	M_4	1055	837	1105	1168		139	— 62	304	— 94	M_6
	SN_4	—1062	— 290	507	847		855	985	785	196	MSN_6
	MS_4	535	— 742	— 598	805		—1075	— 359	811	831	$2MS_6$
	S_4	— 419	706	— 408	121		— 860	873	— 900	1162	$2SM_6$
	Σ_4	— 847	— 14	818	3797		—1511	379	40	1758	Σ_6
	Σ_3	— 708	204	1463	2332		0	0	0	0	

FORM 5-B

	(R cos r)	corr.	R cos r	(R sin r)	corr.	R sin r	R²	tgr
So	....	....	137.120	....	....	....	....	....
Mm	....	....	— 7.134	....	....	2.031	55.0189	— 0.285
MS_f	....	....	3.530	....	....	— 3.645	25.7469	— 1.033
Σo			133.516			— 1.614		
Q_1	— 1.076	— 0.027	— 1.103	— 1.522	— 0.053	— 1.575	3.6972	1.428
O_1	6.818	0.036	6.854	2.063	0.071	2.134	51.5313	0.311
M_1	— 1.221	— 0.084	— 1.305	0.146	— 0.168	— 0.018	1.7033	0.014
K_1	4.517	— 0.005	4.512	0.601	0.049	0.650	20.7806	0.144
J_1	0.488	0.059	0.547	— 0.682	0.114	— 0.568	0.6218	— 1.038
OO_1	0.747	— 0.037	0.710	— 0.123	— 0.076	— 0.199	0.5437	— 0.280
Σ_1	10,273	— 0.059	10.215	0.483	— 0.063	0.424		
μ_2	— 3.644	1.204	— 2.440	— 5.403	+ 2.687	— 2.716	13.3303	1.113
N_2	16.517	— 0.478	16.039	7.436	— 0.971	6.465	299.0457	0.403
M_2	— 76.348	0.190	— 76.158	27.417	1.128	28.545	6614.8580	— 0.375
L_2	2.095	— 1.014	1.081	7.987	— 1.876	6.111	38.5129	5.653
S_2	— 25.038	— 0.043	— 25.081	17.248	0.295	17.543	936.8134	— 0.699
$2SM_2$	0.664	— 0.333	0.331	— 0.587	— 0.230	— 0.817	0.7771	2.468
Σ_2	— 85.753	— 0.472	— 86.228	54.097	1.033	55.131		
MO_3	....	....	— 0.046	....	....	— 0.241	0.0602	5.239
M_3			— 1.128			— 0.392	1.4260	0.348
MK_3			— 0.233			0.525	0.3299	— 2.253
Σ_3			— 1.407			0,107		
MN_4			— 0.587			1.332	2.1188	— 2.269
M_4			2.360			0.308	5.6645	0.131
SN_4			1.783			— 0.080	3.1855	— 0.045
MS_4			— 0.014			1.563	2.4432	—111.6
S_4			— 0.258			0.528		
Σ_4			3.284			3.652		
$2MN_6$			— 0.246			0.190		
M_6			0.096			— 0.147		
MSN_6			0.085			— 0.218		
$2MS_6$			0.356			0.223		
$2SM_6$			— 0.016			0.366		
Σ_6			0.275			0.414		

diurnal constituents since it is still necessary to make corrections due to the interferences of certain small constituents. However, before determining these corrections it is necessary to calculate for zero hour on the 16th day of the series of observations the values of V, u and f for the various constituents. We will say nothing regarding this calculation which is well known. Form 6-D shows the results of the operations carried out.

6. — Additional Refinement of the results

On account of the limitations inherent in an analysis of short period, the constituents the speeds of which are close to one another cannot be separated; they appear in groups which we may consider represented respectively by the main constituent of the group duly corrected in amplitude and in phase.

The groups to be considered are the following :

S_2, K_2 and T_2
K_1 and P_1
Q_1 and ρ_1
J_1 and θ_1
N_2 and ν_2
MS_4, MK_4 and MT_4
$2MS_6$, $2MK_6$ and $2MT_6$

As we know, we may treat each one of these groups as a constituent expressed by :

$$y = f(1 + W)H \cos (V + u + w + qt - g)$$

where $(1 + W)$ and w are the corrective elements of the main constituent of the group. The determination of these elements is quite well known and their values are already given in tables of various publications, such as the one on page 73 of vol. XXXI of this Review. Form 6-D shows the values calculated for obtaining R and r for the groups S_2, N_2 and K_1. The other groups do not require any special treatment since the values of W and w for the group (S_2, K_2, T_2) clearly apply to (MS_4, MK_4, MT_4) and to ($2MS_6$, $2MK_6$, $2MT_6$), because the ratios of the amplitudes within these two last groups, as given by the equilibrium tide, are the same as for (S_2, K_2, T_2). For the same reasons the values of $(1 + W)$ and w for the group N_2 and ν_2 may be used for MN_4, $M\nu_4$ and for $2MN_6$ and $2M\nu_6$ as well as for Q_1 and ρ_1 and J_1 and θ_1; but for this last group, w must be taken with the opposite sign. Finally, the values of W and w for the groups K_1 and P_1 may be used for MK_3 and MP_3.

These corrections are still insufficient, since they affect only the main constituents of each group and in reality, there is still the interference of the constituents of each of these groups on the constituents determined by the analysis, and in the process of refinement under study, we may include other constituents such as $2N_2$ in the semi-diurnal group and π_1, Ψ_1 and φ_1 in the diurnal group. To eliminate the interference of these constituents on all the others, it is necessary first of all to calculate the values of R and r for the first ones. This is perfectly feasible if we accept as valid the equilibrium relations which enable us to draw up table 6-I. In so far as the values of R are concerned, it is not necessary to give any explanation, but as regards the values of w, we must add some comments. For all the constituents which appear in table 6-I, except for $2N_2$ we can write :

$$-r = V + u + w - g \qquad \text{main constituent}$$

$$-r' = V' + u' - g \qquad \text{secondary constituent}$$

whence

$$r' = r + w + (u - u') + (V - V')$$

TABLE 6-I

R			*r*
K_2	S_2	0.272 (f of K_2) (H of S_2)	$(r+w)$ of $S_2 - 2(V+u)$ of $K_1 + 180°$
T_2	S_2	0.059 (H of S_2)	$(r+w)$ of S_2 + V of $K_1 - 12°$
ν_2	N_2	0.194 (fH of N_2)	$(r+w)$ of N_2 — (3V of M_2 — 2V of N_2)
$2N_2$	N_2	0.133 (fH of N_2)	$2(r+w)$ of $N_2 - r$ of M_2
	K_1	0.331 (H of K_1)	$(r+w+u)$ of K_1 + 2V of K_1
π_1	K_1	0.019 (H of K_1)	$(r+w+u)$ of K_1 + 3V of $K_1 - 12°$
φ_1	K_1	0.008 (H of K_1)	$(r+w+u)$ of K_1 + 3V of $K_1 + 12°$
φ_1	K_1	0.014 (H of K_1)	$(r+w+u)$ of K_1 — 2V of $K_1 + 180°$

In so far as the constituent $2N_2$ is concerned, the equilibrium relations show that :

$$\text{arg. } 2N_2 = 2\,(\text{arg. } N_2) - \text{arg. } M_2 \qquad (6a)$$

Consequently, if we admit that in practice this equality remains valid, we may write in place of (6*a*) :

$$\text{phase } 2N_2 = 2\ (\text{phase } N_2) - \text{phase } M_2$$

TABLE 6-II

P_1	π_1	ψ_1	φ_1		K_2	T_2	ν_2	$2N_2$		
1	1	1	1		0	0	0	0	$r+w+u$	K_1
0	0	0	0		— 2	0	0	0	$V+u$	K_1
2	3	— 1	— 2		0	1	0	0	V	K_1
0	0	0	0		0	0	1	2	$r+u$	N_2
0	0	0	0		0	0	2	0	V	N_2
0	0	0	0		0	0	— 3	0	V	M_2
0	0	0	0		0	0	0	— 1	r	M_2
0	0	0	0		1	1	0	0	$r+w$	S_2
0	—12°	12°	180°		180°	—12	0	0		

On the other hand, if on account of the interference of ν_2, N_2 becomes increased by w, that of $2N_2$ will be increased by $2w$ in accordance with the equilibrium relation. Thus, the expression (6*a*) will continue to be valid, which enables us to write the expression of r for $2N_2$ which appears in table 6-I. In this case, in carrying out to completion the calculation of R and r for S_2, N_2, K_2 and M_2 as shown in Form 6-A, we may calculate with the formulae in table 6-I all the values of R and r of the constituents which have an effect which we wish to eliminate in the case of others. In order to understand the calculation of the terms R, it suffices to refer to Form 6-C.

FORM 6-A

	$f(1+W)$	R	H	r
K_1	1.194	4.6	3.9	7°.6
N_2	1.121	18.1	16.1	24.2
S_2	0.862	30.4	35.3	145.4
M_2	....	....	....	160.3

FORM 6-B

K_1	$r + w + u$	— 14.9
K_1	$V + u$	214.7
K_1	V	221.7
N_2	$r + w$	30.4
N_2	V	86.2
M_2	V	311.5
M_2	r	160.3
S_2	$r + w$	128.0
		1.0

FORM 6-C

	P_1	π_1	ψ_1	φ_1		K_2	T_2	ν_2	$2N_2$
f	1.000	1.000	1.000	1.000		1.168	1.000	0.981	0.981
H*	3.9	3.9	3.9	3.9		35.3	35.3	16.1	16.1
H'/H	0.331	0.019	0.008	0,014		0.272	0,059	0.194	0.133
R = fH'	1.3	0.1	0.0	0.1		11.2	2,1	3.1	2.1
r	68,5	278.2		81.7		238.6	337,7	348.3	260.5
cos r	0,367	0.143		0.144		—0.521	0,925	0.979	0.165
sin r	0,930	—0.990		0.990		—0.854	—0,379	—0.203	—0.986
R cos r	0,477	0.014	0	0.014		—5.835	1,943	3.035	—0.347
R sin r	1,209	—0.099	0	0.099		—9.565	—0,796	—0.629	—2.071

(*) H = H (K_1) for the diurnal constituents; H = H (S_2) for K_2 and T_2; H = H (N_2) for ν_2 and $2N_2$.

FORM 6-D

	S_0	*Mm*	MS_f	Q_1	O_1	M_1	K_1	J_1	OO_1
V_0		225.3	311.5	224.5	89.8	245.8	221.7	82.0	173.7
u		....	— 1.8	8.1	8.1	112.9	— 7.0	— 9.9	—27.8
w		....	—17.4	6.2	....	....	—15.5	— 6.2	....
r		164.1	314.0	235.0	17.3	180.8	8.2	313.9	305.0
g		29	246	113.8	115.2	179.5	207.4	19.8	91.2
f		0.932	0.981	1.111	1.111	1.186	1.069	1.106	1.427
$1+W$		....	0.862	1.143	....	....	1.117	1.143	....
$f(1+W)$		....	0.846	1.270	....	....	1.194	1.264	....
R		7.4	4.8	1.9	7.2	1.3	4.6	0.8	0.7
H	137.1	7.9	5.6	1.5	6.5	1.1	3.9	0.6	0.5
	μ_2	N_2	M_2	L_2	S_2	$2SM_2$	MO_3	M_3	MK_3
V_0	263.0	86.2	311.5	356.8	....	48.5	41.3	287.3	173.2
u	— 1.8	— 1.8	— 1.8	— 3.8	....	1.8	6.3	— 2.7	— 8.8
w	....	6.2	....	....	—17.4	—34.8	....	....	—15.5
r	228.1	21.9	159.4	80.0	145.0	292.1	259.0	199.2	113.9
g	129.3	112.5	109.1	73.0	127.6	307.6	306.6	123.8	262.8
c	1.000	1.000	1.000	1.000	1.000	1.000	0.953	0.953	0.953
f	0.981	0.981	0.981	1.280	1.000	0.981	1.092	0.972	1.050
$1+W$		1.145	....	....	0.862	0.724	....	....	1.117
$cf(1+W)$		1.123	0.981	1.280	0.862	0.710	1.041	0.926	1.171
R	3.7	17.3	81.3	6.2	30.6	0.9	0.2	1.2	0.7
H	3.8	15.4	82.8	4.8	35.5	1.3	0.2	1.3	0.6
	MN_4	M_4	SN_4	MS_4	$2MN_6$	M_6	MSN_6	$2MS_6$	$2SM_6$
V_0	37.7	263.0	86.2	311.5	349.2	214.5	37.7	263.0	311.5
u	— 3.6	— 3.6	— 1.8	— 1.8	— 5.4	— 5.4	— 3.6	— 3.6	— 1.8
w	6.0	....	—11.4	—17.4					
r	113.8	7.5	357.4	90.5					
g	153.9	266.9	70.4	22.8					
c	1.333	1.333	1.333	1.333	1.000	1.000	1.000	1.000	1.000
f	0.962	0.962	0.981	0.981	0.944	0.944	0.962	0.962	0.981
$1+W$	1.145	....	1.145	0.862					
$cf(1+W)$	1.468	1.273	1.496	1.126					
R	1.5	2.4	1.8	1.6					
H	1.0	1.9	1.2	1.4					

K_1, MK_3

K_1 $\quad 2V_0 + u = 76.4$

$wf = -16.6$

$wf = 0.125$

$f = 1.069$

$w = -15.5$

$W = 0.117$

N_2, MN_4, $2MN_6$

M_2 $\quad 3V = 934.5$

N_2 $\quad 2V = 172.4$

difference $= 42.1$

$w = 6.2$

$1 + W = 1.143$

S_2, MS_4, MS_f, $2MS_6$

K_1 $\quad V + u = 214°.7$

$w/f = -14°.9$

$W/f = -0.118$

K_2 $\quad f = 1.170$

$w = -17°.4$

$W = -0.138$

$2SM_2$, $2SM_6$ =

$w = 2w(S_2)$

$W = 2W(S_2)$

SN_4, MSN_6

$w = w(S_2) + w(N_2)$

$1 + w = W(S_2) + 1 + W(S_2)$

J_1

$w = -w(N_2)$

$1 + W = 1 + W(N_2)$

Q_1

as for N_2

As to the terms r, we may express them as the multipliers which are found in table 6-II to make the combination of values which are included in Form 6-B. The values of V, *u* and *w* found there are the same as those in Form 6-D.

It still remains to be explained how we may do away with the values of *R cos r* and *R sin r* of the isolated constituents in the analysis, the amounts arising from the values of *R cos r* and *R sin r* calculated in Form 6-C. For this purpose, we shall use the matrix calculation which will make the development very much simpler.

Now, in designating by $\{C\}$ the column vector constituted by the known terms of the systems 5-II, by M the matrices of these systems and by $\{I\}$ the column vector constituted by the unknown terms, any one of these systems may be represented by

$$\{C\} = k\ M.\ \{I\} \qquad (6b)$$

where *k* is the general coefficient of the unknown terms of the systems in table 5-II. It happens however, that in the drawing up of (6*b*) the unknown

TABLE 6-III

General coefficient : 30.384				
	P_1	π_1	ψ_1	φ_1
A + H	—0.695	—0.533	—0.982	—1.000
B + G	—0.099	0.047	—0.514	—0.635
C + F	0.539	0.610	0.301	0.215
D + E	0.945	0.954	0.912	0.900
± (A — H)	0.719	0.846	0.187	—0.019
± (B — G)	0.955	0.999	0.858	0.773
± (C — F)	0.842	0.792	0.954	0.976
± (D — E)	0.327	0.301	0.410	0.437

TABLE 6-IV

General coefficient : 29.856				
	K_2	T_2	ν_2	$2N_2$
A + H	0.708	0.168	—0.708	0.674
B + G	—0.706	—0.941	—0.113	—0.729
C + F	—0.707	—0.498	0.531	—0.693
D + E	0.707	0.766	0.944	0.714
± (A — H)	—0.706	—0.985	—0.706	0.738
± (B — G)	—0.708	—0.339	—0.994	0.684
± (C — F)	0.707	0.867	—0.847	—0.721
± (D — E)	0.707	0.643	—0.331	—0.700

terms which appear in table 6-I are not involved. These unknown terms give rise to matrices which one can see in tables 6-III and 6-IV, which have been constructed in the same manner as 5-II. Consequently, if we designate by $\{I'\}$ the column vector which represents these unknown terms, and by M′ the respective matrix, the equation (6*c*) should be replaced by another one obtained by adding to its second part kM′ {I′} :

$$\{C\} = kM\ \{I\} + kM'\ \{I'\} \qquad (6c)$$

From this we deduce :

$$\{I\} = \frac{M^{-1}\{C\}}{k} - M^{-1} M'\ \{I'\} \qquad (6d)$$

Evidently, $\frac{M^{-1}}{k}$ is any one of the inverse matrices which appear in table 5-III. Consequently, the expression (6*d*) shows that the correction to be applied to {I} is equal to the product divided by k of the matrix of the diurnal species in table 5-II by the matrix 6-III or of that of the semi-diurnal species by 6-IV. The products of the matrices are those which are represented in table 6-V and 6-VI. Thus, the values of the column vector {I′} being known, that is to say the values of *R cos r* and of *R sin r* for the constituents which appear in the table mentioned above, the corrections to apply to *R cos r* and to *R sin r* of each of the isolated constituents in Form 5-A will be equal to the sum of the products of the multipliers of tables 6-V and 6-VI by the respective values of *R cos r* or of *R sin r* as the case may be. In the example quoted, the total corrections are those which appear following the values of *R cos r* and *R sin r* in form 5-B.

TABLE 6-V

	P_1	π_1	ϕ_1	φ_1
Q_1	—0.056	—0.092	0.031	0.060
O_1	0.075	0.126	—0.040	—0.076
M_1	—0.172	—0.270	0.072	0.129
K_1	0.044	—0.922	—0.990	—0.963
J_1	0.123	0.175	—0.082	—0.173
OO_1	—0.078	—0.104	0.042	0.084
	—0.064	—1.087	—0.967	—0.939

TABLE 6-VI

	K_2	T_2	ν_2	$2N_2$
μ_2	—0.068	0.040	0.122	—1.036
N_2	0.065	—0.045	0.017	0.181
M_2	—0.091	0.048	—0.154	—0.096
L_2	0.182	—0.096	0.086	0.076
S_2	—0.012	0.031	—0.066	—0.079
$2SM_2$	0.042	—0.003	—0.035	—0.071
	0.118	—0.025	—0.030	—1.025

It is now necessary to give some explanations on the multipliers K_1, N_2 and S_2 in tables 6-V and 6-VI. We have seen that in order to correct K_1 of P_1, S_2 of K_2 and T_2 and N_2 of ν_2 we have introduced the factors 1 + W and the angles *w*. If we look at fig. 6.1 where we have represented the schematic vectorial combination of K_1 and P_1 we notice that :

$$R \cos r\,(K_1) = R \cos r\,(K_1 + P_1) - R \cos r\,(P_1)$$
$$R \sin r\,(K_1) = R \sin r\,(K_1 + P_1) - R \sin r\,(P_1)$$

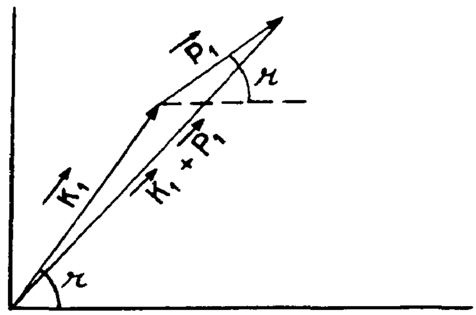

FIG. 6.1

Under these conditions, the corrections for P_1 have been determined by allowing that the factor for correcting P_1 should be equal to —1. Now, by multiplying the matrices, the value would be —0.956, instead of which 0.044 which is given in table 6-V. Consequently, if we have already made the correction by considering the multiplier —1 it is only necessary to add 0.044 which is to be found in table 6-V. We can give the same explanation for values —0.012, 0.031 for S_2 and 0.017 for N_2 in table 6-VI.

The calculation will be completed in Form 6-D where we shall include the final values of R and r calculated from R^2 and tg r from Form 5-B.

7. — Accuracy of the results

It is difficult to know the accuracy of the values obtained for R and r, since that it would be necessary to know the average error of the ordinates y. However, although the procedure is rather long, it is not difficult to express the average errors of R and of r as a function of a mean error of y. That has the advantage of permitting the study of the relative accuracy of the constants obtained for the various constituents as well as for the comparison of this method with others. In the study which we have made we have obtained the following formulae :

$$m_R = \left\{ \begin{array}{ll} \text{L.P.} & 0.105 \\ \text{D.} & 0.114 \\ \text{S.D.} & 0.112 \\ \text{T.D.} & 0.103 \\ \text{Q.D.} & 0.107 \\ \text{6.D} & 0.104 \end{array} \right\} \times m_y; \qquad m_r = \left\{ \begin{array}{ll} \text{L.P.} & 6^\circ.0/R \\ \text{D.} & 6.5/R \\ \text{S.D.} & 6.4/R \\ \text{T.D.} & 6.1/R \\ \text{Q.D.} & 6.0/R \\ \text{6.D} & 6.0/R \end{array} \right\} \times m_y;$$

which shows us that the average errors of the amplitudes are approximately of the same order for all the constituents but the average errors of the phases are inversely proportional to these amplitudes. As the coefficients of these are approximately of the same order, we may say that the constituents of the same amplitude are determined with the same accuracy. However, if we imagine, for example, that the constituents K_1 and OO_1 retain the same relationship as that of the equilibrium tide, we shall have approximately R $(K_1) \approx 33R\ (OO_1)$ and thus m_r for OO_1 will be 33 times greater than m_r for K_1. Consequently, one should not place too much trust in the constant g for the small constituents.

We have made a similar calculation for the *Tidal Institute's* method for the diurnal constituents and we find that $m_R \approx 0.075\ m_y$ and $m_r = 4°.3\ m_y/R$. If we take account of the elimination of the errors already made by the tracing of contour lines, we may say that the small difference of $0.04\ m_y$ and of $2°.2\ m_y/R$ is perfectly justified.

8. — Conclusion

We are persuaded that the study we have just presented enables us to conclude that the semi-graphic method is efficient. We also believe that it may be used for shorter periods. For this purpose it will suffice to bring the "columns" close together. Consequently, we intend to study this possibility and to present the conclusions at the same time as a complete study of the formulae of the errors obtained in the preceding paragraph.

In conclusion, we desire to express to Cdr. G. Lemiere our sincere thanks for the invaluable assistance given us in the calculations of all the tables, and of the analysis itself, and especially in the translation from the original text in Portuguese.