ADJUSTMENT OF AERIAL TRIANGULATION

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1. Introduction

This paper deals with the least square adjustment of an aerial triangulation strip by the Verdin-Moreau method, to which have been introduced some sound simplifications.

It is well known that, in almost all methods of adjustment, corrections are computed for the machine-coordinates; then the corrected machine-coordinates are transformed into ground-coordinates. However, in the Verdin-Moreau method, adjusted transformation elements are found for each stereo-model so that the machine-coordinates x, y and z can be directly transformed into ground-coordinates X, Y and Z, without any previous corrections.

In the original method, the adjusted horizontal ground-coordinates of the points observed along the strip are found in terms of the adjusted values of the azimuth A and the scale denominator K, computed for each stereo-model, and the adjusted heights of these points are obtained in terms of the adjusted values of the general tip Φ and the general tilt Ω, for each pair.

As the computations were somewhat extensive in their original form, the authors tried to simplify them, as described in the Belgian review " Photogrammétrie " (no. 43, March 1956), in an article by A. Verdin and E. Moreau. In fact, they pointed out that the horizontal adjustment is considerably simplified if the strips have a N-S or E-W direction. In addition, the height adjustment can always be simplified if Φ and Ω are small, as they usually are.

It will be shown later on that, if weights are not used, the simplification devised by the authors is a general one and not a special case of strips in a N-S or E-W direction. In this article will be given the complete theory as well as the practical application of the simplified method, which has been in use for some years in Brazil.

To simplify further development, it is convenient to explain how to find Φ for each stereo-model of the strip, both in the aerial-levelling and in the $bz = 0$ method.

To generalize the application of the method, the formulas will be derived in such a way that they can be applied to the three usual bridging methods, i.e., the aero-levelling, the aerial-traverse ($bz = 0$) and the aero-polygon, in which no values of bz are known beforehand (*). Thus the general formulas will be derived to compute the successive values of the general tip Φ for the stereo-models by considering the aero-levelling and the aerial-traverse. In this way the aero-polygon can be looked upon as a particular case of the above-mentioned bridging methods.

It is well known that in both the aerial-levelling and the $bz = 0$ method it is always necessary, in each stereo-model, to modify the tip φ of the rear projecting camera, in order to eliminate the y paralaxes. Accordingly, projector i will have a φ tip in the $[(i-1), i]$ stereo-model, and a φ' tip in the $[i, (i+1)]$ model. The difference $\varphi' - \varphi = \delta\varphi$ is the Φ difference for the two stereo-models, as can be seen in fig. 1.1. If we designate by Φ_{12} the residual tip of the first model 1-2, the tip of the model 2-3 will be :

$$\Phi_{23} = \Phi_{12} + \delta\varphi_2 + \Delta\Phi_2$$

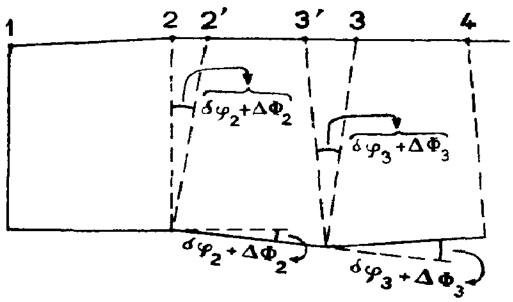

FIG. 1.1

where $\Delta\Phi_2$ is the accidental error in the $\delta\varphi_2$ determination. By extension of the same reasoning, we have :

$$\begin{aligned} \Phi_{23} &= \Phi_{12} + \delta\varphi_2 + \Delta\Phi_2 \\ \Phi_{34} &= \Phi_{23} + \delta\varphi_3 + \Delta\Phi_3 \\ \Phi_{45} &= \Phi_{34} + \delta\varphi_4 + \Delta\Phi_4 \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{1a}$$

In the aero-levelling, the $\delta\varphi$ values are approximately equal to the angle γ between the earth's rays correspondent to the positions of two successive exposures. In the $bz = 0$ method, this angle is also included in γ. In the aero-polygon, we have approximately $\delta\varphi = \gamma$. Thus we can see from expressions (1a) that each value of Φ contains the systematic accumulation of γ in all three methods of bridging. The effect of γ, or of any similar

(*) The aero-levelling is performed by using bz values as given by the statoscope; the aerial-traverse is the Poivilliers method "Cheminement photographique à altitude constante" and the aero-polygon is the conventional method of bridging.

systematic error, can be ignored as they are eliminated in the adjustment process. Thus, in the expressions (1*a*), the values of $\delta\varphi$ are the φ differences read on the instrument's scale and will be zero for the free bridging.

2. Least square absolute orientation of the end pairs

The following development is concerned with a strip in which control points are available only on the first and last stereo-models. Therefore, the transformation formulas will be derived, and they will show which elements are to be determined in function of the ground control.

If x, y and z are the machine-coordinates of any observed point, and if K is the scale denominator of the stereo-model, the lengths on the ground will be Kx, Ky and Kz. If X, Y and Z are the ground-coordinates of the considered point, fig. 2.1 represents the instrument $xO'y$ coordinate system in relation to the ground XOY system for the planimetry, and fig. 2.2 shows the instrument plane xOy which is the datum for the height measurements. From fig. 2.1, the following formulas are immediately obtained for transformation of the plane coordinates :

$$\begin{aligned} X &= P + Kx \cos A + Ky \sin A \\ Y &= Q - Kx \sin A + Ky \cos A \end{aligned} \qquad (2a)$$

or, by putting

$$\begin{aligned} K \cos A &= e \\ K \sin A &= f, \end{aligned} \qquad (2b)$$

$$\begin{aligned} X &= P + ex + fy \\ Y &= Q + ey - fx \end{aligned} \qquad (2c)$$

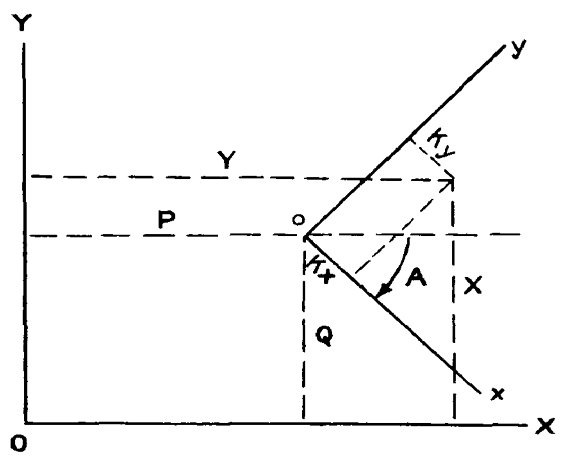

FIG. 2.1

Fig. 2.2 shows that the plane xOy is inclined to the horizontal plane and that the components of the inclination are Φ and Ω. Hence, the true

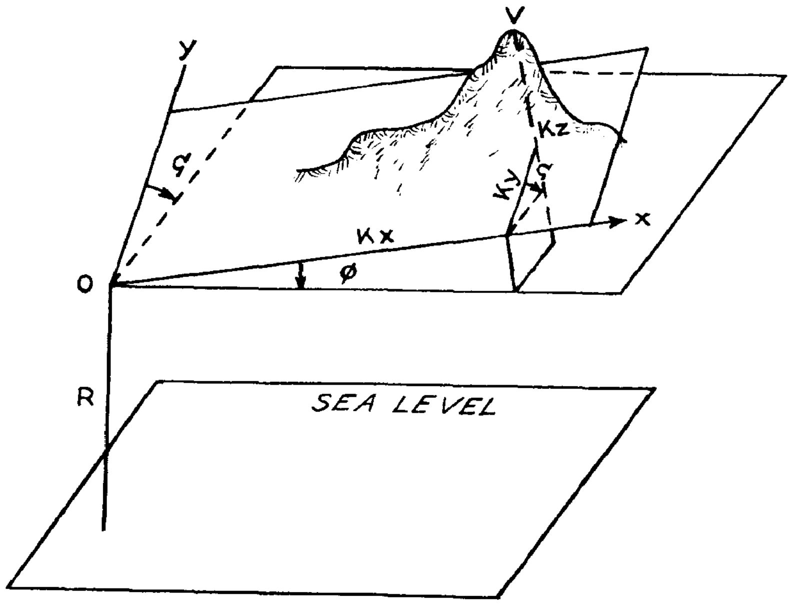

FIG. 2.2

height of a point V is given with sufficient approximation by :

$$Z = R + Kz + Kx \tan \Phi + Ky \tan \Omega$$

Now, as Φ and Ω are small, their tangents can be substituted by their respective arcs and the above expression can be written as follows :

$$Z = R + Kz + K\Phi x + K\Omega y$$

or, by putting

$$K\Phi = E \qquad K\Omega = F, \tag{2d}$$

$$Z = R + Kz + Ex + Fy \tag{2e}$$

Hence, if we determine P, Q, R, e, f. E and F in function of the ground control points, expressions (2*b*), (2*c*), (2*d*) and (2*e*) can be used to find the ground coordinates of any other observed point of the stereo-model.

If more than two planimetric ground control points are known, P, Q, e and f can be determined by the least square method. Similarly, it may be used for R, E and F, if more than three ground control points are known. As the values of Φ and Ω are always very small, the horizontal and vertical adjustment can be treated independently. However, as the reasoning is the same for both problems, they will be treated together.

Let (X_1, Y_1), (X_2, Y_2), etc., be the horizontal ground coordinates of n points and Z_1, Z_2, etc., the true heights of n points, not necessarily the same. Then, if (x_1, y_1), (x_2, y_2), etc., are the horizontal machine-coordinates, and z_1, z_2, etc., are the instrumental heights, we obtain from (2*c*) and (2*e*) :

$$X_1 = P + ex_1 + fy_1 \quad Y_1 = Q + ey_1 - fx_1 \quad Z_1 - Kz_1 = R + Ex_1 + Fy_1$$
$$X_2 = P + ex_2 + fy_2 \quad Y_2 = Q + ey_2 - fx_2 \quad Z_2 - Kz_2 = R + Ex_2 + Fy_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2f)$$

$$X_n = P + ex_n + fy_n \quad Y_n = Q + ey_n - fx_n \quad Z_n - Kz_n = R + Ex_n + Fy_n$$

The sum of each of the above groups of expressions gives us the following expressions, where the Gaussian symbol for summation is used :

$$\begin{aligned}[X] &= nP + e[x] + f[y] \\ [Y] &= nQ + e[y] - f[x] \\ [Z - Kz] &= nR + E[x] + F[y]\end{aligned}$$

Now, dividing by n and putting

$$\begin{aligned}[X]/n &= X_g \quad [x]/n = x_g \\ [Y]/n &= Y_g \quad [y]/n = y_g \\ [Z]/n &= Z_g \quad [z]/n = z_g\end{aligned} \tag{2g}$$

we have

$$\begin{aligned}X_g &= P + ex_g + fy_g \\ Y_g &= Q + ey_g - fx_g \\ Z_g - Kz_g &= R + Ex_g + Fy_g\end{aligned} \tag{2h}$$

Hence, if we introduce in (2f) the values of P, Q and R taken from (2h), and we put

$$\begin{aligned}X - X_g &= X' \quad x - x_g = x' \\ Y - Y_g &= Y' \quad y - y_g = y' \\ Z - Z_g &= Z' \quad z - z_g = z'\end{aligned} \tag{2i}$$

then expressions (2f) may be simplified as follows :

$$\begin{array}{lll} X'_1 = ex'_1 + fy'_1 & Y'_1 = ey'_1 - fx'_1 & Z'_1 - Kz'_1 = Ex'_1 + Fy'_1 \\ X'_2 = ex'_2 + fy'_2 & Y'_2 = ey'_2 - fx'_2 & Z'_2 - Kz'_2 = Ex'_2 + Fy'_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ X'_n = ex'_n + fy'_n & Y'_n = ey'_n - fx'_n & Z'_n - Kz'_n = Ex'_n + Fy'_n \end{array}$$

The groups of expressions in X′ and Y′ are the condition equations for the horizontal adjustment and the group in Z′ the condition equation for the height adjustment. Thus we have the following system of normal equations for the horizontal adjustment :

$$\left\{\begin{aligned}([x'x'] + [y'y'])\,e &= [x'X'] + [y'Y'] \\ ([x'x'] + [y'y'])\,f &= -[x'Y'] + [y'X']\end{aligned}\right. \tag{2j}$$

Similarly, we obtain for the height adjustment the following system of normal equations :

$$\left\{\begin{aligned}[x'x']\,E + [x'y']\,F &= [x'(Z' - Kz')] \\ [x'y']\,E + [y'y']\,F &= [y'(Z' - Kz')]\end{aligned}\right. \tag{2k}$$

The solutions of (2j) and (2k) give us all the terms of (2g), except P, Q and R. Hence values of P, Q and R can be computed afterwards. Forms 2-I and 2-II show the complete solution of the problem. Some readily comprehensible checking operations can be seen there.

Form 2-I

First pair

POINTS	x	y	$x' = x - x_g$	$y' = y - y_g$	X	Y	$X' = X - X_g$	$Y' = Y - Y_g$
PFP 16	1 680.80	- 5 901.10	-1 192.60	-2 372.30	67 704.99	209 166.35	- 225.91	+2 113.36
PFM 33A	4 141.80	- 5 661.00	+1 268.40	-2 132.20	66 153.24	207 936.77	- 1 777.66	+ 883.78
PFP 14	4 049.50	- 1 178.00	+1 176.10	+2 350.80	68 158.17	204 962.43	+ 227.27	-2 090.56
Δ P 15	1 621.50	- 1 375.10	-1 251.90	+2 153.70	69 707.21	206 146.40	+ 1 776.31	- 906.59
SUMS	11 493.60	-14 115.20	0.0	0.0	31 723.61	828 211.95	+ 0.01	- 0.01
MEAN	2 873.40	- 3 528.80	⟵ (x_g, y_g)	MEAN	7 930.90	207 052.99	⟵ (X_g, Y_g)	

$S_1 = [x'x'] + [y'y'] = 26\ 320\ 366.60$ $e = S_2/S_1 = 0.672741$ $K^2 = e^2 + f^2 = 0.64048449$

$S_2 = [x'X'] + [y'Y'] = -\ 17\ 706\ 764.90$ $f = S_3/S_1 = +\ 0.433479$ $K = 0.800303$

$S_3 = -[x'Y'] + [y'X'] = 11\ 409\ 312.27$ $tgA = S_3/S_2 = 0.644347$ $A = 32°\ 47'\ 44''$

CHECK

$X_g = 67\ 930.90$	$Y_g = 207\ 052.99$	$X_c = P + ex + fy$
$-ex_g = +\ 1\ 933.05$	$-ey_g = -\ 2\ 373\ 97$	$Y_c = Q + ey - fx$
$fy_g = +\ 1\ 529.66$	$fx_g = +\ 1\ 245.56$	$[v]_x = [v]_y = 0$
$P = 71\ 393.61$	$Q = 205\ 924.58$	

POINTS	ex	fy	X_c	$v = X - X_c$	ey	-fx	Y_c	$v = Y - Y_c$
	-1 130.74	- 2 558.00	7 704.87	+ 0.12	-3 969.91	+ 728.59	209 165.89	+ 0.46
	-2 786.36	- 2 453.92	6 153.33	- 0.09	+3 808.38	- 1 795.38	207 937.57	- 0.80
	-2 724.26	- 510.64	8 158.71	- 0.54	+ 792.49	- 1 755.37	204 961.69	+ 0.74
	-1 090.85	- 596.08	9 706.68	+ 0.53	+ 925.08	+ 702.89	206 146.75	- 0.35

Last pair

POINTS	x	y	$x' = x - x_g$	$y' = y - y_g$	X	Y	$X' = X - X_g$	$Y' = Y - Y_g$
PFA	27 398.10	- 5 646.40	- 980.05	-2 173.23	50 436.15	197 822.36	- 287.47	+1 898.94
PF 23	29 455.80	- 5 819.20	+1 077.65	-2 346.03	48 970.60	197 040.93	- 1 753.02	+1 117.51
P 19	29 249.30	- 923.40	+ 871.15	+2 549.77	51 246.87	193 816.02	+ 523.25	-2 107.40
PFP 20	27 409.40	- 1 503.70	- 968.75	+1 969.47	52 240.85	195 014.37	+ 1 517.23	- 909.05
SUMS	113 512.60	-13 892.70	0	- 0.02	202 894.47	783 693.68	- 0.01	0
MEAN	28 378.15	- 3 473.17	(x_g, y_g)	MEAN	50 723.62	195 923.42	(X_g, Y_g)	

$S_1 = [x'x'] + [y'y'] = 24\ 426\ 130.94$ $e = S_2/S_1 = -\ 0.676885$ $K^2 = (e^2 + f^2) = 0.64905148041$

$S_2 = [x'X'] + [y'Y'] = -\ 16\ 533\ 671.72$ $f = S_3/S_1 = +\ 0.436896$ $K = 0.805637$

$S_3 = -[x'Y'] + [y'X'] = +\ 10\ 671\ 672.88$ $tgA = S_3/S_2 = 0.645450$ $A = 32°\ 50'\ 25''$

CHECK

$X_g = 50\ 723.62$	$Y_g = 195\ 923.42$	$X_c = P + ex + fy$
$-ex_g = +\ 19\ 208.74$	$-ey_g = -\ 2\ 350.94$	$Y_c = Q + ey - fx$
$-fy_g = +\ 1\ 517.41$	$fx_g = +\ 12\ 398.30$	$[v]_x = [v]_y = 0$
$P = 71\ 449.77$	$Q = 205\ 970.78$	

POINTS	ex	fy	X_c	$v = X - X_c$	ey	-fx	Y_c	$v = Y - Y_c$
	-18 545.36	- 2 466.88	50 437.53	- 1.38	+3 821.96	-11 970.12	197 822.62	- 0.26
	-19 938.19	- 2 542.38	48 969.20	+ 1.40	+3 938.93	-12 869.12	197 040.59	+ 0.34
	-19 798.41	- 403.43	51 247.93	- 1.06	+ 625.04	-12 778.90	193 816.92	- 0.90
	-18 553.01	- 656.96	52 239.80	+ 1.05	+1 017.83	-11 975.06	195 013.55	+ 0.82

Form 2-II

First pair — $K_o = 0.8$ — $K' = K/K_o = 1.000379$*

	Points	x	y	z	$x' = x - x_g$	$y' = y - y_g$	$z' = z - z_g$	Z	$Z' = Z - Z_g$	$K'z'$	$Z' - K'z'$
1	PFP 16	1 680.80	-5 901.10	508.00	-1 192.60	-2 372.30	-10.07	507.99	-10.23	-10.10	-0.13
2	PFM 33A	4 141.80	-5 661.00	539.00	+1 268.40	-2 132.20	+20.93	539.44	+21.22	+21.00	+0.22
3	PFP 14	4 049.50	-1 178.00	499.30	+1 176.10	+2 350.80	-18.77	499.35	-18.87	-18.84	-0.03
4	ΔP 15	1 621.50	-1 375.10	526.00	-1 251.90	+2 153.70	+ 7.93	526.12	+ 7.90	+ 7.96	-0.06
	Sums	11 493.60	-14 115.20	2 072.30	0.00	0.00	0.02	2 072.90	0.02	0.02	0.00
	Means	2 873.40	-3 528.80	518.07	⟵ x_g, y_g, z_g ⟵	mean ⟶ Z_g		518.22			

$a = [x'x'] = +\ 5\,981\,598.14$ $\quad a/b = +30.947670 \quad d/b = +0.002452$

$b = [x'y'] = +\ 193\,281.35$ $\quad -b/c = -\ 0.095031 \quad -k/c = +0.000018$

$c = [y'y'] = +20\,338\,768.46$ $\quad D_\varepsilon = +30.852639 \quad N_\varepsilon = +0.002470$

$d = [x'(Z' - K'z')] = +\ 473.92$

$k = [y'(Z' - K'z')] = -\ 360.43$ $\quad b/a = +\ 0.032312 \quad d/a = +0.000079$

$-c/b = -10.522900 \quad -k/b = +0.001865$

$E = N_\varepsilon/D_\varepsilon = +\ 0.000080$ $\quad D_F = -10.490588 \quad N_F = +0.001944$

$F = N_F/D_F = -\ 0.000185$

CHECK

Points	Ex	Fy	K'z	Z_c	v
1	+0.13	+1.09	508.19	508.49	+0.50
2	+0.33	+1.05	539.20	539.66	+0.22
3	+0.32	+0.21	499.49	499.10	-0.25
4	+0.13	+0.25	526.20	525.66	-0.46
$v = Z_c - Z$;		$[v] = 0$			

$R = Z_g - K'z_g - Ex_g - Fy_g = 518.22 - 518.26 - 0.23 - 0.65 = -0.92$

Last pair — $K_o = 0.8$ — $K' = K/K_o = 1\,007046$*

	Points	x	y	z	$x' = x - x_g$	$y' = y - y_g$	$z' = z - z_g$	Z	$Z' = Z - Z_g$	$K'z'$	$Z' - K'z$
1	PFA	27 398.10	- 5 646.40	736.50	- 980.05	-2 173.23	+28.50	713.20	+25.53	+28.70	-3.17
2	PFP 23	29 455.80	- 5 819.20	735.50	+1 077.65	-2 346.03	+27.50	712.32	+24.65	+27.69	-3.04
3	P19 Δ	29 249.30	- 923.40	679.00	+ 871.15	+2 549.77	-29.00	660.43	-27.24	-29.20	+1.96
4	PFP 20	27 409.40	- 1 503.70	681.00	- 968.75	+1 969.47	-27.00	664.73	-22.94	-27.19	+4.25
	Sums	113 512.60	-13 892.70	2 832.00	0.00	- 0.02	0.00	2 750.68	0.00	0.00	0.00
	Means	28 378.15	- 3 473.17	708.00	⟵ x_g, y_g, z_g ⟵	mean ⟶ Z_g		687.67			

$a = [x'x'] = +\ 3\,819\,206.41$ $\quad a/b = -\ 44.922850 \quad d/b = +0.030336$

$b = [x'y'] = -\ 85\,017.09$ $\quad -b/c = +\ 0.004125 \quad -k/c = -\ 0.001329$

$c = [y'y'] = +20\,606\,924.53$ $\quad D_\varepsilon = -\ 44.918725 \quad N_\varepsilon = +0.029007$

$d = [x'(Z' - K'z')] = -\ 2\,579.05$

$k = [y'(Z' - K'z')] = +\ 27\,388.87$ $\quad b/a = -\ 0.022260 \quad d/a = +0.006752$

$-c/b = +242.385920 \quad -k/b = +0.322157$

$E = N_\varepsilon/D_\varepsilon = -\ 0.000646$ $\quad D_F = 242.363660 \quad N_F = 0.328909$

$F = N_F/D_F = +\ 0.001357$

CHECK

Points	Ex	Fy	K'z	Z_c	v
1	-17.70	-7.66	741.69	714.05	+0.85
2	-19.03	-7.90	740.68	711.47	-0.85
3	-18.90	-1.25	683.78	661.35	+0.92
4	-17.71	-2.04	685.80	663.77	-0.96
$v = Z_c - Z$; $[v] = 0$					

$R = Z_g - K'z_g - Ex_g - Fy_g = 687.67 - 712.99 + 18.33 + 4.71 = -2.28$

(•) K' is used instead of K because gears for K_o were available to transform z readings in natural height differences.

3. Adjustment of the strip

Before the derivation of the condition equations it is possible to introduce a considerable simplification into the development concerning the horizontal adjustment. As we have shown in section 2, if the adjusted values of P, Q, e and f are known for a stereo-model, then the ground-coordinates of every point observed can be computed by the expression (2*a*). Hence it will be very convenient to use the variables e and f, as did VERDIN in his interpolation method. But A and K are independent variables and expressions (2*b*) show that both e and f are functions of A and K. However if we assign equal weights to A and K, it is very easy to prove that e and f also have equal weights and are correlation free, which seems paradoxical.

If we designate by ΔA, ΔK, Δe and Δf, the corrections to be applied to A, K, e and f respectively, we can differentiate the expressions (2*b*) and substitute the corrections for the differentials. Thus we have :

$$\Delta e = - K \sin A \, \Delta A + \cos A \, \Delta K$$
$$\Delta f = K \cos A \, \Delta A + \sin A \, \Delta K$$

Now, if we put

$$\Delta K / K = \Delta \lambda ,$$

we can write

$$\Delta e = - K \sin A \, \Delta A + K \cos A \, \Delta \lambda$$
$$\Delta f = K \cos A \, \Delta A + K \sin A \, \Delta \lambda$$

or, according to (2*b*),

$$\begin{aligned} \Delta e &= - f \, \Delta A + e \, \Delta \lambda \\ \Delta f &= e \, \Delta A + f \, \Delta \lambda \end{aligned} \qquad (3a)$$

If we compute the weight numbers by Tienstra's symbolic method, we have to substitute Δe, Δf, ΔA and $\Delta\lambda$ in (3*a*) by the symbols Q_e, Q_f, Q_A and Q_λ respectively. Thus,

$$Q_e = - f Q_A + e Q_\lambda$$
$$Q_f = e Q_A + f Q_\lambda$$

Hence,

$$Q_{ee} = f^2 Q_{AA} + e^2 Q_{\lambda\lambda} - 2ef Q_{A\lambda}$$
$$Q_{ff} = e^2 Q_{AA} + f^2 Q_{\lambda\lambda} + 2ef Q_{A\lambda}$$
$$Q_{ef} = - ef \, (Q_{AA} - Q_{\lambda\lambda}) + (e^2 - f^2) \, Q_{A\lambda}$$

Now, as A and λ are correlation free and we assign equal weights to A and λ, we have $Q_{AA} = Q_{\lambda\lambda}$ and $Q_{A\lambda} = 0$. Thus we can conclude that

$$Q_{ee} = Q_{ff} \text{ and } Q_{ef} = 0$$

Consequently, e and f also have equal weights and are correlation free. Therefore, it is possible to adjust directly the values of e and f.

Now we are in a position to derive the condition equations. If we look at form 2-I, we see that the values of P, Q, e and f differ in the first and last stereo-models. Thus we can write the literal form of the closing errors :

$$\begin{aligned} w_P &= P_{(n-1),n} - P_{12} \\ w_Q &= Q_{(n-1),n} - Q_{12} \\ w_e &= e_{(n-1),n} - e_{12} \\ w_f &= f_{(n-1),n} - f_{12} \end{aligned} \tag{3b}$$

In addition, form 2-II gives immediately the closing errors :

$$\begin{aligned} w_R &= R_{(n-1),n} - R_{12} \\ w_F &= F_{(n-1),n} - F_{12} \end{aligned} \tag{3c}$$

But the difference $E_{(n-1),n} - E_{12}$ is not a closing error, except in the case of the aero-polygon. In fact, we have from (1*a*) and (2*d*) :

$$\begin{aligned} E_{23} &= E_{12} + \delta E_2 + \Delta E_2 \\ E_{34} &= E_{23} + \delta E_3 + \Delta E_3 \\ E_{45} &= E_{34} + \delta E_4 + \Delta E_4 \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \end{aligned} \tag{3d}$$

where

$$\delta E = K\delta\varphi \tag{3e}$$

Thus, if we add the expressions (2*d*), we obtain :

$$E_{(i-1),i} = E_{12} + [\delta E_j]_2^{i-1} + [\Delta E_f]_2^{i-1}$$

For the whole strip we have $i = n$, and the sum of all values of ΔE will be the closing error. Hence

$$w_E = E_{(n-1),n} - E_{12} - [\delta E] \tag{3f}$$

Now the ground-coordinates X, Y and Z can be considered free from errors. Therefore, if we differentiate the expressions (2*c*) and (2*e*), the results will be as follows :

$$\begin{aligned} \Delta P &= - x\Delta e - y\Delta f \\ \Delta Q &= y\Delta e - x\Delta f \\ \Delta R &= - z\Delta K - x\Delta E - Y\Delta F \end{aligned} \tag{3g}$$

These errors have a cumulative effect; consequently if we add all the values for the whole strip, the sum of ΔP will be equal to w_P. With the same reasoning we can write the following condition equations for the horizontal adjustment.

$$\begin{aligned} -[x\Delta e] - [y\Delta f] &= w_P \\ [y\Delta e] - [x\Delta f] &= w_Q \end{aligned} \tag{3h}$$

The other two equations are obviously

$$\begin{aligned} [\Delta e] &= w_e \\ [\Delta f] &= w_f \end{aligned} \tag{3i}$$

For the height adjustment we have, from the last expression of (3*g*) :

$$-[z\Delta K] - [x\Delta E] - [y\Delta F] = w_R \tag{3j}$$

and the other two will be :

$$[\Delta E] = w_E \qquad [\Delta F] = w_F \tag{3k}$$

All the condition equations are now derived and the adjustment problem can be resolved in a special manner. The method of solution that we have chosen, adjustment in two groups, was described by WRIGHT and HAYFORD in their book, "Adjustment of Observations", (1906). The method was successfully used by the authors to adjust a central point polygon. But the so called "solution in two groups" was generalized by TIENSTRA as follows :

> " Every problem of adjustment may be divided into an arbitrary number of phases, provided that in each succeeding phase the co-factors resulting from the preceding phase(s) are used ".

This possibility was considered by TIENSTRA as the principal property of the least square method. We shall use this property as the means of obtaining some important simplifications, in the same way as WRIGHT and HAYFORD succeeded in doing in the adjustment of the central point polygon.

The first phase of the adjustment can be worked out by considering only equations (3*i*) and (3*k*). Each one of these equations is independent of the others, thence they may be adjusted independently. Thus a first group of corrections is found in the usual manner :

$$\overline{de} = w_e / (n - 2) \qquad \overline{df} = w_f / (n - 2) \qquad \overline{dE} = w_E / (n - 2) \qquad \overline{dF} = w_F / (n - 2) \tag{3l}$$

where n is the number of photographs in the strip. It is easy to understand that these first corrections eliminate any systematic error, such as the above-mentioned one resulting from the curvature of the earth.

Now a new correction is needed to take into account the condition equations (3*h*) and (3*j*). Hence the total corrections can be written as follows :

$$\Delta e = de + \overline{de} \qquad \Delta f = df + \overline{df} \qquad \Delta E = dE + \overline{dE} \qquad \Delta F = dF + \overline{dF} \tag{3m}$$

These values can be introduced in (3*h*), (3*i*) and (3*k*), and the results will be :

$$- [x\,de] - [y\,df] - \overline{de}[x] - \overline{df}[y] = w_P$$
$$[y\,de] - [x\,df] + \overline{de}[y] - \overline{df}[x] = w_Q$$
$$[de] = 0$$
$$[df] = 0$$

for the horizontal adjustment, and

$$
\begin{aligned}
-[z\,\Delta\,K]-[x\,dE]-[y\,dF]-\overline{dE}[x]-\overline{dF}[y] &= w_R\\
[dE] &= 0\\
[dF] &= 0
\end{aligned}
$$

for the height adjustment. But if we put

$$
\begin{aligned}
w_P+\overline{de}[x]+\overline{df}[y] &= w'_P\\
w_Q-\overline{de}[y]+\overline{df}[x] &= w'_Q\\
w_R+[z\,\Delta\,K]+\overline{dE}[x]+\overline{dF}[y] &= w'_R
\end{aligned}
\qquad (3n)
$$

the final condition equations will be :

$$
\begin{aligned}
-[x\,de]-[y\,df] &= w'_P\\
[y\,de]-[x\,df] &= w'_Q\\
[de] &= 0\\
[df] &= 0
\end{aligned}
\qquad (3o)
$$

for the horizontal adjustment, and

$$
\begin{aligned}
-[x\,dE]-[y\,dF] &= w'_R\\
[dE] &= 0\\
[dF] &= 0
\end{aligned}
\qquad (3p)
$$

for the height adjustment. If we designate C_1, C_2, C_3 and C_4 as the correlative factors for the horizontal adjustment and C'_1, C'_2 and C'_3 as the correlative factors for the height adjustment, we can write the correlative equations as follows :

$$
\begin{aligned}
de_2 &= -x_2C_1+y_2C_2+C_3\\
de_3 &= -x_3C_1+y_3C_2+C_3\\
&\dots\dots\dots\\
&\dots\dots\dots\\
df_2 &= -y_2C_1-x_2C_2+C_4\\
df_3 &= -y_3C_1-x_3C_2+C_4\\
&\dots\dots\dots\\
&\dots\dots\dots
\end{aligned}
\qquad (3q)
\qquad\qquad
\begin{aligned}
dE_2 &= -x_2C'_1+C'_2\\
dE_3 &= -x_3C'_1+C'_2\\
&\dots\dots\dots\\
&\dots\dots\dots\\
dF_2 &= -y_2C'_1+C'_3\\
dF_3 &= -y_3C'_1+C'_3\\
&\dots\dots\dots\\
&\dots\dots\dots
\end{aligned}
\qquad (3r)
$$

Now, as

$$[de]=[df]=[dE]=[dF]=0,$$

we can add each group of the above expressions for the $(n-2)$ stereo-models, and write the following results :

$$
\begin{aligned}
(n-2)C_3-C_1[x]+C_2[y] &= 0 \qquad & (n-2)C_2-C_1[x] &= 0\\
(n-2)C_4-C_1[y]-C_2[x] &= 0 \qquad & (n-2)C_3-C_1[y] &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
C_3 &= C_1[x]/(n-2)-C_2[y]/(n-2)\\
C_4 &= C_1[y]/(n-2)+C_2[x]/(n-2)\\
C'_2 &= C'_1[x]/(n-2)\\
C'_3 &= C'_1[y]/(n-2)
\end{aligned}
$$

If we introduce these values in (3*q*) and (3*r*), and we put

$$x_i - [x] / (n-2) = a_i$$
$$y_i - [y] / (n-2 = b_i \qquad (3s)$$

the correlative equations will read as follows :

$$\begin{aligned} de_2 &= -a_2C_1 + b_2C_2 \\ de_3 &= -a_3C_1 + b_3C_2 \\ &\dots\dots \\ &\dots\dots \\ df_2 &= -b_2C_1 - a_2C_2 \\ df_3 &= -b_3C_1 - a_3C_2 \\ &\dots\dots \\ &\dots\dots \end{aligned} \qquad (3t) \qquad \begin{aligned} dE_2 &= -a_2C_1' \\ dE_3 &= -a_3C_1' \\ &\dots\dots \\ &\dots\dots \\ dF_2 &= -b_2C_1' \\ dF_3 &= -b_3C_1' \\ &\dots\dots \\ &\dots\dots \end{aligned} \qquad (3u)$$

Then the normal equations will be :

$$\Big[\ a_i^2 + b_i^2\ \Big]_2^{n-1} C_1 = -w_P'$$
$$\Big[\ a_i^2 + b_i^2\ \Big]_2^{n-1} C_2 = -w_Q' \qquad (3v)$$

for the horizontal adjustment, and

$$\Big[\ a_i^2 + b_i^2\ \Big]_2^{n-1} C_1' = -w_R' \qquad (3x)$$

for the height adjustment. Expressions (3*v*) and (3*x*) show how amazing is the final simplification. Indeed, as all the correlative factors are obtained in the same manner, we solve a simple equation of the form $ax = b$, and the coefficient is the same for the three correlative factors.

Form 3-I shows a complete solution for the horizontal adjustment, and form 3-II shows the height adjustment. The formulas to be used for the computation of Δe and Δf and the successive computation of e, f, P and Q are given on the bottom of form 3-I, and those used to find the values of ΔK, ΔE and ΔF and the successive values of R are shown on the bottom of form 3-II. As the height adjustment follows the horizontal adjustment the values of K can be derived from expression (2*b*) which gives

$$K = \sqrt{e^2 + f^2} \ (*)$$

where e and f are taken from form 3-I. But it is also seen in form 3-II that values of K′ are found in the same way as those seen in form 2-II, where a foot-note explains the reason for its use.

(*) BARLOW's *tables can be used to compute* K.

Form 3-I

PAIR	x y	a = x - x_o b = y - y_o	$10^6 aC_1$ $10^6 bC_1$	$10^6 bC_2$ $-10^6 aC_2$	$\Delta e 10^6$ $\Delta f 10^6$	e f	$-x\Delta e$ $-y\Delta e$	$-y\Delta f$ $x\Delta f$	P Q
1/2	- -	- -	- -	- -	- -	-0.672741 +0.433479	- -	- -	71.393.61 205.924.57
2/3 N_2	4 481.30 - 3 576.00	-11 489.32 - 189.38	-434 - 7	+ 2 -112	- 809 + 192	-0.673550 +0.433671	+ 3.63 - 2.89	+ 0.69 + 0.86	71 397.93 205 922.55
3/4 N_3	6 889.10 - 3 472.30	- 9 081.52 - 85.68	-343 - 3	+ 1 - 88	- 719 + 220	-0.674269 +0.433891	+ 4.95 - 2.49	+ 0.76 + 1.52	71 403.64 205 921.58
4/5 N_4	9 145.40 - 3 384.10	- 6 825.22 + 2.52	-258 + 0	- 0 - 67	- 635 + 244	-0.674904 +0.434135	+ 5.81 - 2.14	+ 0.83 + 2.23	71 410.28 205 921.67
5/6 N_5	11 474.90 - 3 532.50	- 4 495.72 - 145.88	-170 - 5	+ 1 - 44	- 546 + 262	-0.675450 +0.434397	+ 6.27 - 1.93	+ 0.92 + 3.01	71 417.47 205 922.75
6/7 N_6	13 614.20 - 3 358.10	- 2 356.42 + 28.52	- 89 + 1	- 0 - 23	- 466 + 289	-0.675916 +0.434686	+ 6.34 - 1.56	+ 0.97 + 3.93	71 424.78 205 925.12
7/8 N_7	15 852.40 - 3 498.00	- 118.22 - 111.38	- 4 - 4	+ 1 - 1	- 380 + 306	-0.676296 +0.434992	+ 6.02 - 1.33	+ 1.07 + 4.85	71 431.87 205 928.64
8/9 N_8	18 494.30 - 3 277.40	+ 2 523.68 + 109.22	+ 95 + 4	- 1 + 25	- 283 + 340	-0.676579 +0.435332	+ 5.23 - 0.93	+ 1.11 + 6.29	71 438.21 205 934.00
9/10 N_9	20 497.40 - 3 323.00	+ 4 526.78 + 63.62	+171 + 2	- 0 + 44	- 206 + 357	-0.676785 +0.435689	+ 4.22 - 0.68	+ 1.18 + 7.32	71 443.61 205 940.64
10/11 N_{10}	22 778.10 - 2 854.90	+ 6 807.48 + 531.72	+257 + 20	- 5 + 66	- 125 + 397	-0.676910 +0.436086	+ 2.85 - 0.35	+ 1.13 + 9.04	71 447.59 205 949.33
11/12 N_{11}	24 967.70 - 3 407.60	+ 8 997.08 - 20.98	+340 - 1	+ 0 + 88	- 37 + 398	-0.676947 +0.436484	+ 0.92 - 0.12	+ 1.36 + 9.94	71 449.87 205 959.15
12/13 N_{12}	27 482.10 - 3 569.00	+11 511.48 - 182.38	+435 - 7	+ 1 +112	+ 59 + 416	-0.676888 +0.436900	- 1.62 + 0.21	+ 1.48 +11.43	71 449.73 205 970.79
Σ = (n - 2) x_o Σ = (n - 2) y_o	175 676.90 -37 252.90	- 0.08 - 0.08	0 0	0 0	-4 147 +3 421		+44.62 -14.21	+11.50 +60.42	+ 56.12 + 46.21
x_o y_o	+15 970.62 - 3 386.62								

$e_{(n-1),n}$ = - 0.676885 $f_{(n-1),n}$ = + 0.436896 $P_{(n-1),n}$ = 71 449.77 $Q_{(n-1),n}$ = 205 970.78

$-e_{12}$ = + 0.672741 $-f_{12}$ = - 0.433479 $-P_{12}$ = - 71 393.61 $-Q_{12}$ = - 205 924.58

w_e = - 0.004144 w_f = + 0.003417 $\overline{de}$ [x] = - 66.23 $\overline{de}$ [y] = + 14.04

$\overline{df}$ [y] = - 11.59 $-\overline{df}$ [x] = - 54.64

$$\overline{de} = \frac{w_e}{n-2} = -0.000377 \qquad \overline{df} = \frac{w_f}{n-2} = +0.000311 \qquad w'_P = -21.66 \qquad w'_Q = +5.60$$

$$\Sigma(a_i^2 + b_i^2) = 573\ 913\ 097.40 \qquad C_1 = \frac{-10^6 w'_P}{\Sigma(a_i^2+b_i^2)} = +0.03774 \qquad C_2 = \frac{-10^6 w'_Q}{(a_i^2+b_i^2)} = -0.00975$$

Formulas : $\Delta e_i = a_i C_1 + b_i C_2 + \overline{de}$; $e_{i,(i+1)} = e_{(i-1),i} + \Delta e_i$; $P_{i,(i+1)} = P_{(i-1),i} - x_i \Delta e_i - y_i \Delta f_i$

$\Delta f_i = b_i C_1 - a_i C_2 + \overline{df}$; $f_{i,(i+1)} = f_{(i-1),i} + \Delta f_i$; $Q_{i,(i+1)} = Q_{(i-1),i} - y_i \Delta e_i + x_i \Delta f_i$

Form 3-II

PAIR	K^2	K	K'	ΔK'	z	$10^6 K\delta\varphi$	$10^6 a C'_1$	$10^6 \Delta E$	$10^6 E$	$10^6 b C'_1$	$10^6 \Delta F$	$10^6 F$	-ΔK'z	-ΔEx	-ΔFy	R
1/2	0.64048449	0.800303	1.000378	-981	511.0	- 8 676	-89	- 7 759	+ 80	- 1	+ 139	- 185	+0.50	+ 34.77	+0.50	- 0.92
2/3	0.64174014	0.801087	1.001359	-904	543.3	+ 9 904	-71	+10 839	-7 679	- 1	+ 139	- 46	+0.49	- 74.67	+0.48	+ 34.85
3/4	0.64290008	0.801810	1.002263	-833	552.5	- 7 620	-53	- 6 667	+3 160	0	+ 140	+ 93	+0.46	+ 60.97	+0.47	- 38.85
4/5	0.64396861	0.802477	1.003096	-750	577.0	+ 2 583	-35	+ 3 554	-3 507	- 1	+ 139	+ 233	+0.43	- 40.78	+0.49	+ 23.05
5/6	0.64493346	0.803077	1.003846	-688	591.7	+ 3 468	-18	+ 4 456	+ 47	0	+ 140	+ 372	+0.41	- 60.66	+0.47	- 16.81
6/7	0.64581436	0.803627	1.004534	-604	657.8	-11 007	- 1	-10 002	+4 503	- 1	+ 139	+ 512	+0.39	+158.56	+0.49	- 76.59
7/8	0.64659432	0.804111	1.005138	-523	636.7	+ 1 704	+19	+ 2 729	-5 499	+ 1	+ 141	+ 651	+0.33	- 50.47	+0.46	+ 82.85
8/9	0.64727309	0.804533	1.005666	-458	680.2	- 6 319	+35	- 5 278	-2 770	0	+ 140	+ 792	+0.31	+108.19	+0.47	+ 33.17
9/10	0.64786284	0.804899	1.006124	-400	731.5	+ 8 093	+53	+ 9 152	-8 048	+ 4	+ 144	+ 932	+0.29	-208.47	+0.41	+142.14
10/11	0.64837815	0.805219	1.006524	-309	734.7	+ 4 426	+70	+ 5 502	+1 104	0	+ 140	+1 076	+0.23	-137.37	+0.48	- 65.63
11/12	0.64877552	0.805466	1.006833	-213	694.0	- 8 352	+90	- 7 256	+6 606	- 1	+ 139	+1 216	+0.15	+199.41	+0.50	-202.29
12/13	0.64905141	0.805637	1.007046						650			+1 355				2.23
Σ						-11 796	0	- 730		0	1 540		+3.99	- 10.52	+5.22	

$E_{(n-1),n} = -\ 0.000646$
$-E_{12} = -\ 0.000080$
$-[K\delta\varphi] = +\ 0.011796$
$w_E = +\ 0.011070$

$\overline{dE} = \dfrac{w_E}{n-2} = +\ 0.001006$

$F_{(n-1),n} = +\ 0.001357$
$-F_{12} = +\ 0.000185$
$w_F = +\ 0.001542$

$\overline{dF} = \dfrac{w_F}{n-2} = +\ 0.000140$

$R_{(n-1),n} = -\ 2.28$
$-R_{12} = 0.92$
$dE\ [x] = 176.73$
$[xK\delta\varphi] = -\ 170.66$
$dF\ [y] = -\ 5.22$
$[z\ \Delta K'] = -\ 3.99$
$w'_r = -\ 4.50$

Formulas : $\Delta E_i = \overline{dE} + K\delta\varphi_i + a_i C'_1$; $\Delta F_i = \overline{dF} + b_i C'_1$; $E_{i,(i+1)} = E_{(i-1),i} + \Delta E_i$; $F_{i,(i+1)} = F_{(i-1),i} + \Delta F_i$

$R_{i,(i+1)} = R_{(i-1),i} - \Delta K'_i\ z_i - \Delta E_i\ x - \Delta F_i\ y_i$; $C'_1 = -\ w'_r + \Sigma\ (a^2 + b^2) = +\ 0.0078 \times 10^{-6}$

4. Conclusion

We think that the practical example worked in the previous section is sufficient to show how to achieve an easy least square adjustment of a strip if the simplification introduced is used. However, if one or more control points are known along the strip, the problem becomes more complicated. It is possible, though, to combine the least square adjustment with some kind of graphical adjustment which will take into account the above-mentioned control points. This method is used in the Netherlands, and is described in the Belgian review "Photogrammétrie" (No. 41-1955), in an article by D. LESNE and F. PEETERS.

In a very recent pamphlet published by the "Deutsche Geodätische Kommission " a highly interesting work by Hans BERTRAM (died 1945) was presented by Prof. Dr.-Ing. habil Rudolf. FÖRSTNER. In this paper "Beitrag zur Ausgleichung räumlicher Polygonzüge bei der Luftbild-triangulation" (1963), a least square method of adjustment is derived in which the variables are exactly the same as those adopted in the Belgian method. The main difference between these two methods is that in the Belgian method the ground coordinates are computed directly in terms of the adjusted transformation elements whereas in BERTRAM's method the ground coordinates are computed in terms of the machine coordinates after their correction. The BERTRAM procedure allows an important simplification in the solution of the normal equations. In fact these formulas are similar to our ($3v$) and ($3x$).

BERTRAM's paper shows diagrams of the curves of absolute errors in x, y and z; these curves are, as usual, irregular. This irregularity is a consequence of the accidental errors in the relative orientation and of the double accumulation of these errors in the bridging process. Thus it is possible to draw a useful conclusion which is valid for every method where control points exist only on the end stereo-pairs : the best results will be obtained if the relative orientation procedure is sufficiently accurate. Numerical orientation would be advisable.

We have adapted the method to be used for the adjustment of strips in closed circuits without jeopardizing the simplicity.

We feel that in the near future analytical aerial-triangulation will be almost generalized but that this will not eliminate the universal plotters, which have a very long life-span. Thus, wherever these instruments are at present in existence in an organization, their use will be continued. Therefore, we think the adjustment method described here will be serviceable for some years to come.