

# PRINCIPLES OF INERTIAL NAVIGATION FOR THE USE OF AIR AND MARINE VEHICLES AND FOR MISSILES

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## I. — GENERAL

### I.1 - Introduction and definition

The fundamental principle of inertia allows an extremely simple and concise definition of a process for which the navigator has lacked only the means of implementation since the time when, as he did not have gyroscopic resources at his disposal, the possibility of computing the direction of the local vertical by self-contained and internal means always stood out of his reach.

The basis of inertial navigation is the measurement of accelerations as absolute quantities in amount and in direction with reference to axes which may be either bound to the earth or assumed to be fixed in inertial space, i.e. pointing towards three stars and originating from the centre of the earth. This measurement allows the speed vector to be plotted and considered as the sum of speed changes and then, through a new integration, the path travelled to be computed as the sum of primary arcs. This procedure, then, differs essentially from ordinary navigational methods which measure only relative quantities. Since the custom was to consider marine and aerial vehicles as moving with a constant speed, the suggestion of a double integration of acceleration often comes up against the primary objection that there is nothing to integrate, since this acceleration would be constantly null. In fact the usual accelerations of air and sea vehicles are, when en route, smaller than or equal to a few thousandths of  $g$ ; this shows the importance of initial values for angles and distances introduced into the system.

### I.2 - Historical background

The history of inertial navigation goes back to the first efforts to determine the vertical direction by internal means either at sea or in the air, i.e. to the earliest work on the use of pendulums or of artificial levels with fluid or bubbles. More exactly its origin may be assigned to the time of the development of the first gyro-compasses which in fact outlined the horizontal accelerations caused by the motion of the vehicles. At about

the same time ANSCHUTZ in Germany and Elmer SPERRY in the United States around 1900 conceived directional instruments on the principle of the pendulous gyroscope. It is to the credit of Professor Maximilian SCHÜLER that he presented to the Anschütz Industry in 1923, along with his theory, well known under the name Schüler Tuning, a new device consisting of gyroscopes and accelerometers. In 1924, the American ABBOT, and then in 1932 the Frenchman ESNAULT-PELTERIE, had already developed three-axes-platforms stabilized by gyroscopes and equipped with accelerometers and integrators. The principle of a closed-loop-servo-control of a platform by accelerometers was outlined as early as 1940 by various authors. By this time the improvement of gyroscopic engineering had resulted in a gyroscopic compass compensated for yaw and roll and for oscillations of the platform. But it was during World War II that in Germany, on the occasion of the development of a guidance system for V2 rockets, the first simple model of an inertial guidance system was evolved.

Since then the research work at the M.I.T. to which Dr. DRAPER has linked his name, has allowed the United States to lead the way to entire navigational systems which the extraordinary progress made in the art of gyroscopic engineering has contributed to render progressively effective, precise and operational.

### 1.3 - Notations used

$\varphi, G$	geographical coordinates : latitude and longitude
$R_v$	true course
$C_v$	true heading
$x, y$	paths covered along a great circle and cross tracks
$Z$	vertical paths
$V$	linear speed of the vehicle
$a$	angular speed of the vehicle
$R$	earth's radius
$\alpha$	oblateness of the meridian ellipse
$U$	angular rate of earth's rotation
$\Omega$	angular speed of the gyroscope wheel
$C$	main moment of inertia of the wheel
$C\Omega$	main angular momentum
$K$	torque coefficient
$\theta$	pitch inclination
$\alpha$	roll inclination
$T$	period of oscillation, Schüler T period = 84.4 minutes
$\omega$	Schüler pulsation
$w$	angular drift rate of the vertical gyros
$z$	angular drift rate of the directional gyros
$g$	gravitational acceleration
$J_e$	transport acceleration
$J_r$	relative acceleration
$J_c$	Coriolis acceleration.

**II. — RETENTION OF THE VERTICAL**

**II.1 - Role of the vertical on a spherical earth**

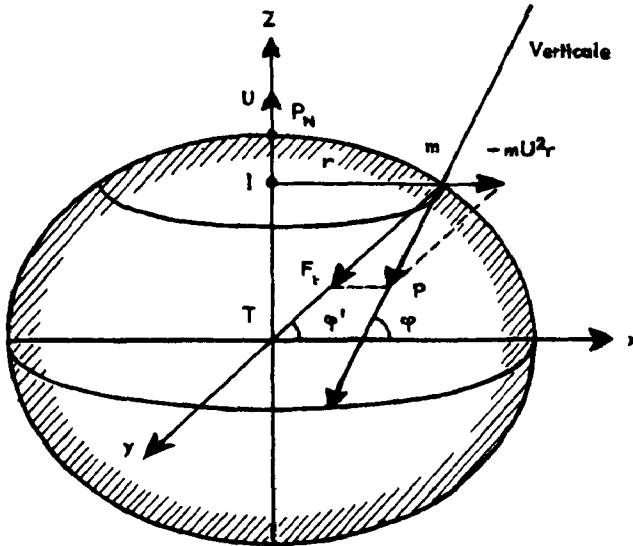
On the first analysis we shall deal only with a spherical earth whose shape is geometrically sufficient to cover the present needs of ordinary slow speed in maritime navigation. In the case of a moving vehicle on the surface of the earth the fundamental law is that the whole of the external forces, as well as the reaction of the supports, and the sum of the inertial transport forces, the relative and the Coriolis forces, forms a vector system equivalent to zero at any instant.

This law is written :

$$\Sigma (\vec{F} - m\vec{J}_e - m\vec{J}_r - m\vec{J}_c) = 0 \tag{1}$$

For relative axes fixed to the earth, the earth's attraction  $F_t$  and the inertial transport forces which are centrifugal inertial forces of the type  $-mU^2r$ , are both included in the weight of the system which may be considered as an external force included in the  $F$  vectors. The law is then written :

$$\Sigma (\vec{F}' - m\vec{J}_r - m\vec{J}_c) = 0 \tag{2}$$



provided that the action of celestial bodies other than the earth is neglected, and this is sufficient except in the case of tides or for spatial probes.

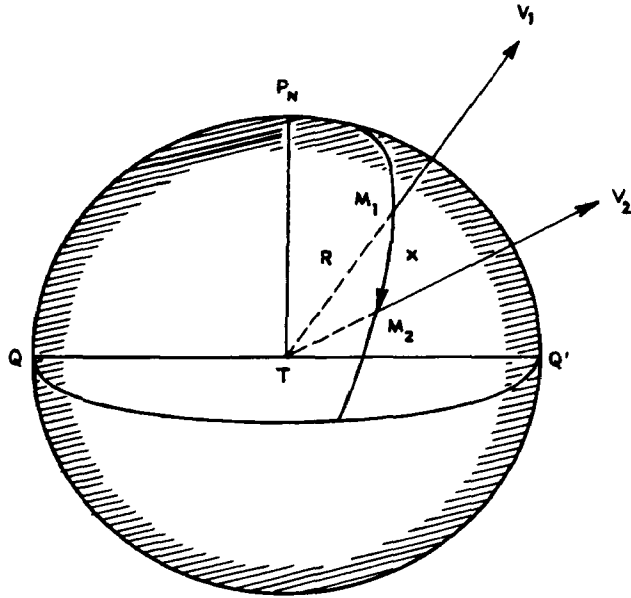
We shall therefore consider the astronomical vertical and not the geocentric vertical, and it is known that the maximum angle between these two verticals is  $12'$  at  $45^\circ$  latitude. It is likewise known that the angle between this vertical and the terrestrial equator or — and this is equivalent — with the polar axis, defines the astronomical latitude.

If a vehicle is travelling along a specific great circle, for example a meridian, the arc of the meridian covered is equal to the angle through which the vertical has turned :

$$\theta = \frac{x}{R} = \frac{\Delta\varphi}{R} \quad (3)$$

Usually to measure this angle it is necessary to be provided with two elements :

- 1) the memory of an initial vertical;
- 2) the continuous knowledge, at any time, of the local vertical.



In fact, on the surface of a spherical earth, the position of a vehicle is fully determined by the place where the instantaneous vertical penetrates the sphere. The same is true for an aircraft wishing to fix its position with relation to axes bound to the earth. It will, however, require — in addition to the latitude and longitude of the point where this vertical crosses the surface — a third coordinate which will be the height  $Z$  measured along this vertical. Conversely, the position of a ballistic missile can be defined by three rectangular coordinates  $X, Y, Z$  measured on three axes fixed in inertial space irrespective of the earth's motion.

## II.2 - Initial requirements

The apparent simplicity of the principle of the use of inertia, as mentioned in sub-section 1, immediately gives rise to initial conditions from which the first difficulties of application arise.

- 1) The accelerometers, which are devised for measuring the components of propulsive acceleration, have measuring axes which must be accurately

fixed with reference to selected axes of reference, that is to say either with respect to initial direction fixed in space, or to axes bound to the observer such as the local vertical on the earth.

2) The angles between these measuring axes and those of reference must therefore be known.

As far as horizontal navigation is concerned this condition should be satisfied by providing the accelerometers with a platform locked to horizontal, that is to say rotating with the earth and with the displacement of the vehicle.

One solution would be to link the accelerometer equipment to the structure of the craft. In that case it would be necessary to know at any given moment the angular elements of the attitude of the vehicle with respect to the axes of reference.

3) In all cases where the platform or accelerometer equipment is not situated in a horizontal plane it must be possible in the output signal to separate the proportion of acceleration due to propulsion from that due to gravity. This means that for an inclination  $\theta$ , the indication of the accelerometer should be corrected by the quantity  $g \sin \theta$ . This adjustment, which must be made with an accuracy comparable to that of the measurement, i.e. around  $10^{-5} g$ , is particularly difficult to carry out on account of changes in the field of gravity with the height, level and shape of the geoid.

Particularly when a vertical axis accelerometer is used for measuring vertical paths will it be noticed that the twice integrated acceleration is equivalent to a vertical path of the order of  $\frac{1}{2}gt^2$ , i.e. about 36 000 nautical miles per hour. Errors due to inaccurate adjustment would therefore not be acceptable for the needs of three-dimensional navigation and this measurement remains limited to the field of missiles having high acceleration over a very short period of time.

In the same way it can be noted that a  $1^\circ$  difference in the horizontal position of a platform supporting the accelerometers introduces into the measurement an error due to gravity of  $g \sin \theta$ , i.e. about  $\frac{10}{60}$  m/s<sup>2</sup>, which on the path covered in one hour involves an error of 600 nautical miles.

### II.3. - The possibilities of a pendulum - the Schüler pendulum

Since a system constituted by a simple or compound pendulum can measure only the apparent and not the true vertical, it cannot be used to measure the change in the vertical. The centre of mass  $G$  in a short pendulum whose centre of support undergoes a horizontal acceleration  $\gamma$ , remains behind the vertical of the centre of support. In 1923, Professor SCHÜLER worked out the conditions under which a pendulum along the vertical at rest would always remain aligned along the local vertical whatever the acceleration  $\gamma$ . It will remain unchanged if the angular

acceleration of the pendulum is equal to the angular change in the local vertical. If  $\mathcal{S}$  is the arc of a great circle covered in a period of time  $t$ , the angular displacement is :

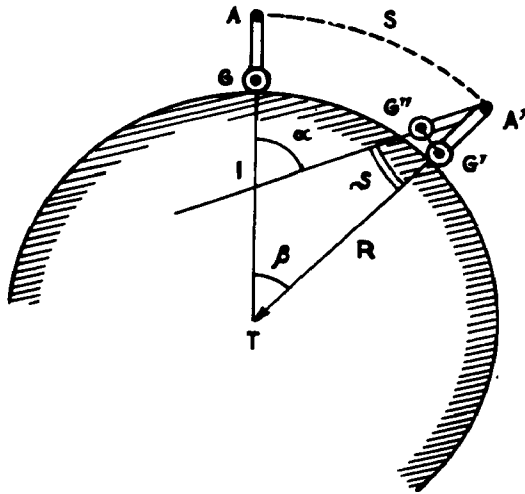
$$\alpha = \frac{S}{R}$$

The pendulum lag will be defined by the deviation :

$$\mathcal{S} = \beta - \alpha$$

with :

$$\beta = \frac{S}{AI}$$



If, in a compound pendulum, we put :

$$AI = \frac{I}{ma} = \lambda \quad (I = \text{moment of inertia; } a = \text{radius of gyration})$$

we have :

$$\mathcal{S} = S \left( \frac{1}{\lambda} - \frac{1}{R} \right) \tag{4}$$

The condition  $\mathcal{S} = 0$  entails that  $\lambda = R$ . The pendulum indicating the vertical should have a radius equal to the earth's radius. This condition is impossible to realise mechanically : for on one hand, in the case of a compound pendulum whose radius is 10 cm, the distance from the centre of support to the centre of mass would be less than  $10^{-3}$  microns, and on the other hand, in this case the back-lash torque would be less than the friction torque.

However if such an ideal pendulum existed, the vertical indicated by this pendulum would not be fixed but would oscillate around the local vertical with a long period — the so-called *Schüler period*. In fact the angular displacement of the true vertical would be measured for a deviation

$$\theta = \frac{-g \sin \theta}{R}$$

And the law of oscillation would be :

$$\frac{d^2 \theta}{dt^2} = \frac{-g \sin \theta}{R} \tag{5}$$

or for small angles :

$$\frac{d^2 \theta}{dt^2} = \frac{-g}{R} \theta$$

This is the equation of a pendulous movement of period :

$$T = 2\pi \sqrt{\frac{R}{g}} \tag{6}$$

equal to the oscillation period of a simple pendulum with an arm equivalent to the earth's radius, whose value is 84.4 minutes — the so-called Schuler period.

If such a compound pendulum could be built it would oscillate without damping about the local vertical of the centre of support for 84.4 minutes, whatever the accelerations applied to this centre.

#### II.4 - Possibilities of gyroscopes - Abstract of gyroscopic properties

Gyroscopes are pendulous systems which allow the retention of a fixed direction in space.

If, with respect to axes bound to the earth, a wheel is spun round its axis with an uniform rate, the fundamental law of dynamics may be written :

$$\Sigma (\vec{F} - m\vec{J}_c) = 0 \tag{7}$$

There is equilibrium between the external forces applied to the wheel (including its weight and its frictions) and the Coriolis forces, since the relative inertial forces  $\Sigma_i - m\vec{J}_r$  are null as they are only centrifugal forces of the type  $-m\Omega^2 r$ , which cancel out by reason of symmetry.

The Coriolis forces are due to the transporting rotation of the earth which, at latitude  $\varphi$ , is expressed by  $U \cos \varphi$  in the horizon, and by  $U \sin \varphi$  along the vertical.

If  $C\Omega$  is the angular momentum along the gyroscope axis  $OZ$ , deviated by an angle  $\theta$  from the terrestrial polar axis, the only disturbing torque of Coriolis force is caused by the component  $U \sin \theta$  of the earth's rotation rate :

$$\mathcal{M} = C\Omega \cdot U \cdot \sin \theta \tag{8}$$

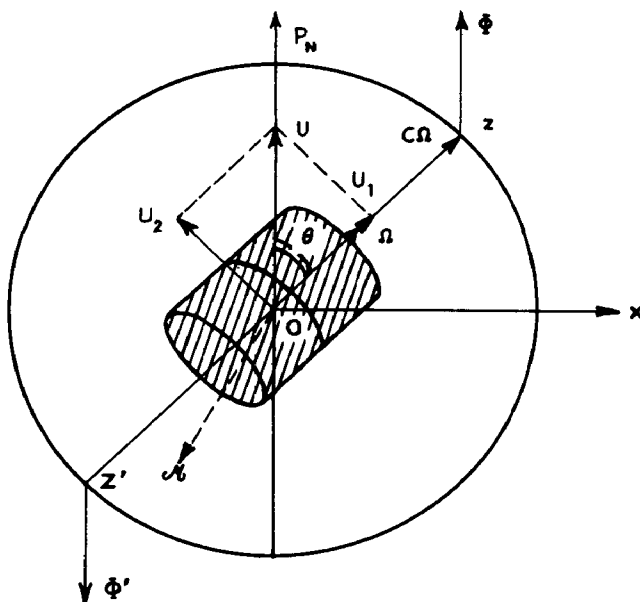
which can be substituted by a torque of the same axis caused by two equal forces  $\Phi$  and  $\Phi'$  applied to both ends of axis  $ZZ'$  of the gyroscope, in a direction parallel to the polar axis :

$$\mathcal{M} = C \cdot \Omega \cdot U \cdot \sin \theta = \Phi \times ZZ' \cdot \sin \theta$$

Whence :

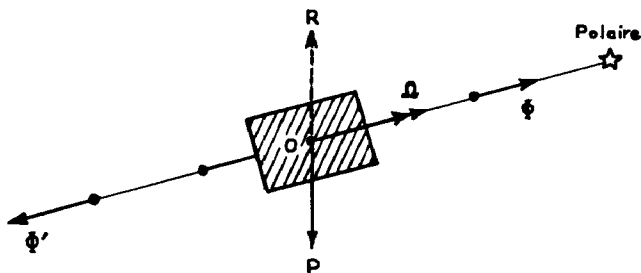
$$\Phi = \frac{C \cdot \Omega \cdot U}{ZZ'} \tag{9}$$

These two forces, called *fictitious magnetic forces*, are constant in amplitude and direction and independent of  $\theta$ . This occurs as if the axis of the gyroscope were behaving like a magnet placed in a magnetic field parallel to the polar axis.



#### RETENTION OF THE EARTH'S POLAR AXIS

Consequently if, originally, the gyroscope points in such a way that the positive sense of its spin vector  $\Omega$  is directed towards the North Pole (that is approximately towards the Pole Star), and if its weight  $P$  is counter-balanced by the reaction of the supports, the momentum of Coriolis forces is null. The gyroscope remains in apparent equilibrium with respect to the earth, its axis pointing towards the North Pole.



Therefore we have at our disposal a system capable of keeping in memory a specific initial vertical, as for instance the one aligned with the polar axis. If another means of realising the local vertical at any time is afforded, then the angle formed by these two verticals, i.e. the *latitude*, can be measured.



**II.5 - Retention of a fixed direction in inertial space**

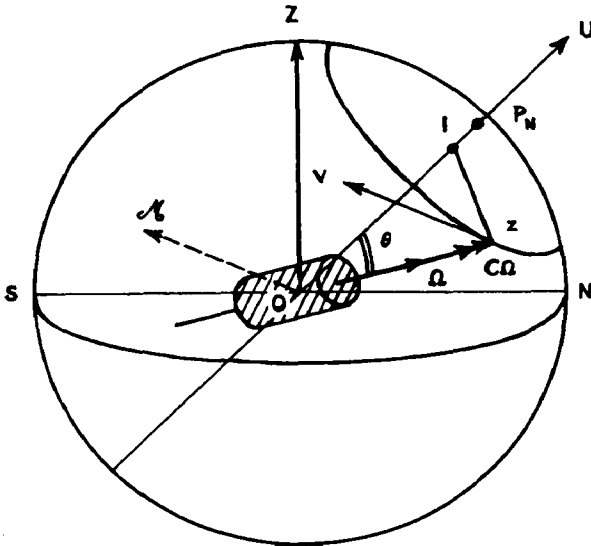
Let us suppose that the aforementioned ideal gyroscope is pointing not at the pole but at a star situated at an angular distance  $\theta$  from the polar axis. It is subject to its weight counter-balanced by the reaction of the supports, and to the fictitious magnetic torque :

$$\mathcal{M} = C\Omega \cdot U \cdot \sin \theta$$

whose axis is perpendicular to the plane containing the world's axis and the gyroscope's axis.

According to the theorem of angular momentum, point  $C\Omega$  in the axis will move with a linear speed  $v$  equipollent with  $\mathcal{M}$ , i.e. tangent to the celestial trajectory of the star at point  $z$

$$v = \mathcal{M} = C\Omega \cdot U \sin \theta \tag{8}$$



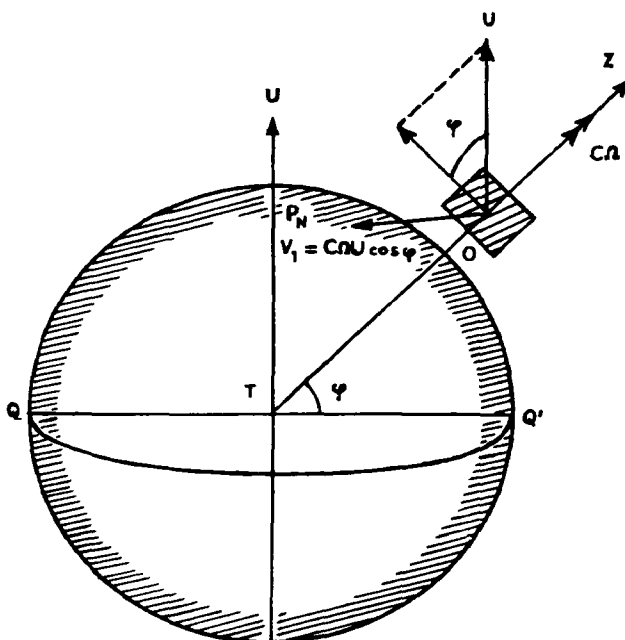
Point  $z$  of the axis, the extremity of angular momentum, will rotate in the direction of diurnal motion on the parallel with an angular speed  $U'$  such as :

$$U' = \frac{V}{Iz} = \frac{C.\Omega.U. \sin \theta}{C.\Omega \sin \theta} = U \tag{9}$$

The axis of the gyroscope will precess in the direction of diurnal motion, about the world's axis with the angular speed of the earth's rotation. Its extremity will remain pointing at the star, i.e. its axis will keep a fixed direction in inertial space and it will show an apparent motion in altitude and azimuth with respect to axes bound to the earth.

**SPECIAL CASES**

a) If the axis of the gyroscope is pointed at the zenith (local vertical), the axis  $\mathcal{M}_1$  of the fictitious magnetic torque, or of Coriolis inertial forces, remains perpendicular to the meridian (plane ZOP), and therefore stays

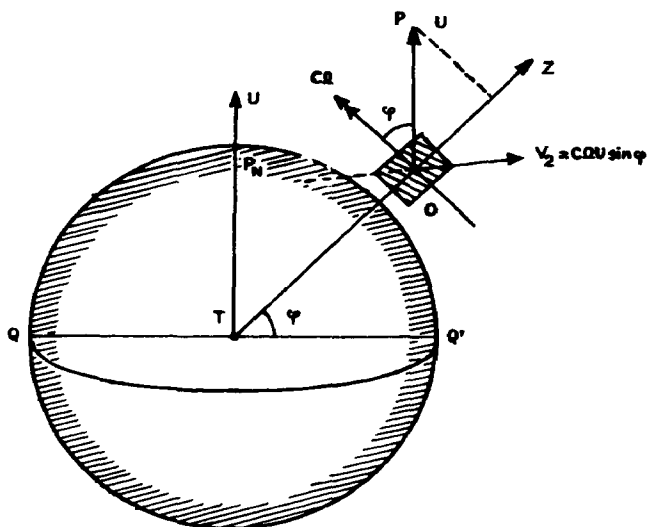


horizontal, as does the precession speed  $v$  applied to the extremity of the angular momentum :

$$v_1 = C \cdot \Omega \cdot U \cdot \cos \varphi \quad (10)$$

b) If the axis is pointed in the horizon, towards the meridian, the axis  $\mathcal{M}_2$  of the fictitious magnetic torque is also horizontal as is the speed of precession  $v_2$  whose value is :

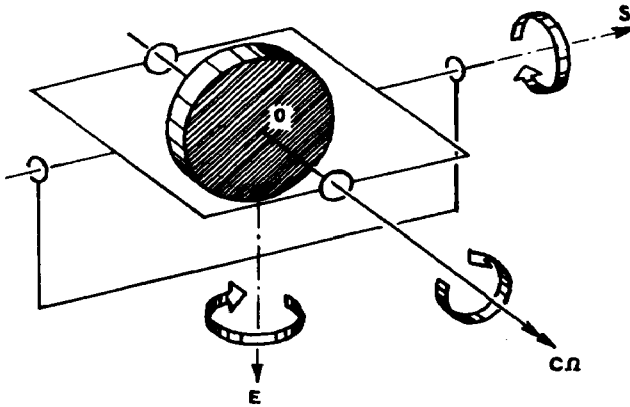
$$v_2 = C \cdot \Omega \cdot U \cdot \sin \varphi \quad (11)$$



### III. — VARIOUS TYPES OF GYROSCOPES

#### III.1 - Freedom of a gyroscope

The gyroscope which maintains a fixed direction in inertial space is an ideal instrument whose spin axis is free to move in any plane and which is only subject to Coriolis inertial forces caused by earth's rotation. That is to say, the reaction of the supports is exactly cancelled by the weight and all the disturbing torques caused by torsion, friction and by dynamical, mechanical or thermal unbalance are considered as null. We shall call gyroscopes of inertial class the instruments in which the disturbing torques are reduced to very small values, and for which "drifts", i.e. precessions due to these torques, are smaller than 1/10 of a degree per hour.



As for the behaviour of the ideal gyroscopes, which will be considered below, we shall assume that they are contained in a case mounted on gimbals attached to the vehicle's framework. Their movement under the effect of disturbing torques, due, for instance, to the reaction of the gimbals occasioned by a change in the relative orientation of the casing, will be sensed on three axes: the  $OC$  axis along which is the angular momentum  $C\Omega$ , the *input* axis or sensing axis around which the disturbing torque  $OE$  is felt, and the *output*  $OS$  around which occurs the gyroscopic precession. Without dealing with the various complex support systems at present being used in modern gyroscopic engineering, mention must be made of floated gyros, which consist of floating the casing in a liquid, of the fluoro-carbon type, of similar density, thus partially eliminating lack of balance by bringing the centre of buoyancy and the centre of mass close together, and leading to a mode of indifferent equilibrium. Finally the dynamic balance is often obtained by means of viscous damping which alters the period of oscillation. The aim of this mode of suspension is to obtain gimbal axes with a light load, the primary role of these gimbals being that of centring.

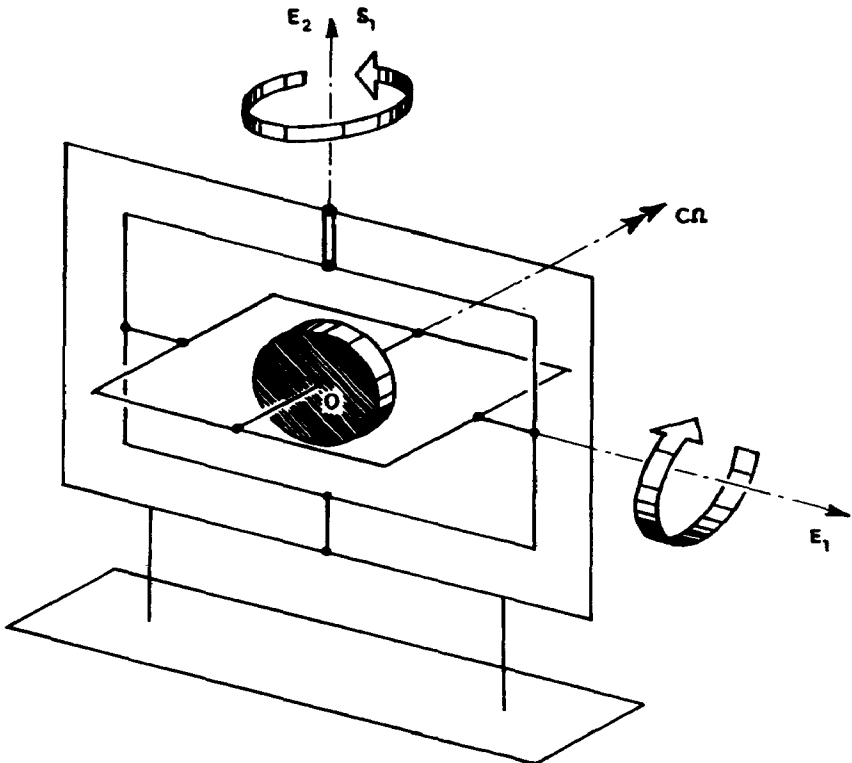
a) *Free gyroscopes*

These are gyroscopes, of two degrees of freedom, suspended on dual-axis gimbals. They are sometimes classed as *amplitude gyros* which can measure directly the angular deviation of the sensing axis with respect to a reference direction (Directional gyros - Horizon gyros), or as speed gyros or *gyrometers* which measure the precession speed of the sensing axis (yaw, roll and pitch indicators).

b) *Single-degree-of-freedom gyroscopes*

This is a gyroscope whose rotor axis can only rotate around a single axis (the output axis) when a disturbing torque is applied to the input axis. These gyros are called "captive" as they usually operate as zero-measuring devices. Any rotating couple applied to the input axis gives rise to a precession about the output axis, the angular speed of which is detected by a *pick-off*, or cancelled by means of a signal transmitted by a *torque motor* to the gimbal of the input axis, or by means of a viscous damping, the effect of which is a speed integration. The reaction of the gyroscope therefore depends on the dynamic characteristics.

It is recalled that the expression *torque motor* means a unit composed of a stator and a rotor between which is applied a torque dependent on their relative position and on the current fed to the stator. This current can be accurately defined by a detector or by a computer programme. In the last case it permits the precession of the gyroscopic axis so that its fixed direction, with respect to the chosen reference axes, is restored.



c) *Two-degree-of-freedom gyroscopes*

These are free gyroscopes having two input and two output axes. They are often used as zero-measuring devices, but their dynamic equilibrium is dependent on amplitude and not on speed. Within the limits of a small angular displacement of the sensing axis, the integral of the angular speed of the output gimbal axis in relation to the casing is proportional to the angular displacement of this casing in relation to fixed axes. If, by means of a *pick-off*, the amplitude deviation is measured, an error signal is fed to a servo-motor which restores the casing to its initial position with respect to the sensing axis. In this case the adopted fixed reference is this initial position, and a platform linked to the casing can thus be restored to the horizontal. For this function the gyro itself plays only the indirect role of a zero-indicator and the dynamic characteristics of the rotor have no influence.

### III.2 - Special cases

a) *Floated single-degree-of-freedom integrating gyroscopes*

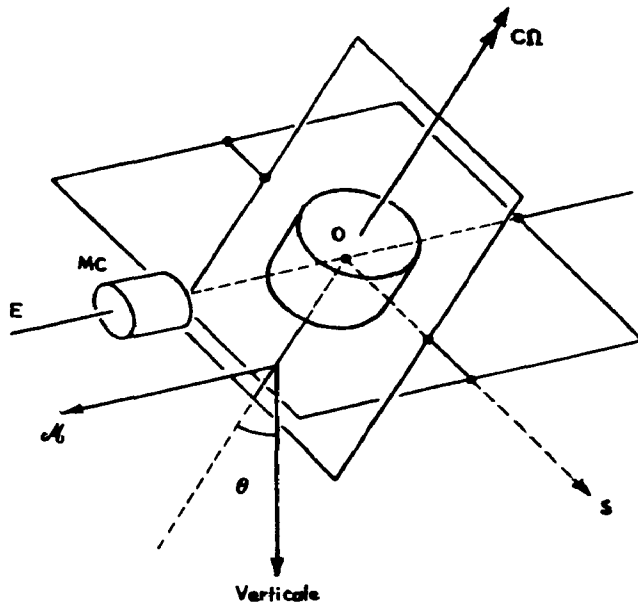
In the case of a single-degree-of-freedom, the rotation of the output axis may be damped by a viscous liquid. Dynamic equilibrium is attained when the gyroscopic reaction torque is balanced by the damping torque of the viscous liquid. The output axis deviation pick-off signal is then proportional to the integral of the input angular speed. Thus an amplitude reference is obtained provided the output axis has a small deviation range around the reference axis.

b) *Erected gyroscopes*

These are free gyroscopes which are generally not of the inertial class, i.e. which cannot by themselves retain for a long time a fixed direction relative to the earth, but which are restored more or less rapidly to this position by a spurious torque intentionally supplied by a lack of balance or a load, i.e. by a pendulum. This pendulum ceases its effect as soon as the sensing axis has reached a position of equilibrium which according to the particular case may be either the vertical, or the horizontal and the meridian.

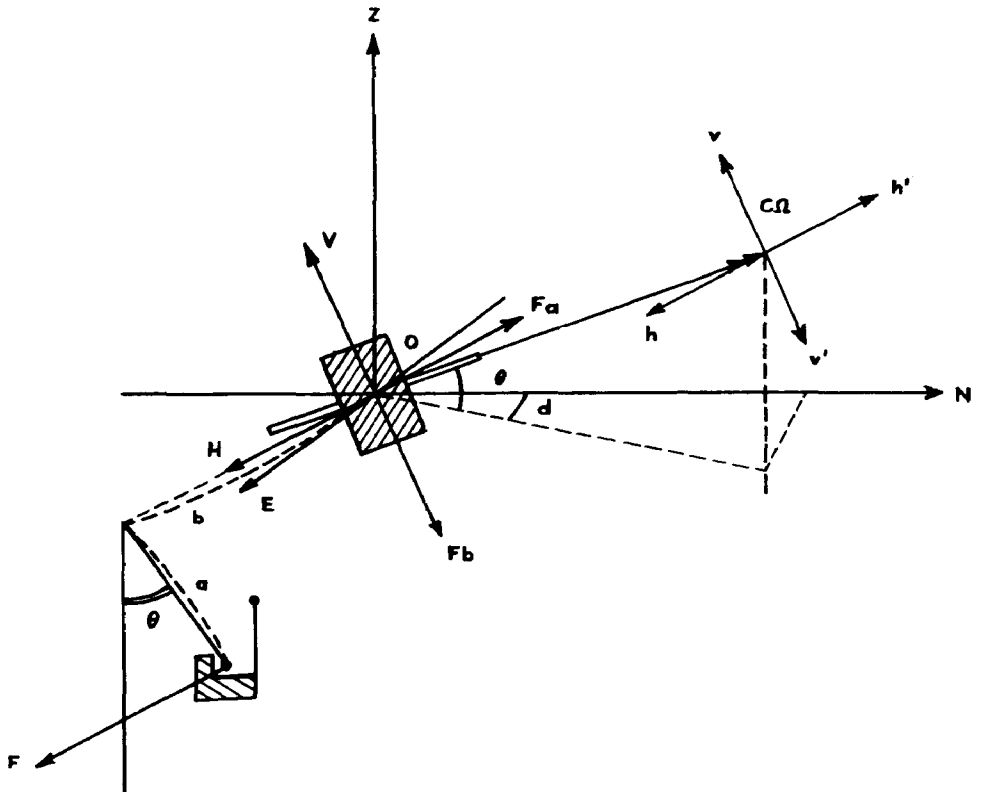
The two types of erected gyroscopes are the aircraft vertical reference gyros and the gyroscopic compasses on ships. The former are made up of a vertical axis gyroscope whose input axis can be fed through a torque motor with a driving precession signal, of the type  $g \sin \theta$  provided either by an accelerometer or by a pendulous system.

It is thus a question of a "monitored" gyro whose monitoring rate is slow enough to keep the gyro unaffected by attitude accelerations for a short time. For long duration propulsive accelerations, for instance during lengthy turnings, the indicated vertical differs from the true vertical and it is often preferable to cut off the monitoring.



c) Gyroscopic compasses

Gyroscopic compasses used in maritime navigation are free gyros whose gyroscopic axis is forced to maintain a stable position by means of solid



or liquid loads, with a lever arm  $a$  giving rise, at an inclination  $\theta$ , to an horizontal axis torque  $Fa = pa \sin \theta$  which, combined with the horizontal axis torque  $H = C\Omega U \cdot \sin \varphi \cdot \cos \theta$ , due to the earth's rotation and always directed towards the west, restores the gyroscopic axis to the horizon. Under the effect of the torque  $V = C\Omega U \cos \varphi \sin d$  having a vertical axis due to the earth's rotation at latitude  $\varphi$  for a deviation  $d$  from the meridian, the axis of the wheel rises east of this meridian and falls west of it. The damping of the motion which leads to a stable position is obtained by shifting the load pivot  $p$  slightly to the east, thus forming a vertical axis torque  $F_b$ , of the type  $p \cdot b \sin \theta$  acting in the opposite sense to torque  $V$ . Finally the gyro axis will be stabilized at the angle :

$$\theta_0 = \frac{C\Omega \cdot U \cdot \sin \varphi}{pa} \tag{12}$$

with a deviation :

$$d_0 = \frac{b}{a} \cdot \tan \varphi \tag{13}$$

after about two oscillatory periods following the starting of the wheel, whatever the initial position.

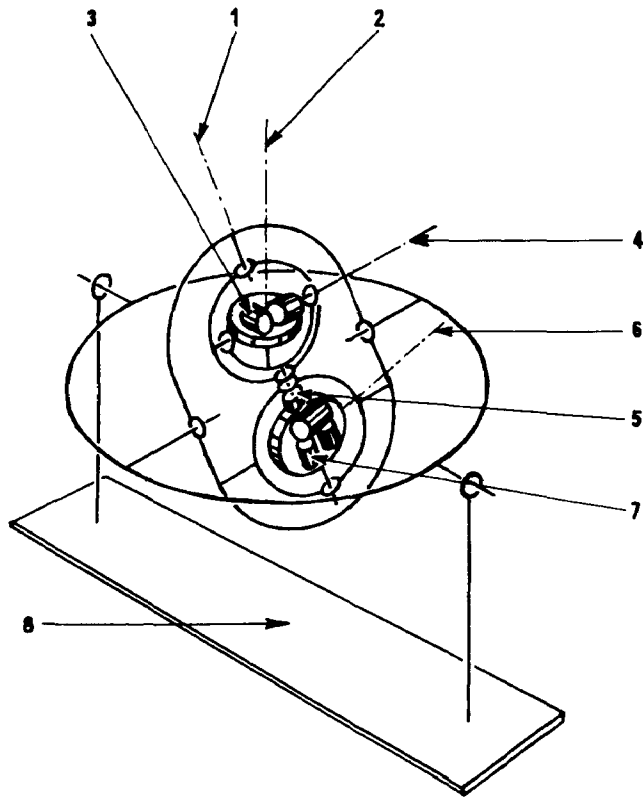
#### IV. — HORIZONTAL NAVIGATION PLATFORMS

Maritime or aerial navigation of vehicles primarily concerns the investigation of two horizontal components of the propulsive acceleration along axes selected, for instance, in the direction of the geographic north and east. The double integration of these accelerations will lead as a matter of course to the determination of the northing path  $\varphi_2 - \varphi_1$ , and to the easting path  $(G_2 - G_1) \cos \varphi$ , that is to say, finally to the geographical latitude and longitude.

##### IV.1 - Astronomic or geometric-type platform

In the early stages of the proposed systems for navigation it appeared that the simplest of these systems would consist in the use of the property of permanence of the spinning axis of the gyroscope to measure the latitude directly, and then the longitude indirectly, by comparing the Greenwich meridian to the local meridian defined by the local vertical and the polar axis. In this device a block of three floated gyroscopes retains, as accurately as possible, three fixed directions in inertial space : the polar axis, the reference easterly semi-axis at Greenwich, which is mechanically rotated at the earth's rotation speed  $U$ , and the northerly semi-axis in the equatorial plane which follows the motion.

The other part of the instrument and the framework of the ship rotate freely about the block of gyroscopes, and decoupling is effected by servo-mechanism. Two additional suspension axes support a platform containing the two accelerometers whose axes are initially adjusted along



the reference OX and OY directions, generally the local east and the local north. This platform thus defines the initial vertical and, by means of signals from the accelerometers, keeps in memory this vertical, using feedback loops of integrators which produce the correction signals corresponding to the displacement. This device allows the reading of the latitude and a sidereal clock provides the angle between the Greenwich and local meridians, that is to say the longitude.

Such a type of platform requires large dimensions, a great number of suspension axes (five gimbals) and various servos. In spite of the advantages of the simplicity of this device, which is that of an automatic astro-compass, the direct measurement of angles entails, at the first order, the drift of the spatial block gyroscopes and thus involves significant cumulative errors. It seems that the system has been abandoned because of the mechanical complications of the frame and because even in maritime navigation the single platform locked to the vertical has now become the general practice. It could however be considered for an automatic maritime astro-compass.

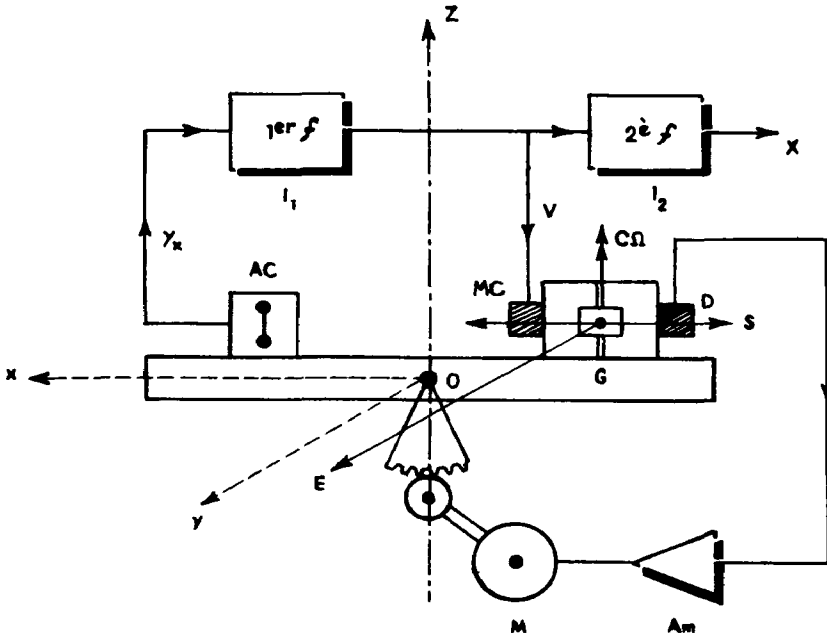
#### IV.2 - Single platform for the local vertical

##### a) Principle of a single rotation axis platform - Schüler tuning

In the first analysis it is assumed that the table in question can only oscillate in the plane of the figure around the *oy* axis. It carries, an



accelerometer *Ac*, a gyroscope *G*, whose sensing axis *GE* is parallel to *oy*, with a pick-off *D* and a torque motor *Mc*, the accelerometer being connected to two integrators *I*<sub>1</sub> and *I*<sub>2</sub>.



The operation is as follows : the accelerometer measures an acceleration  $\gamma_x$  integrated the first time by *I*<sub>1</sub> which gives a speed *V* whose signal is fed to torque motor *Mc*. This results in a shift of the gyro detected by pick-off *D*. After being amplified, the pick-off signal is used to rotate the table through angle  $\theta$ , but at the same time it applies a torque to the gimbals of the gyro which precesses around the output axis *GS* with a speed  $\frac{d\theta}{dt}$ . Equilibrium is obtained when the gyro reaction torque  $C \cdot \Omega \cdot \frac{d\theta}{dt}$  is equal and opposite to the control torque generated by *Mc*.

We then have :

$$C \cdot \Omega \cdot \frac{d\theta}{dt} = K \cdot V = K \cdot \frac{dx}{dt}$$

*K* being the torque constant.

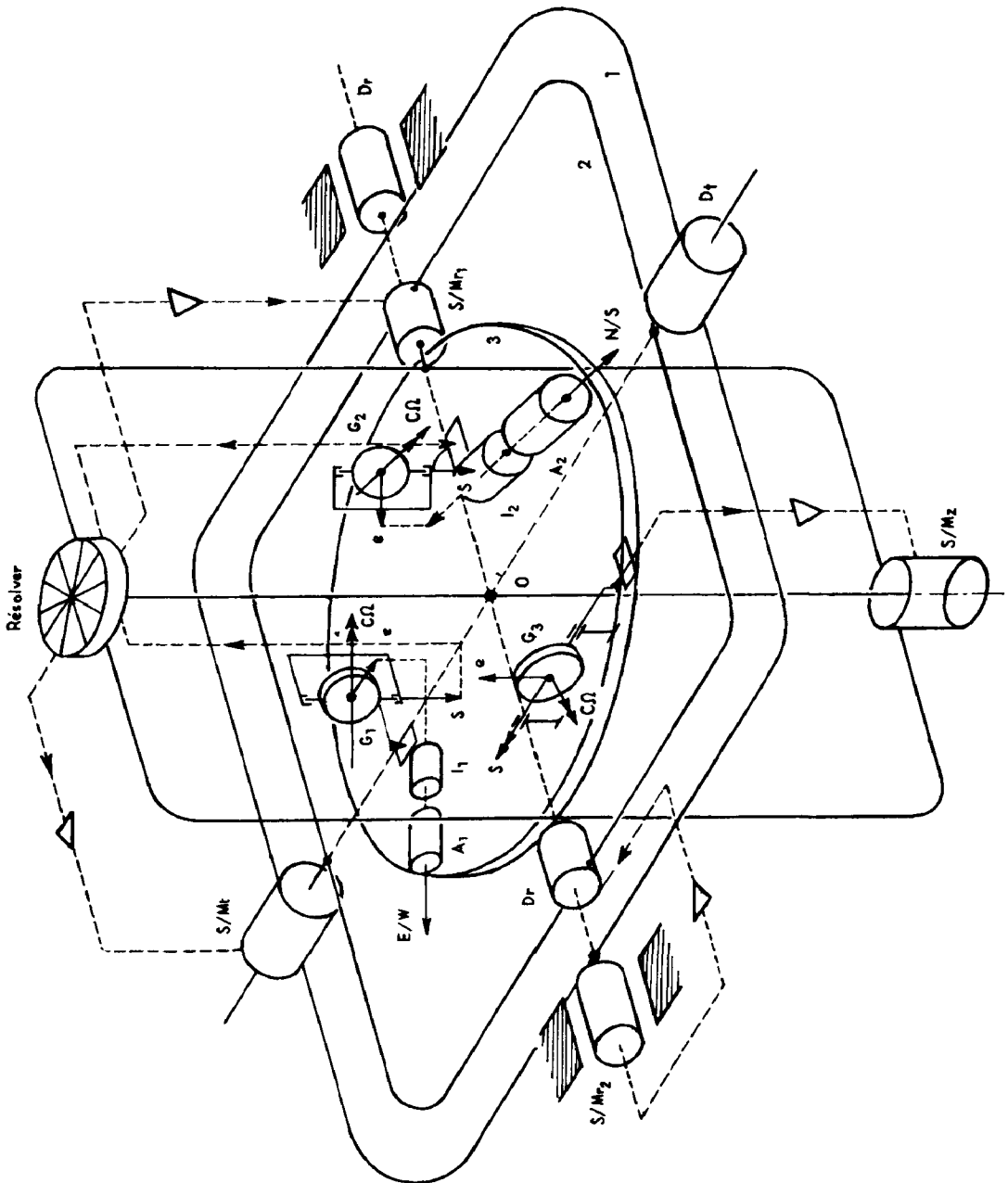
Integrating :

$$C \cdot \Omega \cdot \theta = K \cdot x$$

*x* being the displacement.

Assuming that on a spherical earth the path of the vehicle is a great circle, the vertical will have been rotated from  $\frac{x}{R}$ . In order that the platform should remain horizontal we must have  $\theta = \frac{x}{R}$  that is :

$$C \Omega = K \cdot R \tag{14}$$



Such a device operates exactly like a Schüler pendulum which, whatever the propulsive acceleration  $\gamma_x$ , continuously tracks the local vertical. The platform set to zero inclination while at rest remains steadily horizontal. This condition must then necessarily be achieved at the outset, but if there is an initial inclination  $\theta_0$  the acceleration measured at rest is  $-g \sin \theta_0$ . The equation is then :

$$C \cdot \Omega \cdot \theta = -K \iint g \sin \theta \cdot dt^2 \quad (\theta \text{ being a small angle})$$

which expresses the equation for an undamped pendulum whose actual period is :

$$T = 2\pi \sqrt{\frac{C \Omega}{Kg}} = 2\pi \sqrt{\frac{R}{g}} = 84.4 \text{ minutes} \quad (15)$$

It can be seen that the indicated vertical oscillates with the platform with an 84-minute period about the true vertical with amplitude  $\theta_0$ . If the earth were flat, with  $R = \infty$ , the oscillatory period would be infinite, but there would be no compensation since  $\theta = \frac{x}{R}$  would be constantly zero.

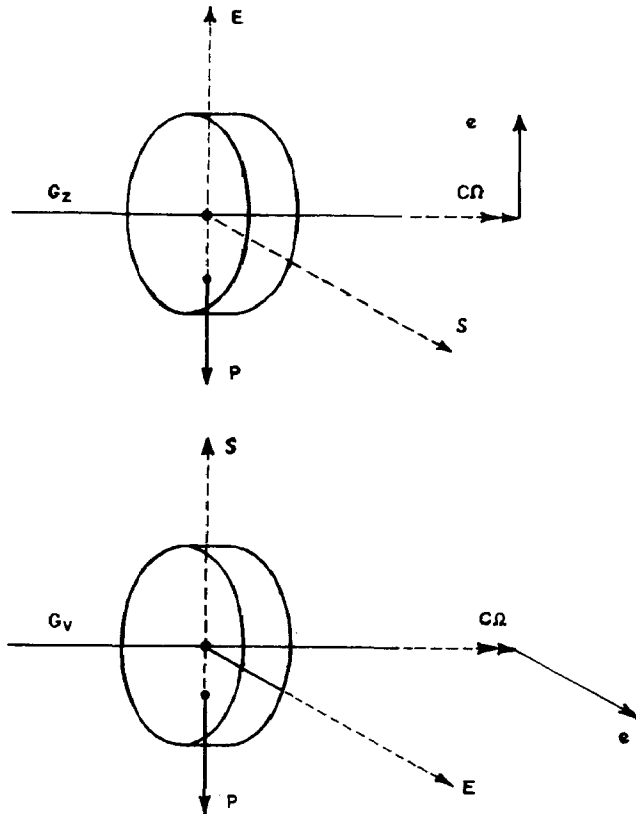
Taking into account the Schüler oscillation the second integrator will constantly provide the arc of path covered  $x$ . The oscillation amplitude may be reduced by adding a damping term proportional to the amplitude, but in doing so an error is introduced on the vertical, which may be neglected or corrected provided the acceleration does not last more than a fraction of the period, and likewise a positional error proportional to the time and to the damping coefficient  $A$ . It is therefore necessary to make the initial horizontal adjustment of the platform as accurately as possible, and this may be done by using the precession of the gyroscope at rest, and adjusting the platform until no signal is detected. Another procedure consists in using an erecting load system for vertical reference gyros, and then once under way to cut off the erecting device.

*Note.* — Emphasis must be placed on the fact that, in this preliminary discussion, the dynamic effects of inertia, the static effects of friction and of stiffness for the whole of the gyro, gimbals, accelerometers, servos, as well as the drifts and integration errors involved, have all been neglected. These phenomena are considerable drawbacks in the manufacture of inertial systems, and these obstacles have only been overcome by tremendous technological efforts, thus explaining the slow improvement in efficiency and accuracy. Furthermore the earth has been considered as spherical; in fact the vertical tracked is the astronomical vertical which does not pass through a fixed point. But, at rest, the flattening of the earth, i.e.  $\frac{1}{297}$ , introduces an error of only three seconds of arc on the vertical supplied by Schüler tuning.

#### b) Two-axis platform

The vertical is defined by two rectangular axes of rotation  $ox$  and  $oy$  of the platform, situated in the horizontal plane. There should therefore be available two stabilization units made up of two single-degree-of-freedom gyroscopes  $G_1$  and  $G_2$ , whose input axes  $G_{1e}$  and  $G_{2e}$  are horizontal,

rectangular and respectively aligned along, for instance, the geographical north and east. The output axes  $G_1S_1$  and  $G_2S_2$  of these gyros are vertical and their axes of rotation are horizontal. The gyro  $G_1$  with a North/South input axis is connected through an integrator  $I_1$  to an accelerometer  $A_1$  whose measuring axis is East/West. Similarly the torque motor of gyro  $G_2$  with an East/West input axis will be fed by accelerometer  $A_2$  having a North/South measuring axis after integration in the integrator  $I_2$ . The pick-offs mounted on the gimbals of gyros  $G_1$  and  $G_2$  supply signals which after amplification are fed into a sine-cosine resolver which sends the appropriate signals to both the roll and pitch servo motors  $SMr$  and  $SMt$ . In such a block diagram of installation these servo motor axes must coincide with the directions of the accelerometer axes assumed to be directed North/South and East/West. This orientation of the platform is obtained by means of a third single-degree-of-freedom gyroscope  $G_3$ , whose input axis  $G_3e$  is vertical in order to pick-off any change in azimuth of the platform; the rotation signal thus worked out is fed, after amplification, to an azimuth servo motor. In fact, the pitch and roll axes rotate with the vehicle heading and do not coincide with the meridian directions which are those of the accelerometers' axes; the specific function of the sine and cosine resolver mounted on the vertical axis  $OZ$  of the platform will thus be to distribute, after amplification, signals of the type  $\Delta\theta \sin C_V - \Delta\beta \cos C_V$  to the pitch and roll servo motors. In this operation each gyroscope detects



the platform shift about its input axis in such a way that the platform angular speeds are always very small about the output axis, which is the same as the input axis of the other gyroscope.

The directions of the angular momentum of gyros  $G_1$  and  $G_2$  may be either vertical or horizontal. It is however essential that their input axis be horizontal and pointing along fixed directions on the horizon. Nevertheless it is desirable to select axes of rotation in a horizontal direction for gyroscopes  $G_1$  and  $G_2$  called *vertical reference gyros*, because the effect of an eventual load unbalance would remain nearly parallel to the angular momentum. On the contrary the gyroscope  $G_3$ , called the *azimuth reference gyro* or *directional gyro*, whose output axis is horizontal, undergoes a significant disturbing effect since the torque of unbalance is perpendicular to this output axis and thus increases this gyro's drift. This drift is therefore particularly critical in the case of the azimuth reference gyroscope.

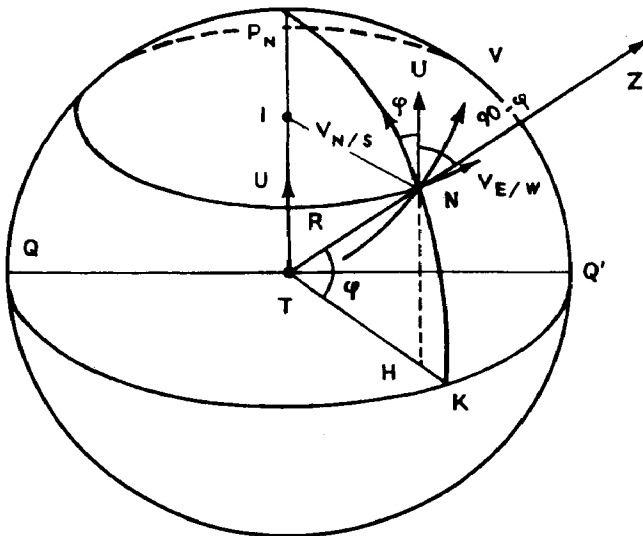
IV.3 - Transport speed of axes bound to the earth

The theoretical fixity in inertial space of vertical and azimuth reference gyros results in apparent effective speeds with respect to axes bound to the earth, in fact the local vertical and the North/South line. These speeds must be corrected by controlled precessions applied to the gyroscopes' input gimbal axes by means of torque motors. These driving speeds result in angular rotations which are on one hand the components of the earth's rotation rate  $U$ , and on the other hand components of the primary horizontal displacement along a loxodrome tangent to a great circle. It is known that these elementary rotations are, along the parallel :

$$\frac{dG}{dt} \cdot \cos \varphi \text{ or } G' \cos \varphi$$

and along the meridian :

$$\frac{d\varphi}{dt} \text{ or } \varphi'$$



Putting therefore :

$$\begin{aligned} x' &= V_{N/S} = R \cdot \varphi' = V \cdot \cos R_v \\ y' &= V_{E/W} = R \cdot G' \cos \varphi = V \cdot \sin R_v \end{aligned}$$

The components of these transport speeds along the geographical axes and along the vertical are therefore :

$$\left. \begin{aligned} H_{N/S} &= U \cdot \cos \varphi + \frac{V_{E/W}}{R} = (U + G') \cos \varphi \\ H_{E/W} &= \frac{V_{N/S}}{R} = \varphi' \\ H_z &= U \cdot \sin \varphi + \frac{V_{E/W}}{R} \cdot \tan \varphi = (U + G') \sin \varphi \end{aligned} \right\} \quad (16)$$

It is exactly the same as if the movement in longitude were equivalent to a modification in the earth's rate.

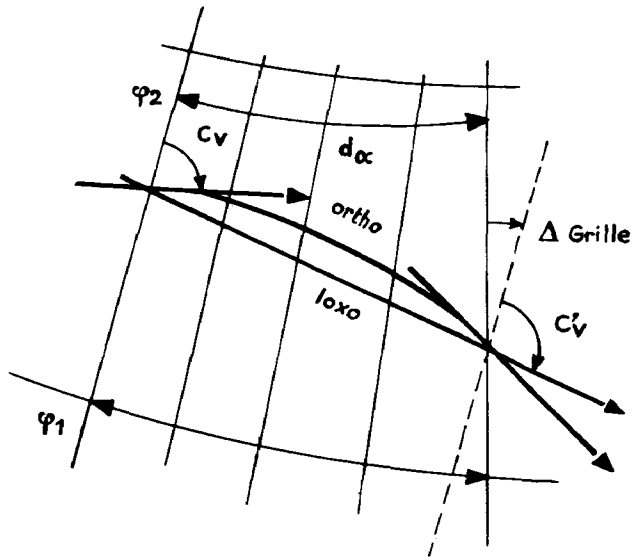
*Note.* — Free gyros, which are called directional gyros, are used in aerial navigation. Their spin axis is nearly horizontal and will hold rather roughly a fixed direction in space.

If the earth had no motion, the path travelled on a spherical earth by an aircraft with a heading controlled by the axis of the directional gyroscope would be a great circle. As the earth is spinning it is necessary, in order to follow an orthodrome, to precess the gyro at angular speed  $U \cdot \sin \varphi$ . If we wish to follow a loxodrome the gyro should be precessed at this speed  $U \cdot \sin \varphi$  in addition to the angular speed with respect to the meridian, that is :

$$U \cdot \sin \varphi \left( 1 + \frac{V_{E/W}}{R \cdot U \cdot \cos \varphi} \right)$$

or :

$$U \sin \varphi + \frac{V}{R} \cdot \sin R_v \cdot \tan \varphi \quad (17)$$



V being the aircraft speed and  $R_V$  the true course. Such are the "Polar Path" type gyros which are controlled by two corrections previously worked out : firstly a precession  $U \sin \varphi_0$  for a mean latitude and secondly a convergency precession  $\frac{V}{R} \sin R_V \cdot \tan \varphi_0$ . For high latitudes the convergency correction is so large that a "grid heading" must be followed, which makes a constant angle with the meridian selected as a reference. On the Mercator projection the grid convergency is zero; on a stereo-polar projection it is equal to one; it is  $\sin \varphi$  on the earth, and equal to  $n$  on a Lambert conical projection. Therefore the grid correction to be applied is :

$$\Delta_{\text{grid}} = \frac{n}{\sin \varphi} \cdot \frac{V}{R} \cdot \sin R_V \cdot \tan \varphi \quad (18)$$

#### IV.4 - Compensation of transport speeds applied to the gyros

The angular transport speeds  $H_{N/S}$ ,  $H_{E/W}$ ,  $H_Z$  are extracted from a computer and allow signals to be worked out and applied respectively to the torque motors of vertical reference gyros  $G_1$  and  $G_2$  and to the torque motor of azimuth reference gyro  $G_3$ , in order to cause these gyros to precess about their input axes, and to achieve their steadiness with respect to the local vertical and the horizontal North/South line.

The torque motor of vertical reference gyro  $G_1$  whose input axis is N/S receives signal  $(U + G') \cos \varphi$ , for which  $U$  is programmed in the computer,  $G'$  is extracted from integrator  $I_1$  following accelerometer  $A_1$  with an E/W axis. The torque motor of gyroscope  $G_2$  whose input axis is E/W receives a signal proportional to  $\varphi'$  extracted from integrator  $I_2$  following N/S axis accelerometer  $A_2$ . Again, gyroscope  $G_3$  is precessed by signal  $(U + G') \sin \varphi$  in which  $U$  is provided by the computer,  $G'$  by integrator  $I_1$ , and  $\sin \varphi$  is taken from the computer after measurement of  $\varphi$  by the second integrator following the N/S axis accelerometer  $A_2$ .

#### IV.5 - Corrections to accelerometers

Emphasizing again that only a spherical earth is concerned for the moment and that vertical speed is negligible, the general law of relative mechanics applied to the outputs of single sensing axis accelerometers is :

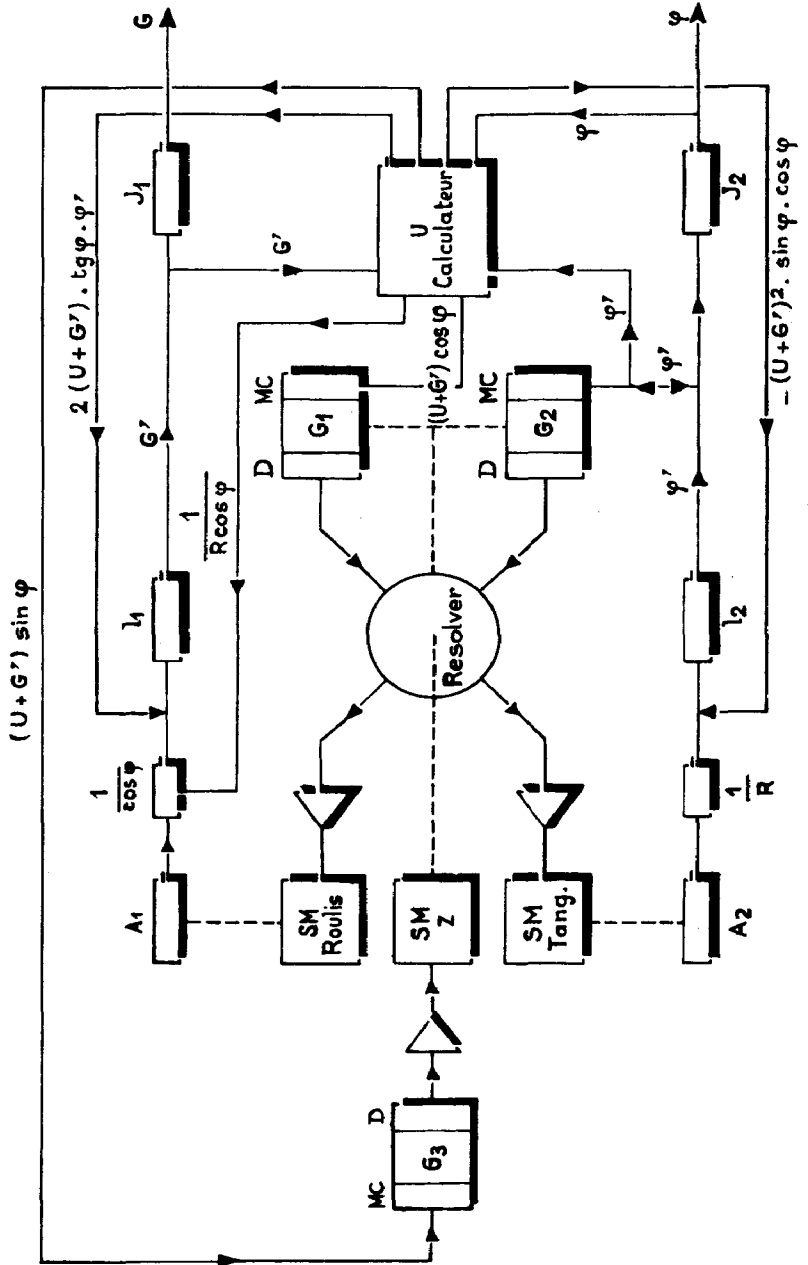
$$\text{indicated } \gamma = \text{relative } \gamma + \text{transport } \gamma + \text{Coriolis } \gamma \quad (19)$$

The two relative components of speed are :

$$\begin{aligned} V_{N/S} &= R \cdot \varphi' \\ V_{E/W} &= R \cdot G' \cos \varphi \end{aligned}$$

The two relative components of acceleration are :

$$\left. \begin{aligned} \gamma_{r N/S} &= R \cdot \varphi'' \\ \gamma_{r E/W} &= R \cdot G'' \cdot \cos \varphi \end{aligned} \right\} \quad (20)$$





The total transport rotation in each component of acceleration may be considered as the sum of the earth's rotation rate  $U$  and of the instantaneous rotations due to path  $G'$  along the parallel, and as path  $\varphi'$  along the meridian.

a) *Corrections for the N/S sensing axis accelerometer  $A_2$*

Applying equation (19) we have :

$$\text{indicated } N/S \ \gamma = R \cdot \varphi'' + (U + G')^2 \cdot R \cdot \sin \varphi \cos \varphi + 0 \quad (21)$$

Indeed the transport accelerations of centripetal form are due on one hand to the displacement in longitude  $(U + G')$  and on the other hand to the displacement in latitude  $\varphi'$ . The first is  $R(U + G')^2 \cos \varphi$ , and its projection on the N/S track is  $R(U + G')^2 \cos \varphi \cdot \sin \varphi$ . The second is normal to the N/S path and its projection is zero.

The Coriolis acceleration is caused by rotation  $(U + G')$  but its projection on the N/S track is zero.

b) *Corrections for the E/W sensing axis accelerometer  $A_1$*

Equation (19) gives :

$$\text{indicated } E/W \ \gamma = R G'' \cos \varphi + 0 - 2(U + G') \cdot R \varphi' \cdot \sin \varphi \quad (22)$$

The centrifugal accelerations due to displacements in longitude and latitude are orthogonal to the E/W path, and so their projections on the track followed are zero.

The Coriolis component due to rotation  $(U + G')$  is projected in true value on the E/W path; it is expressed by :

$$- 2(U + G') \cdot R \varphi' \cdot \sin \varphi \cdot \cos 0.$$

c) *Working out of corrections*

The above-mentioned computer used for stabilization of gyroscopes also provides corrections to accelerometers. It takes :  $U$  from a programme,  $\varphi'$  from the output of the first N/S integrator  $I_2$ ,  $\varphi$  from the output of the second N/S integrator  $J_2$ ,  $G' \cos \varphi$  from the output of the first E/W integrator  $J_1$ ,  $G$  from the output of the second E/W integrator  $J_1$ ;  $\sin \varphi$  and  $\cos \varphi$  are worked out in the computer. (Changes in latitude of a ship, or even long-range aircraft, are relatively small).

d) *Final readings*

Equations (21) and (22) define the horizontal components of accelerations relative to the earth which are picked-off after corrections at the accelerometer's output :

$$\varphi'' = \frac{\text{indicated } N/S \ \gamma}{R} - (U + G')^2 \sin \varphi \cdot \cos \varphi \quad (23)$$

$$G'' = \frac{\text{indicated } E/W \ \gamma}{R \cdot \cos \varphi} + 2(U + G') \cdot \varphi' \cdot \tan \varphi \quad (24)$$

$\varphi$  and  $G$  are deduced by a double integration with respect to time.

IV.6.1 - FLATTENING CORRECTIONS

If the earth is considered as a revolution ellipsoid with a flattening  $\alpha = \frac{a-b}{a} = \frac{1}{297}$ , the constant value of the sphere's radius R must be replaced in formulae (23) and (24) by the main curvature radii of this ellipsoid at the place under consideration, i.e. for the meridian section ellipse, by the approximate value :

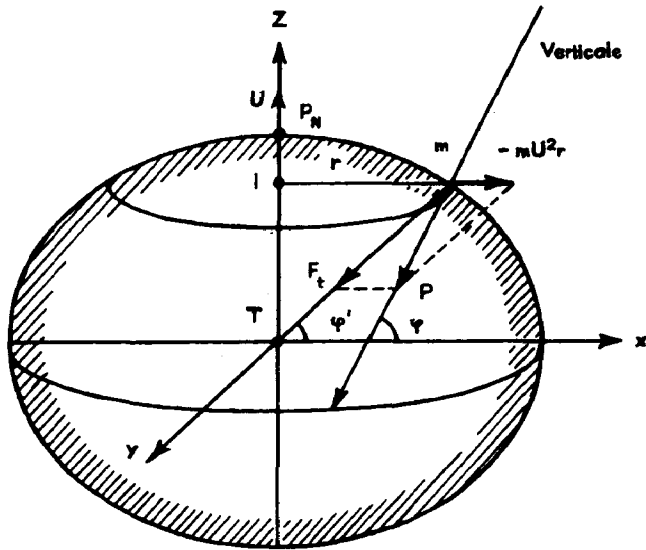
$$r_1 = a (1 - 2 \alpha \cdot \cos 2 \varphi) \tag{25}$$

and, in the normal plane, by the value :

$$r_2 = a (1 + 2 \alpha \cdot \sin 2 \varphi) \tag{26}$$

$a$  being the equatorial radius and  $b$  the polar radius. It is known that these differ from one another by about 21 480 metres; the  $1^\circ$  arc of meridian differs by 1 125 metres between the pole and the equator. It is therefore remarkable that the direction of the vertical thus determined is that of the main normal to the ellipsoid which is the geometrical direction of gravity. Consequently, if we consider the actual weights resulting from the earth's attracting force  $F_0$  and from the centrifugal inertial force  $m U^2 r$ , we have thus already taken into account the centripetal acceleration appearing in the correction term of equation (21) :  $(U + G')^2 \sin \varphi \cdot \cos \varphi$ . We may then neglect the  $U^2$  term and the correction term is reduced to :

$$\varphi'' = \frac{\text{indicated N/S } \gamma}{R} - (G'^2 + 2 U \cdot G') \sin \varphi \cdot \cos \varphi \tag{27}$$



IV.6.2 - CORRECTION FOR ALTITUDE Z

If the horizontal movement occurs at altitude Z, (aircraft) radius R must be replaced by  $(R + Z)$ ; Z being assumed to be known with sufficient accuracy by, for instance, an altimeter.

Notes —

a) At altitude  $Z$  the Schüler condition :

$$C\Omega = K \cdot R \quad (14)$$

is not fulfilled. The torque constant  $K$  must be adjusted according to the equation

$$C\Omega = K \cdot R \left( 1 + \frac{Z}{R} \right).$$

At an altitude of 15 000 metres the error in the Schüler condition is about  $\frac{15}{64}$ , i.e.  $\frac{1}{4}$ . It can be shown that the error in the vertical is very small but that the positional error is proportional to time or to the path travelled. This correction may be pre-computed and applied to the position indicated for each altitude.

b) The small change in  $g$  with elevation, according to NEWTON's law :

$$g_z = g_0 \frac{R^2}{(R + Z)^2} \quad (28)$$

does not change the Schüler condition but causes small variations in the period

$$T_0 = 2\pi \sqrt{\frac{R}{g_0}}$$

which becomes :

$$T = 2\pi \cdot \frac{R + Z}{R} \sqrt{\frac{R + Z}{g_0}} = T_0 \left( 1 + \frac{Z}{R} \right).$$

Finally the approximate value :

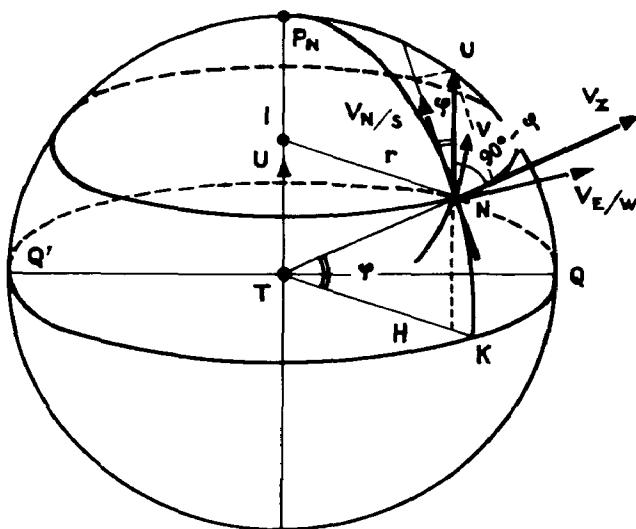
$$g_z \approx g_0 \left( 1 - \frac{2Z}{R} \right) \quad (29)$$

is adopted for  $g$ .

## V. — NON-MARITIME VEHICLES

### V.1 - Case of a vertical speed $V_z$

This is applicable to missiles, V.T.O.L. aircraft and aircraft with a rapid rate of climb. A third accelerometer with an OZ upward sensing axis, perpendicular to the platform, must be added to the two accelerometers with N/S and E/W sensing axes located on the platform. Its signals will then be processed by two successive integrations as in the case of the other two accelerometers. Consequently new corrections to the horizontal accelerometers are required, because of the effect of new transport and Coriolis accelerations caused by  $V_z$ .



### V.2 - Corrections to N/S accelerometer

A transport tangential acceleration of the type  $V \cdot \frac{d\alpha}{dt}$  must be taken into account, that is :

$$-V_N \cdot \frac{V_z}{R} = -\varphi' \cdot V_z$$

If we consider the astronomical vertical, the term in  $U^2$  is neglected, and the final output of the accelerometer becomes :

$$\varphi'' = \frac{\text{indicated N/S } \gamma}{R} - \varphi' \cdot \frac{V_z}{R} - (G'^2 + 2U \cdot G') \sin \varphi \cos \varphi \quad (30)$$

### V.3 - Corrections to E/W accelerometer

We must consider :

- a) a transport tangential acceleration of the type :

$$-V_E \cdot \frac{V_z}{R} = -G' \cdot V_z \cdot \cos \varphi$$

- b) a Coriolis acceleration, projected in true value on the E/W axis, due to rate  $U$  and speed  $V_z$  :

$$-2U \cdot \cos \varphi \cdot V_z$$

The total correction to acceleration indicated on the E/W axis is then :

$$G'' = \frac{\text{indicated E/W } \gamma}{R \cos \varphi} - G' \cdot \frac{V_z}{R} - 2U \cdot \frac{V_z}{R} + 2(U + G') \varphi' \cdot \tan \varphi \quad (31)$$

### V.4 - Correction to OZ accelerometer

As previously the fundamental law expressed by equation (19) is :

$$\text{relative } \gamma = \text{indicated } \gamma - \text{transport } \Sigma \gamma - \text{Coriolis } \Sigma \gamma$$

a) The transport accelerations are in two parts :

1 — the sum of centripetal accelerations due to  $V_{N/S}$  and  $V_{E/W}$ , i.e.

$$-\frac{V_{N/S}^2}{R} \text{ and } \frac{V_{E/W}^2}{R},$$

which are projected in true value on the OZ axis, i.e. again

$$R(\varphi'^2 + G'^2 \cos^2 \varphi).$$

2 — The gravitational component, i.e. about

$$g_0 \left(1 - \frac{2Z}{R}\right).$$

b) The component of Coriolis acceleration between rotation rate  $U$  and  $V_{E/W}$ , projected on  $\gamma_z$

$$2 \cdot U \cdot \cos \varphi \cdot V_{E/W} = 2 U \cdot R \cdot G' \cos^2 \varphi$$

Summarizing, we have :

$$\begin{aligned} \text{relative } \gamma &= \text{indicated } \gamma + R(\varphi'^2 + G'^2 \cos^2 \varphi) + \\ &+ 2 \cdot U \cdot R \cdot G' \cos^2 \varphi + g_0 \left(1 - \frac{2Z}{R}\right) \end{aligned} \quad (32)$$

We can see that after double integration these relative accelerations lead to the fixing of the aircraft or missile by means of two spherical coordinates which are the geographical coordinates of the point on the earth having, for instance, the missile at the zenith and a rectilinear coordinate which is altitude  $Z$  on the vertical of this point. However measurement of this elevation by inertial systems is only interesting for missiles because the term  $g_0 \left(1 - \frac{2Z}{R}\right)$ , representing gravity, remains uncertain. Finally, element  $V_z$  should preferably be measured by a device independent of the loop, for example, by a barometric or radio altimeter.

## VI. — MAGNITUDES OF CERTAIN DIMENSIONS AND PARAMETERS

Between 1950 and 1963 gyroscopes used in inertial navigation have been reduced in weight and size by as much as ten times in some cases, from the range of (17 kg - 4 kg) to (5 kg - 0.250 kg) in their weight and from (20 cm - 8 cm) to (10 cm - 5 cm) in their size. In subminiaturized gyros, designed for missiles, the rotor made of beryllium weighs no more than 150 g and has a 3 cm diameter and a 1 cm width. The weight and the dimensions of the platforms were reduced and at the same time their behaviour became more reliable. Thus the 1950 platform, of 700 kg and 50 cubic feet, was liable to have one failure per 50 hours during laboratory tests; at the present time one can expect aircraft platforms of 50 kg and 3 cubic feet to have only one failure per 2 000 operational hours.

Considerable progress has also been made in accelerometers, which now reach a minimum measuring bias of  $10^{-5} g$  in maritime navigation

and  $10^{-4} g$  in aerial navigation. Their linearity has been brought to figures better than  $10^{-4}$  in the measuring range. Finally, by means of amazing technological advances, and through unconventional solutions to the problems of gyroscopic gimbals, the mean wander of gyroscopes has been considerably reduced. This drift is now as small as one minute of arc per hour and is often smaller than one hundredth of a degree. The angular momentum, which in the past exceeded  $5 \times 10^7 g/cm^2/s$  has been generally reduced to about  $10^5 g/cm^2/s$  for vertical reference gyros, and to  $5 \times 10^6 g/cm^2/s$  for directional gyros. The spin rate averages about 20 000 r.p.m. i.e. an angular value of  $2 \times 10^3$  radians/second.

The earth's rotation rate  $U$ , which is slightly faster than the mean sidereal movement, is  $7.29 \times 10^{-5}$  radians/second, i.e. about  $15''.0438$  per hour at the equator. (The 84.4-minute Schuler period corresponds to a pulsation of  $\frac{2\pi}{T} = \omega$ ,  $\omega$  being  $1.24 \times 10^{-3}$  radians/second). The earth's rotation rate set in computations must, at the latitude of the vehicle, be known to better than  $2 \times 10^{-4}$  part of its value. The angular speeds of ships or aircraft along the meridian or parallel, i.e.  $\varphi'$  and  $G' \cos \varphi$ , may be considered as instantaneous rotations and are comparable to the earth's rotation components on the horizon and about the vertical.

For a seagoing ship, with a speed of 36 knots along the equator, this rotation speed is about  $\frac{1}{100}$  of minute of arc per second; for the earth it is about  $U = \frac{1}{4}$  of minute per second, i.e. about 25 times greater. In aerial navigation an E/W speed of 360 knots at the equator may be considered as equivalent to a rotation rate of  $\frac{1}{10}$  of minute per second, i.e. about the half or third part of that of the earth.

In any case the torque motor of vertical reference gyros must be able to develop a precession on the input axes greater than  $15^\circ/\text{hour}$ .

The average propulsion acceleration of a ship at sea when under way is generally less than  $10^{-3} g$ . At the extreme limit it can reach when manoeuvring a value of  $\frac{1}{20} g$ , which corresponds to an increase of speed of one knot per hour, i.e.  $0.50 m/s^2$ . It should be noted that if the ship starts from rest, a constant acceleration of  $10^{-3} g$  generates at the end of an hour an effective speed of  $\frac{3\,600 g}{10^{-3}} = 36 m/s$  or 72 knots.

The pitch and roll movements of a ship give rise to accelerations more than 100 times larger but, with a period of 4 to 12 seconds, which cannot be compared with the Schuler period.

In an aircraft at cruising speed the most usual average acceleration is about  $\frac{1}{100}$  of  $g$ . It can reach a maximum of  $3g$  and lies between 0 and  $\pm \frac{g}{10}$ .

We may obtain an idea of the magnitude of the corrections to indicated accelerations for a 36-knot ship, along N/S and E/W paths at 45° latitude, by assuming  $g$ , as above, to have an approximate value of 10 m/s<sup>2</sup>.

*Correction to N/S accelerometer :*

$$\Delta\varphi'' = G'^2 \cdot \sin \varphi \cdot \cos \varphi + 2 U \cdot G' \cdot \sin \varphi \cos \varphi \quad (27)$$

$$\text{a) } G'^2 \cdot \sin \varphi \cdot \cos \varphi \# \frac{1}{2} \times 10^{-5} g$$

$$\text{b) } 2 U \cdot G' \cdot \sin \varphi \cdot \cos \varphi \# \frac{1}{4} \times 10^{-3} g$$

*Correction to E/W accelerometer :*

$$\Delta G'' = -2 (U + G') \varphi' \cdot \tan \varphi \quad (24)$$

$$2 (U + G') \varphi' \cdot \tan \varphi \# \frac{1}{2} \times 10^{-4} g$$

For a 360-knot aircraft, for the same corrections in horizontal navigation, we have the following approximate values :

$$\text{a) } \Delta\varphi'' \# \frac{1}{2} \cdot 10^{-3} g \quad \text{for : } G'^2 \cdot \sin \varphi \cdot \cos \varphi$$

$$\text{b) } \Delta\varphi'' \# \frac{1}{3} \cdot 10^{-2} g \quad \text{for : } 2 U \cdot G' \cdot \sin \varphi \cdot \cos \varphi$$

$$\Delta G'' \# \frac{1}{3} \cdot 10^{-3} g \quad \text{for : } -2 (U + G') \cdot \varphi' \cdot \tan \varphi$$

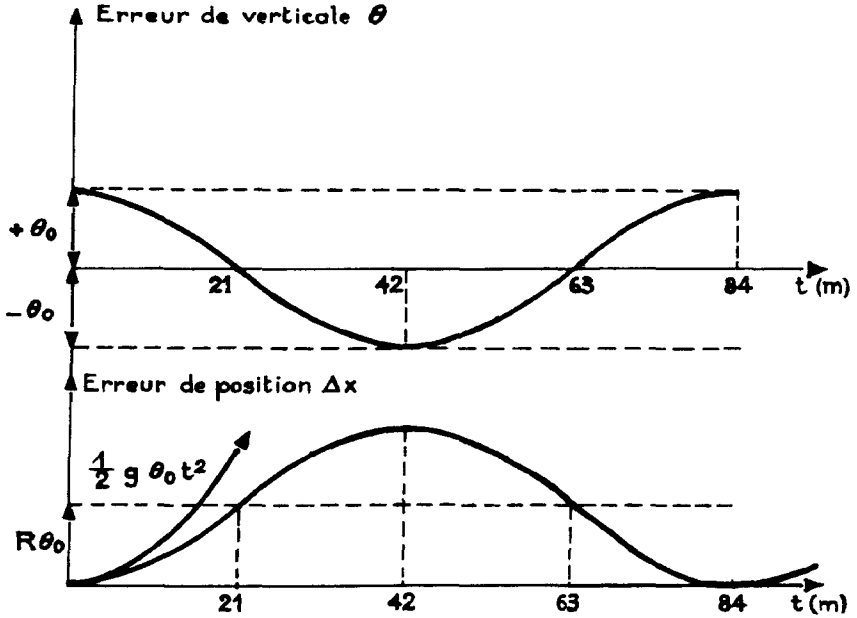
If we should apply a gravity correction to an inclined or vertical axis accelerometer, this correction is close to  $g$  in the case of vertical acceleration, i.e. about 10<sup>3</sup> times the horizontal corrections. The unfitness of vertical axis accelerometers for conventional aircraft flight can thus be seen. On the contrary, in the field of propulsive motion of missiles which ranges from 0 to 10  $g$ , the horizontal acceleration corrections are no longer of great importance.

## VII. — PLATFORM BEHAVIOUR

The ideal platform, considered in sub-section IV.2, initially set rigorously horizontal, would remain in this position if any disturbance was immediately counter-acted by the Schüler effect. In fact, this compensation does not occur immediately, and the deviation may increase during a quarter of a period (or 21 minutes) to be limited, on the average, during a period of 84 minutes, to the initial value differing from zero.

If the initial deviation is  $\theta_0$ , in 84 minutes the platform oscillates between  $+\theta_0$  and  $-\theta_0$  following the law :

$$\theta = \theta_0 \cos \omega t \tag{33}$$



with :

$$\omega^2 = \frac{g}{R} \quad \text{Schüler pulsation : } \omega = \sqrt{\frac{g}{R}} = \frac{2\pi}{T}$$

The positional error is not  $\Delta x = \frac{1}{2} \cdot g \cdot \sin \theta \cdot t^2$ , which would correspond at time  $t$  to a gravitational acceleration  $g \sin \theta$ , but :

$$\Delta x = R \theta_0 (\cos \omega t - 1) \tag{34}$$

since the initial positional error  $R\theta_0$  is known at the time origin, and necessarily corrected.

Within the range of the first 21 minutes, and for small angles  $\omega t$ , we may write :

$$\Delta x \approx R \cdot \omega^2 \cdot \frac{t^2}{2} \cdot \theta_0 \approx \frac{1}{2} \cdot g \cdot t^2 \cdot \theta_0 \tag{35}$$

For  $\theta_0 = 1'$  of arc, for example, the error in the vertical is  $\pm 1'$ , and the positional error is about 2 200 metres. The maximum value of the positional error is  $2 R \cdot \theta_0$  for  $t = 42$  minutes.



**VIII. — NOTES ON ERRORS**

The causes of errors having an influence on the indicated position are very numerous. They originate in mechanical defects or in measurement errors, and must be compensated or corrected with the same precision as that required for measurement. In this paper it is not possible to give a detailed analysis of these errors, but it seems essential to mention the most important among them since the operation of inertial navigation is, in fact, conditioned by the compensation of these errors.

First of all, mention must be made of causes of errors on the indicated position due to vertical deviations. These deviations in general cause a constant positional error equal to the vertical error, and an oscillatory error whose mean value is cancelled out during the duration of a Schuler period. Secondly, the integration errors, which obviously affect most seriously long trips, should be considered. These errors are generally due to wanders of the platform caused by stabilizing gyro drifts, or by errors in the first accelerometers. If the speed drift is  $\epsilon$  the platform shifts by  $\epsilon \cdot t$  and the measured acceleration is in error by  $-g \cdot \epsilon \cdot t$ ; the propulsive speed error is then  $g \epsilon \frac{t^2}{2}$ , and the positional error  $g \epsilon \frac{t^3}{6}$  within the limits of a time period small enough to make the sinusoidal oscillation of small amplitude. An elementary list of the principal errors is given below :

a) *Platform errors :*

- Initial tilt error  $\theta_0$
- Initial alignment error  $z_0$
- Inaccuracy of Schuler condition

b) *Acceleration errors :*

- Accelerometers {
  - Scale factor corrections
  - Bias or null error
  - Linearity

c) *Speed error :*

- Inaccuracy of first integrators {
  - Bias
  - Linearity
- Vertical reference gyros wander rate
- Directional gyros wander rate

d) *Errors due to damping term :*

- Inertial Doppler loop.

### VIII.1 - Platform errors

#### VIII.1.1 - INITIAL ADJUSTMENT

This error, mentioned in section VII, is oscillatory, of the type  $\Delta x = R \cdot \theta_0 (\cos \omega t - 1)$  (34); its maximum value is  $2R \cdot \theta_0$ , or two nautical miles for  $\theta_0 = 1'$  of arc.

#### VIII.1.2 - INACCURATE SCHÜLER CONDITION

If, instead of having the relationship :

$$C\Omega = K \cdot R \quad (14)$$

We have :

$$C\Omega = K \cdot R (1 + \varepsilon) \quad (36)$$

the error on the vertical is :

$$\Delta\theta = \frac{-\Sigma \cdot a}{\omega} \sin \omega t \quad (37)$$

with  $a = (U + G') \cos \varphi$  in, for instance, E/W navigation.

The positional error is :

$$\Delta x = R (1 + \varepsilon) \Delta\theta + \varepsilon \cdot a \cdot t \cdot R \quad (38)$$

An integration error building up with time and speed is added to the error on the vertical limited to  $\frac{-\varepsilon a}{\omega}$ . Consequently the condition  $C \cdot \Omega = K \cdot R$  must be realised to better than  $10^{-3}$ , and therefore the spin rate  $\Omega$  must be crystal-controlled, the torque constancy must be compensated for temperature, and radius  $R$  must be accurate to within  $10^{-3}$ .

#### VIII.2 - Acceleration errors

If the error on the measured acceleration is  $\eta g$ , the error on the vertical is of the type :

$$\Delta\theta = \eta (1 - \cos \omega t) \quad (39)$$

It may reach a value of  $2\eta$ ; the positional error is limited to the value  $\Delta x = 2 R \eta$ .

For  $\eta = 10^{-4}$  we have :  $\Delta x = 10^{-4} \times 2 \times 6 \times 10^3 \text{ km} \approx 1\,200 \text{ metres}$ .

#### VIII.3 - Speed errors

##### VIII.3.1 - FIRST INTEGRATORS' INACCURACY

If, at the first integrator output, the error is  $\Delta V = \mu \cdot R$ , this error is similar to the acceleration error  $\eta g$ . The error on the vertical is of the type :

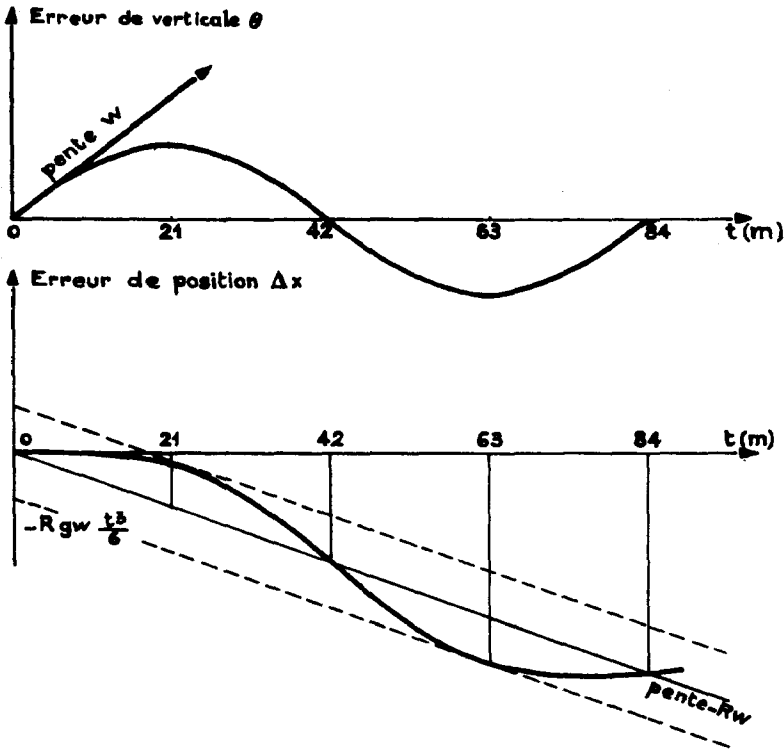
$$\Delta\theta = \frac{\mu}{g} (1 - \cos \omega t) \quad (40)$$

whose amplitude is  $\frac{\Delta V \cdot \omega}{g}$  or  $\frac{\Delta V}{R\omega}$ ; the positional error is limited to  $\pm \frac{\Delta V}{\omega}$ .

VIII.3.2 - DRIFT OF VERTICAL REFERENCE GYROS

If the drift of a vertical reference gyro is constant and equal to  $w$  the platform oscillates with the Schuler period according to the sine-law :

$$\Delta\theta = \frac{w}{\omega} \sin \omega t \tag{41}$$



Initially, the accelerometer being ineffective, the amplitude of the error on the vertical is  $\frac{w}{\omega}$ . The action of the accelerometer is to give to the oscillation a sine-form and, at time  $t$ , this oscillation will be :

$$\Delta\theta = -w \left( t - \frac{\sin \omega t}{\omega} \right) \tag{42}$$

There subsists therefore a speed error  $wt$  on the vertical, which results in an error on the indicated speed  $gw \frac{t^2}{2}$  and, if the period of time is short, in a positional error of  $gw \frac{t^3}{6}$  for small angles  $\omega t$ .

The positional error oscillates with an 84-minute period, no longer about the zero value, but about a value  $R \cdot w \cdot t$ , defined by a constant slope  $R \cdot w$ . For long periods of time, longer than 42 minutes, in maritime navigation for instance, the error is unlimited. If the drift of the vertical reference gyro is  $0.1^\circ$  per hour, the speed error is very large and is of the order of six knots.

On the contrary, for short-range ballistic missiles, the positional error increases very slowly during the first minutes according to the expression

$$-g \cdot w \cdot \frac{t^3}{6}.$$

### VIII.3.3 - WANDER OF DIRECTIONAL GYROS

This drift affects both the orientation of the speed vector, which no longer remains in the vertical propulsion plane, and the amplitude of this vector. The indicated position is affected by a double error, but the error is similar to that produced in a dead-reckoning fix by the drift  $z$  of a gyroscopic compass over a path  $X$ , i.e. about  $z \cdot \frac{X}{2}$ .

### VIII.4 - Errors due to a damping term

Generally, for a damping factor  $A$  assumed to be constant, the error mentioned is proportional to the indicated acceleration  $A (\gamma - g \theta)$ . There is a positional error proportional to speed  $V$  and to factor  $A$ . The time constant of decrease of vertical amplitude is  $\tau = \frac{2 C \cdot \Omega}{A \cdot g}$ . (The equivalent damping torque is  $A \cdot V$ ).

The oscillation of this vertical is :

$$\theta = \frac{A \cdot V}{C \cdot \Omega} \cdot \cos \omega t \quad (43)$$

The oscillatory positional error is :

$$\Delta x = -R \cdot \frac{A \cdot V}{C \cdot \Omega} (1 - \cos \omega t) \quad (44)$$

Its limits are  $\pm \frac{2 R \cdot V}{g \tau}$  which remain small if  $\tau$  is large, i.e. if the damping is poor. On the contrary a short term damping leads to large errors. If a speed measurement by Doppler is introduced in the loop, we have a comparison resulting in the difference  $\Delta V$  of the inertia speed minus the Doppler speed. We thus have a means of damping the platform oscillations while limiting positional errors; the error on the vertical tends towards zero without effect on the speed which, remaining corrected, does not entail positional errors.

### VIII.5 - Conclusions

The foregoing list, relating only to principal errors, is far from being exhaustive. Computer and servo errors, errors due to inaccurate acceleration corrections, pick-off errors, errors due to the oblateness of the earth, etc., must also be added.

To summarize this brief investigation we may say that the accelerometer measurement errors always have a significant influence; but for long duration navigation, for instance in maritime navigation, the influence of the vertical reference gyros' drift is predominant. On the contrary for missiles and short-range aircraft the zero and linearity errors due to an eventual damping term (not often necessary in this case) are the most critical.

## IX. — DEFICIENCIES OF GYROSCOPES

### IX.1 - Present performances

In the main the reliability and accuracy of inertial navigation is dependent upon the production of practically no-wander gyroscopes. This nominal drift is, however, particularly difficult to define. For a given gyroscope it is possible to observe and to define a standard value of gyro drift rate under certain conditions of starting and length of operation, and after this calibration even to compensate this average drift by means of mechanical balance or signals fed to the torque motor.

Neglecting the steady drift rate due to vehicle motion, which is corrected by signals, we must also determine the standard value of random shifts about the average drift from one day to another, one temperature to another, one starting to another, etc. Briefly, the quality of a gyroscope is an essentially variable concept. The cancelling of errors at the end of the 84-minute period is of purely academic interest. However present technology, having achieved extraordinary advances in the form of liquid, gaseous or electro-static suspensions, has made available gyroscopes whose random

drift remains less than  $\frac{1}{100}$  of a degree per hour.

### XI.2 - Causes of drift

Random drifts are due to transient, permanent or periodic precession torques of various origins : unbalance, friction, vibrations, rubbing and thermal torques. Among the most important causes of mechanical drift the following phenomena may be mentioned :

a) A lack of balance around the gimbals' horizontal axes which gives rise to a disturbing gravity torque on the input axis of directional gyros; the resulting drift modifies the directional alignment of the platform. It is

sufficient to know that a space of 1/4 micron between the centre of gravity and the centre of support involves a drift rate of 1° per hour. In classical-support modern gyros, this drift is now less than 0.1° per hour.

b) Torques caused by frictions in the gimbals' horizontal and vertical axes. A disturbing torque on an inner gimbal causes an error on the vertical.

c) Imperfection in the geometry of the framework or of the sensing element or a non-homogeneous due to metal impurities, are causes of periodic disturbing torques.

d) Cross-coupling effects, which appear as disturbing accelerations not aligned with the accelerometer measuring axis, and for gyroscopes, as unequal stiffness of the gimbals which give various responses in different directions. The disturbing torque is proportional to the square of the spurious acceleration. The drift due to these effects has been reduced to less than 1/100 of a degree per hour.

e) The aniso-elasticity effects, due to vibrations, which entail unsymmetrical acceleration components on the gyro output axis. The aniso-elastic component has also been reduced to a standard value of less than 2/100 of a degree per hour per  $g^2$ .

f) Finally the pick-offs, integrators and accelerometers, which indirectly receive signals, are affected by the noise superimposed on these signals. H.F. noises affect the differential terms; L.F. noises affect integrated signals. It may however be assumed that for small acceleration variations the H.F. noise remains small.

Gyroscopic calibration is an extremely delicate operation requiring special means of testing. After a certain number of starts a statistical error may be found, and then compensated by electrical or mechanical means. Random drift from day to day must then be determined under different acceleration and attitude conditions, i.e. during turnings and manœuvres.

Summarizing, although new technological designs have met the accuracy requirements of medium length navigation, and although the solution of inertial systems is excellent over a short period of time, it seems nevertheless that cumulative errors remain significant in the case of lengthy navigation without resetting.

Whilst, for projectiles and missiles, true inertia is sufficient for a few minutes' duration of flight, the needs of long range aerial navigation and above all, long range maritime navigation, can apparently only be met by a periodical resetting of the shifted platforms, i.e. through monitoring with an external long period aid. In aerial navigation this is the case of the Doppler system which is an excellent distance and drift integrator and which lends itself to the combination of Doppler and inertial speeds in a damping loop. Celestial navigation, and chiefly directional gyro control by automatic tracking of a low altitude celestial body, is another means which nowadays seems to be the indispensable complement to long duration inertial navigation and particularly to maritime navigation.

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