HARMONIC ANALYSIS OF TIDES FOR 7 DAYS OF HOURLY OBSERVATIONS

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1. — INTRODUCTION

In 1961 the International Hydrographic Bureau distributed to States Members the pamphlet " Análise Harmónica da Maré — 7 dias " (Harmonic Analysis of Tides — 7 Days), published by the " Diretoria de Hidrografia e Navegação " (Brazilian Directorate of Hydrography and Navigation). The pamphlet was my work, and I believed I had produced something useful from the surveyor's point of view. However the results obtained with this method were sometimes discouraging.

In 1963 Captain Langeraar, Hydrographer of the Royal Netherlands Navy, wrote me that very poor results were obtained by using the above-mentioned method, and requested my comments on the matter. I replied by stating that I was very pleased to learn that the Netherlands Hydrographic Department was interested in studying this short period method of analysis. I am now pleased because, due to the theoretical research done by the Netherlands tidal experts, the Senior Civil Hydrographic Officer W. C. Wernink and Engineer Lieutenant M. Hendrikse of the Royal Netherlands Navy, one can see that the fundamentals of the method hold good, and that the poor results obtained arose from some numerical errors in the distributed pamphlet. In fact the excellent work done by the Netherlands Hydrographic Office now allows rectification of these errors. A good agreement was then found between the real tidal curve and the predicted curve obtained by using the harmonic constants from the analysis.

It would be very desirable to make corrections to this pamphlet, for without them the users will certainly run into the same difficulties. I thought, however, that it would be better to take this opportunity to present a new edition of this work, in which I can develop several more modern theoretical concepts.

2. — FUNDAMENTAL IDEAS

Seven days is the shortest period that can be analysed by classical methods to obtain reliable results without having recourse to many hypothetical relationships. In fact the analysis of a 7-day period allows us to obtain reliable values of the harmonic constants for M_2, S_2, K_1, O_1, M_4 and MS_4, but the directly derived constants are not free from contributions of Q_1 and N_2 which cannot be isolated in such a short-period analysis. Hence some assumption must be adopted in order to reduce the effect of these constituents.

If we designate respectively by R and $-r$ the amplitude and the phase of a constituent corresponding to a definite instant, we can write :

$$R = fH$$
$$-r = V + u - g$$

where $(V + u)$ is the astronomical argument of the constituent for the Greenwich meridian at the instant corresponding to $-r$; f and u are respectively the factor and the nodal angle, and H and g are the harmonic constants to be obtained from the analysis. Hence, if we find $-r$ and R, we shall readily obtain :

$$H = R/f \tag{2a}$$
$$g = V + u + r \tag{2b}$$

But we need to transform these expressions to allow the representation of groups of constituents with relatively close speeds. In fact the constituents belonging to the groups : (S_2, K_2, T_2), (K_1, P_1) and (N_2, ν_2) cannot be separated in any short-period analysis, and they appear in the form of a single constituent having amplitude and phase slightly different from the principal constituent of their respective group. Each of these groups, as will be seen later, can be represented by its principal constituent; it is only necessary to change the corresponding values of f and u respectively into :

$$f\ (1 + W)$$

and

$$u + w$$

where w and W are small quantities. Consequently equations (2a) and (2b) become respectively :

$$H = R/f\ (1 + W) \tag{2c}$$
$$g = V + u + w + r \tag{2d}$$

The values of R and r are not given directly by the analysis. R and r are obtained in function of other previously computed elements. To derive the expressions of these new elements let us consider the height of one constituent at the instant corresponding to phase $-r$:

$$R \cos (-r)$$

If we reckon the time from the instant at which the phase is $(-r)$ and if the latter increases by $q°$ per hour, the height of the oscillation t hours later will be :

$$R \cos (qt - r)$$

or, by expanding,

$$R \cos r \cos qt + R \sin r \sin qt \tag{2e}$$

The values of R cos r and R sin r are computed by analysis, and R and r can be computed by :

$$R = \sqrt{(R \cos r)^2 + (R \sin r)^2} \tag{2f}$$

and

$$r = \tan^{-1} [(R \sin r) / (R \cos r)] \qquad (2g)$$

The harmonic constants are then obtained with (2*c*) and (2*d*).

As the hourly heights of the tide are measured on the marigram from a datum S_0 below the mean sea level, we may express this height by :

$$y = S_0 + \sum_c R \cos r \cos qt + \sum_c R \sin r \sin qt \qquad (2h)$$

where $\sum_c$ represents the sum of all the constituents.

It is interesting to give a general idea of the analysis operations up to the determination of R cos *r* and R sin *r* for each constituent. A summary of these operations follows : the first step is to combine the hourly heights to obtain pairs of linear functions of both R cos *r* and R sin *r*, which will be designated by X_1, Y_1, X_2, Y_2, X_4 and Y_4. Combinations will be chosen so that, in these functions, the greatest numerical factors of R cos *r* and R sin *r* correspond to constituents having the same subscripts as these functions. For example the coefficients of R cos *r* and R sin *r* for the diurnal constituents must be largest in functions X_1 and Y_1, those for the semi-diurnal constituents largest in functions X_2 and Y_2, etc. As the observations to be analysed cover a period of 7 days, the combinations of hourly heights for each day will give a numerical value for each function X_n and Y_n.

The next step is to combine the numerical values of X_n and Y_n to allow us :

a) to isolate each constituent approximately;

b) to separate the unknowns R cos *r* and R sin *r* into two independent groups of linear equations;

c) to obtain, in each equation, a high value of the coefficient of one unknown so that it will be considerably greater than the other coefficients.

A special combination of the equations will then permit us to form two independent systems with *m* linear equations and *m* unknowns. One of these systems gives R cos *r*, and the other R sin *r*. The solution of these systems gives the unknowns for *m* constituents (6 for a 7-day period). From R cos *r* and R sin *r*, R and *r* may be computed from expressions (2*f*) and (2*g*), and the harmonic constants from (2*c*) and (2*d*).

It is logical, since we are considering only 6 constituents and neglecting the effect of many others, that the present method of analysis should be only an approximation, which is, however, sufficiently accurate for practical applications.

After this brief description of the entire analysis we are in a position to study the details.

3. — THE DAILY PROCESS

DOODSON calls the daily process the combination of hourly heights for each day to obtain the numerical values of functions X_n and Y_n, where the constituents of species *n* are approximately isolated. These combinations can be formed by multiplying the hourly heights y_t by a special set of whole multipliers D_t and making the sum. This may be expressed mathematically by :

$$\sum_t D_t y_t = F_n \tag{3a}$$

where F_n is either function of X_n or Y_n corresponding to day d. It is extremely important to point out that subscript t *indicates the hours corresponding to the ordinates selected for the combination.* In addition D_t represents whole numbers, positive or negative, also corresponding to hours t, appropriately chosen to isolate approximately any group of constituents of the same species. It is possible to find these multipliers in sine and cosine tables prepared according to speeds q of solar constituents S_1, S_2, etc. However it is preferable to follow the more general symbolic method devised by A. ARAGNOL, which can also be used to find multipliers for more elaborate methods.

Let $-r_0$ be the phase of a constituent at the time the observations were begun. By assuming this instant to be the *zero* hour of the *zero* day, the phase at t hours of day d (equal to the number of days reckoned from the first day) will be :

$$qt + \rho d - r_0$$

where ρ is the daily speed. The height of the tide at that instant will be :

$$y_t = S_0 + \sum_c R \cos (qt + \rho d - r_0)$$

But S_0 can be considered as a constituent having $q = \rho = r_0 = 0$. Thus we may write :

$$y_t = \sum_c R \cos (qt + \rho d - r_0)$$

But as

$$\cos x = (e^{ix} + e^{-ix}) / 2$$

it may be written :

$$y_t = \sum_c R \left[e^{i(qt+\rho d-r_0)} + e^{-i(qt+\rho d-r_0)}\right] / 2$$

or :

$$y_t = \sum_c \frac{R}{2} \left[e^{i(\rho d-r_0)}\, e^{iqt} + e^{-i(\rho d-r_0)}\, e^{-iqt}\right] \tag{3b}$$

But as R is considered as constant during the period covered by the observations, and as $(\rho d - r_0)$ during one day is also constant, we have from (3*a*) and (3*b*) :

$$\sum_t D_t y_t = \sum_c \frac{R}{2} \left[e^{i(\rho d-r_0)} \sum_t D_t e^{iqt} + e^{-i(\rho d-r_0)} \sum_t D_t e^{-iqt}\right] \tag{3c}$$

As will be seen later it is always possible to choose multipliers D_t and corresponding hours t so that we have :

$$\sum_t D_t e^{\pm iqt} = k e^{\pm i\omega} \tag{3d}$$

where k and ω will be the constants for each constituent. Therefore, substituting in (3*c*) and returning to the trigonometrical form, we obtain :

$$\sum_t D_t y_t = \sum_c k R \cos (\rho d - r_o + \omega) \tag{3e}$$

We then see that the constituent's contribution to the result of a combination depends on the value of k. If we wish to eliminate a group of constituents it is necessary that for these constituents $k = 0$, which results in the condition :

$$\sum_t D_t \, (e^{\pm iq})^t = 0 \tag{3f}$$

If we put :

$$e^{\pm iq} = z \tag{3g}$$

the following equation results :

$$\sum_t D_t \, z^t = 0 \tag{3h}$$

We shall choose equations of this form, according to the needs of the analysis, in which the powers of z are the hours corresponding to the ordinates to be combined and where the coefficients D_t of z are whole numbers.

Let us first show that it is always possible to obtain the simultaneous elimination of all constituents having speeds nq_o ($n = 0, 1, 2, 3, 4, ...$) by a simple subtraction of ordinates. If θ is an hour selected so that :

$$\theta = 360°/q_o \tag{3i}$$

it follows from ($3g$) :

$$z^\theta = e^{\pm in360°} = 1$$

Hence :

$$z = e^{\pm inq_o} \tag{3j}$$

is one solution of the binomial equation :

$$1 - z^\theta = 0 \tag{3k}$$

Since we can consider 1 as z^0, from the above equation we have the whole multipliers $D_o = 1$ and $D_\theta = -1$, so that if we combine the ordinates at zero and θ hours, according to ($3a$), we have from ($3f$) :

$$e^0 - e^{\pm in360°} = 0$$

which means that this combination therefore eliminates all constituents nq_o, as well as S_0, for which $n = 0$.

From a practical point of view we shall consider the solar constituents having speeds $q = n15°$, and in more elaborate procedures, the lunar constituents having approximately $q = n14°\!.5$ and O_1 with speed $13°\!.9$. Thus we shall have :

$$\theta = 360°/15° = 24 \quad \text{for the solar constituents}$$
$$\theta = 360°/14°\!.5 \approx 25 \quad \text{for the lunar constituents}$$
$$\theta = 360°/13°\!.9 \approx 27 \quad \text{for } O_1 .$$

The value of θ for O_1 would be nearer 26 than 27 but the latter figure is used because it is more appropriate for expansion in factors, as will be shown later, and the resulting error is negligible.

If we wish to isolate a group of constituents of subscript n, it is necessary to be able to combine hourly heights so as to eliminate all constituents with a subscript differing from n. It is easy to see that the combinations expressed in binomial form in equation ($3k$) do not satisfy the needs of the analysis.

The expression ($3j$) is one of the solutions of ($3k$), but we can obtain many other solutions if we expand the left-hand side of ($3k$) as a product of polynomials. Any value of z which cancels one of the polynomials will be a root of ($3k$). The several values of z fulfilling this condition will be found by resolving the equations formed by making the chosen polynomials equal to zero. As we shall show, the values of z indicate subscripts n of the eliminated constituents, and the coefficients of z^t in the chosen polynomials will be the multipliers D_t to be used in ($3a$) to obtain elimination.

Several expansions of ($3k$) can be made as a product of the polynomials for $\theta = 24, 25, 27$, as indicated below. Thus we have :

$$
\begin{aligned}
1-z^{24} &= (1-z^8)(1+z^8+z^{16}) \\
&= (1+z^4)(1-z^4)(1+z^4+z^8)(1-z^4+z^8) \\
&= (1+z^4)(1+z^2)(1-z^2)(1+z^2+z^4)(1-z^2+z^4)(1-z^4+z^8) \\
&= (1+z^4)(1+z^2)(1+z)(1-z)(1+z+z^2)(1-z+z^2)(1-z^2+z^4)(1-z^4+z^8) \\
&= (1+z^{12})(1-z^{12}) \\
&= (1+z^{12})(1+z^6)(1-z^6) \\
&= (1+z^{12})(1+z^6)(1+z^3)(1-z^3) \\
1-z^{25} &= (1-z^5)(1+z^5+z^{10}+z^{15}+z^{20}) \\
1-z^{27} &= (1-z^9)(1+z^9+z^{18})
\end{aligned}
$$

All the factors of the above expansion are shown in table 3-I (second column).

Let us now see how to obtain the expressions of k and ω from table 3-I. According to ($3d$) and ($3g$) we may write the following identity :

$$\sum_t D_t z^t \equiv \sum_t D_t e^{\pm iqt} \qquad (3l)$$

To make these computations clear, let us select a polynomial from table 3-I. If we choose polynomial (13) as the left-hand side of ($3l$) we have :

$$1 - z^{12} = -z^6 (z^6 - z^{-6})$$

According to ($3g$) and the following general expression :

$$2i \sin x = e^{ix} - e^{-ix} \qquad (3m)$$

we have :

$$1 - z^{12} = -2ie^{\pm 6iq} \sin 6q$$

but

$$ie^{\pm 6iq} = e^{\pm i\,(6q+90^\circ)}$$

Hence :

$$1 - z^{12} = 2e^{\pm i\,(6q+90^\circ)} \cos\,(6q + 90^\circ)$$

Thus, from this expression and from ($3l$), we have :

$$\sum_t D_t e^{\pm iqt} = 2e^{\pm i(6q+90^\circ)} \cos\,(6q + 90^\circ) \qquad (3n)$$

This expression shows that by means of a combination made according to polynomial (13) the summations indicated by ($3l$) will be transformed into an expression of the form ($3d$) in which $k = 2 \cos\,(6q + 90^\circ)$ and $\omega = 6q + 90^\circ$. These expressions are given on line (13) of table 3-I.

k is a function of q. If k is not zero, the expression will not be zero, and will be equal to the contribution of the constituent having a speed q, which is not cancelled by the chosen combination. The value of this contribution is determined, as previously explained, by the value of coefficient k.

A similar expansion can be made for polynomial (16) of table 3-I :

$$
\begin{aligned}
1 + z^2 + z^4 &= (1 - z^6) \,/\, (1 - z^2) \\
&= z^2\,(z^3 - z^{-3}) \,/\, (z - z^{-1})
\end{aligned}
$$

Hence, according to (3g) and (3l) :

$$1 + z^2 + z^4 = e^{\pm 2iq} \sin 3q \ / \sin q$$

Consequently :

$$\sum_t D_t e^{\pm iqt} = e^{\pm 2iq} \sin 3q \ / \sin q \qquad (3o)$$

In this case we have $k = \sin 3q / \sin q$ and $\omega = 2iq$, expressions which are given in table 3-I for combination (16).

TABLE 3-I

No.	Polynomials	k	ω	S_n = 0 for n =
(1)	$1 + z$	$2 \cos 0.5\,q$	$0.5\,q$	12
(2)	$1 + z^2$	$2 \cos q$	q	6
(3)	$1 + z^3$	$2 \cos 1.5\,q$	$1.5\,q$	4
(4)	$1 + z^4$	$2 \cos 2q$	$2\,q$	3
(5)	$1 + z^6$	$2 \cos 3q$	$3\,q$	2, 6
(6)	$1 + z^{12}$	$2 \cos 6q$	$6\,q$	1, 3, 5, 7
(7)	$1 - z$	$2 \cos (0.5\,q + 90°)$	$0.5\,q + 90°$	0
(8)	$1 - z^2$	$2 \cos (q + 90°)$	$q + 90°$	0
(9)	$1 - z^3$	$2 \cos (1.5\,q + 90°)$	$1.5\,q + 90°$	0, 8
(10)	$1 - z^4$	$2 \cos (2\,q + 90°)$	$2\,q + 90°$	0, 6
(11)	$1 - z^6$	$2 \cos (3q + 90°)$	$3\,q + 90°$	0, 4, 8
(12)	$1 - z^8$	$2 \cos (4q + 90°)$	$4\,q + 90°$	0, 3, 6
(13)	$1 - z^{12}$	$2 \cos (6\,q + 90°)$	$6\,q + 90°$	0, 2, 4, 6, 8
(14)	$1 - z^{24}$	$2 \cos (12\,q + 90°)$	$12\,q + 90°$	0, 1, 2, 3, ...
(15)	$1 + z + z^2$	$\sin 1.5\,q / \sin 0.5\,q$	q	3
(16)	$1 + z^2 + z^4$	$\sin 3\,q / \sin q$	$2q$	4, 8
(17)	$1 + z^4 + z^8$	$\sin 6\,q / \sin 2\,q$	$4q$	2, 4, 8
(18)	$1 + z^8 + z^{16}$	$\sin 12\,q / \sin 4\,q$	$8\,q$	1, 2, 4, 5, 7, 8
(19)	$1 - z + z^2$	$\cos 1.5\,q / \cos 0.5\,q$	q	4
(20)	$1 - z + z^4$	$\cos 3\,q / \sin q$	$2\,q$	2
(21)	$1 - z^4 + z^8$	$\cos 6\,q / \cos 2\,q$	$4\,q$	1, 5, 7
(22)	$1 + z^5 + z^{10} + z^{15} + z^{20}$	$\sin 13.5\,q / \sin 4.5\,q$	$9\,q$	Lunar series (nearly)
(23)	$1 + z^9 + z^{18}$	$\sin 12.5\,q / \sin 2.5\,q$	$10\,q$	O_1 (nearly)

Expressions (3n) and (3o) as well as all those obtained with the polynomials from table 3-I are of the type (3d).

By using the values of k from table 3-I it is possible to compute the contributions of any constituent to the chosen combination. In the case of solar constituents we have $q = n15°$, and it is easy to find subscripts corresponding to the solar constituents eliminated by the chosen combinations. It suffices to compute n by resolving the trigonometric equations obtained by making $q = n15°$ and $k = 0$. The last two combinations of table

3-I give nearly zero values respectively for the speeds of the lunar constituents and of O_1.

When the values of k are given by expressions of the type :

$$(\sin aq) \,/\, (\sin bq)$$

it is impossible to make $q = 0$ to find the contributions of S_0, since we obtain the indeterminate form 0/0. In this case it becomes necessary to write the above expression as follows :

$$\frac{(\sin aq) \,/\, aq}{(\sin bq) \,/\, bq} \cdot \frac{a}{b}$$

and to take its limit, which is a/b, when q tends towards zero. Thus a/b will be the contribution.

Table 3-I gives us the contributions for very simple combinations. However we may choose combinations which are represented by the product of several polynomials. In this case, the right-hand side of expression (3*d*) will be equal to the product of various values of $ke^{\pm i\omega}$ and the result will be expressed by :

$$\sum_t D_t e^{\pm iqt} = J e^{\pm i\eta} \tag{3p}$$

where

$$J = k_1 \cdot k_2 \cdot k_3 \cdot \ldots$$

and

$$\eta = \omega_1 + \omega_2 + \omega_3 + \cdots \tag{3q}$$

Returning to the trigonometrical form, we may write, in place of (3*e*), the general expression :

$$\sum_t D_t y_t = \sum_c JR \cos (\rho d - r_o + \eta) \tag{3r}$$

or, according to (3*a*) :

$$F_n = \sum_c JR \cos (\rho d - r_o + \eta) \tag{3s}$$

which is, like the hourly ordinates, an harmonic expression since J and η are constants which depend only on the adopted combination.

In order to find the most convenient multipliers it is helpful to draw three important conclusions from the theory we have just developed :

a) as all the polynomials of table 3-I can be represented by expressions similar to (3*d*), it is always possible, according to the needs of the elimination, to multiply any of those polynomials by another even if they do not belong to the same expansion in factors;

b) as J is given by the product of the values of k, if $k = 0$ for one constituent or group of constituents, then the combination used is free from contributions of this constituent or group;

c) when a solar constituent of subscript n is eliminated, the contributions of all non-solar constituents having the same subscript are considerably reduced, as their speeds differ only slightly from those of the solar constituents having the same subscript.

In this method of analysis only the 24 hourly heights of the same day are used in each combination for the daily process. Thus the highest power of the product of the chosen polynomials must not exceed 23.

We are now in a position to find all the hourly multipliers D_t used for the approximate isolation of a group of constituents having the same subscript. In the method just described we use only the combinations which eliminate exactly the groups of solar constituents. Let us begin by finding the hourly multipliers to compute function X_1. In this function we wish the largest contributions to come from the diurnal constituents, with small contributions from the other constituents. In addition, only constituents having subscripts $n = 0$, 1, 2 and 4 are considered. In table 3-I we see that combination (13) would be sufficient to eliminate all the constituents having even subscripts. However, to increase the contribution of the diurnal constituents and to operate with all the daily ordinates, combinations (3), (11) and (15) are also used. The product of the four polynomials will contain all powers of z, from 0 to 23, and the values of D_t which are the coefficients of z^t equal to $\pm$ 1, as shown in table 3-II. In this table we see the entire set of multipliers used to obtain all the daily values of functions X_n and Y_n, and table 3-III gives the various polynomials from table 3-I used to find these functions. In the third and fourth columns of this table we find the expressions of J and of η obtained from formulas ($3q$) and table 3-I.

TABLE 3-II

Daily multipliers

functions	HOURS																						
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
X_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X_1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
Y_1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1
X_2	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
Y_2	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
X_4	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1	1	0	-1	-1	0	1
Y_4	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1

TABLE 3-III

Funct.	Polynomials used	J	η
X_0	(1) (2) (4) (18)	$\sin 12q/\sin 0.5q$	$11.5q$
X_1	(15)(3) (11) (13)	$4 \sin^2 3q \sin 6q/\sin 0.5q$	$11.5q$
Y_1	(15) (3) (5) (13)	$2 \sin^2 6q/\sin 0.5q$	$11.5q + 90°$
X_2	(15) (9) (11) (6)	$8 \sin^2 1.5q \sin 3q \cos 6q/\sin 0.5q$	$11.5q$
Y_2	(15) (3) (11) (6)	$4 \sin^2 3q \cos 6q/\sin 0.5q$	$11.5q + 90°$
X_4	(8) (9) (5) (6)	$2 \sin q \sin 12q/\cos 1.5q$	$11.5q$
Y_4	(15) (9) (5) (6)	$\tan 1.5q \sin 12q/\sin 0.5q$	$11.5q + 90°$

The values of J are obtained from the formulas given in table 3-III, using speeds corresponding to the constituents. These values are given in table 3-IV, where are also seen the contributions from Q_1 and N_2 to be used afterwards.

TABLE 3-IV

Values of J

	X_0	X_1	Y_1	X_2	Y_2	X_4	Y_4
S_0	+24.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
K_1	- 0.0664	+15.3466	+15.2807	+ 0.0274	+ 0.0660	+ 0.0048	+ 0.0273
O_1	+ 1.8094	+14.5682	+16.2770	- 0.6187	- 1.6191	- 0.1133	- 0.6913
S_2	0.0000	0.0000	0.0000	+15.4548	+15.4548	0.0000	0.0000
M_2	- 0.8440	+ 1.6925	+ 0.0901	+15.0277	+15.8491	+ 0.2820	+ 0.8001
MS_4	- 0.4281	- 0.0024	+ 0.0458	- 0.8585	- 0.0288	+13.6040	+16.1240
M_4	- 0.8519	- 0.0197	+ 0.1840	- 1.7087	- 0.0910	+13.1646	+16.0006
Q_1	+ 2.8211	+14.0834	+16.6655	- 0.8717	- 2.3820	- 0.1623	- 1.0315
N_2	- 1.2066	+ 2.6311	+ 0.2154	+14.7065	+15.9599	+ 0.4158	+ 1.2042

In table 3-III we see that, for all functions X_n, we have $\eta = 11.5q$, and for Y_n, $\eta = 11.5q + 90°$. Hence we have from (3*s*) :

$$X_n = \sum_c J_x R \cos (\rho d - r_o + 11.5q) \tag{3t}$$

and

$$Y_n = -\sum_c J_y R \sin (\rho d - r_o + 11.5q) \tag{3u}$$

We can now explain the first step in the practical application of the method of analysis. In form 3-1 we see the hourly heights; the simplest way to indicate the daily process is to say that these heights are elements of a matrix (*) $\|y_{dt}\|$ to be multiplied by the transposed matrix of the hourly multipliers of table 3-II, designated by M^T. If F_n represents either one of the functions X_n or Y_n corresponding to day *d*, we may then write :

$$\| F_n \| = \| y_{dt} \| \cdot M^T \tag{3v}$$

where $\| F_n \|$ is the matrix seen in form 3-2.

During the multiplication indicated in (3*v*) it is possible to carry out a verification. Table 3-II shows that all the multipliers are ± 1, with the exception of X_4 where several are zero. Thus, all positive and negative products can be added independently, and the sum of the partial totals, regardless of the signs, will be equal to X_0, except for X_4 where the omitted heights must be added to the result.

(*) For the matrix notation we shall use :
$(\,)$ = row vector;
$\{\,\}$ = column vector;
$\|\ \|$ or capital letters M, G, etc. = matrices;
exponent T : to indicate transposition.

FORM 3-1

Place : Aratú Harbour (Brazil)

Period : 29 July to 12 August 1947

dates \ hours	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2-8	106	155	196	227	236	219	180	129	74	37	24	39	74	116	164	204	231	229	202	158	108	61	38	42
3-8	74	121	169	210	237	236	207	152	98	53	32	21	41	80	131	179	217	233	218	181	134	81	47	36
4-8	52	93	141	189	226	240	227	186	141	83	41	20	27	60	107	156	197	226	227	200	156	109	63	42
5-8	47	76	119	165	206	234	239	218	170	117	67	35	29	50	90	136	179	215	228	215	181	138	91	58
6-8	45	60	101	145	187	220	237	231	193	147	97	51	40	46	73	114	156	195	221	225	205	169	125	86
7-8	58	59	78	115	156	195	222	230	215	179	135	90	58	48	62	93	129	166	195	209	202	180	142	106
8-8	76	74	70	91	122	160	190	214	217	201	161	121	83	64	65	82	110	140	167	191	199	191	167	133

FORM 3-2

Date	X_0	X_1	Y_1	X_2	Y_2	X_4	Y_4
2	3249	-247	5	-1145	1065	20	75
3	3188	-300	- 32	-1416	668	11	24
4	3209	-267	- 69	-1533	219	-21	25
5	3303	-213	- 83	-1469	- 211	-31	21
6	3369	-209	- 59	-1289	- 605	-52	-15
7	3322	- 68	-142	- 932	- 888	-34	-50
8	3289	7	-105	- 477	-1015	-46	-69

4. — THE MONTHLY PROCESS

Here we should not speak of a monthly process since we have only one week of observations. But we shall retain the term to indicate that the operations occur in the same way as if a series of 29 days were being analysed.

To understand properly the signification of the monthly process it is necessary to start from expression (3*s*) which represents any of the daily numerical values of X_n and Y_n obtained by the daily process.

Expression (3*s*) shows that F_n is represented by a sum of harmonic terms having ρ as their daily change of phase. According to the values of ρ, the contributions of the constituents under consideration can accomplish approximately 0, 2, 3 or 4 cycles during one lunation (29.53 days). The values of ρ corresponding exactly to these cycles are given by :

$$\rho = \pm 360^\circ\, p \,/\, 29.53 = \pm 12^\circ\!.19 p \qquad (p = 0,2,3,4)$$

But for values of ρ which are not multiples of 12°.19 the values of p will not be whole numbers and must be found from :

$$p = 29.53\rho \,/\, 360^\circ \tag{4a}$$

and the following table can be drawn up for the values of ρ corresponding to the constituents considered in the analysis :

TABLE 4-I

Constituents	Daily speed ρ	Cycles p per 29.53 days
S_2 and S_0	0°	0
K_1	0.9856	0.08
O_1	— 25.3671	2.08
M_2	— 24.3815	2.00
MS_4	— 24.3815	2.00
M_4	— 48.7630	4.00
Q_1	— 38.4321	3.15
N_2	— 37.4465	3.07

Though Q_1 and N_2 cannot be directly isolated by the monthly process, the amount of their influence over the other constituents can be approximately known by using the equilibrium relationships. Hence, to compute the corrections it is necessary to keep these constituents until the formation of the final equations.

It is possible to combine the daily values of F_n to obtain new functions F_{np} where the new subscript p indicates the number of cycles per month of the constituents having the greatest contribution to F_{np}. As the subscript n already indicates that we should have the largest contributions from components having subscript n, we can see that after this second filtering we must have an approximate isolation of the constituents themselves.

Although the daily process produces a good isolation of the several species of tides, the number of days is so limited that a good isolation of the constituents is impossible by the simple combination of the daily values of F_n. In fact table 4-I shows us that, except for M_4, all the other constituents accomplish only a fraction of a cycle per week. However it will be shown that the functions obtained from the monthly process as applied to the daily values of X_n and Y_n (n being the same) can be combined so that a good isolation of each of the constituents of subscript n is possible.

As in the daily process, the first step of the monthly process is to combine functions F_n in the same way as the hourly heights, which will be expressed by :

$$\sum_d D_d F_n = F_{np}$$

To find the whole multipliers D_d we can start from expression (3s). If the days are reckoned from the central day ($d = 3$), this expression can be transformed as follows :

$$F_n = \sum_c JR \cos \left[\rho (d - 3) + 3\rho - r_0 + \eta\right]$$

or, expanding :

$$F_n = \sum_c \left[JR \cos (3\rho - r_0 + \eta) \cos \rho (d - 3) - \right.$$
$$\left. - JR \sin (3\rho - r_0 + \eta) \sin \rho (d - 3)\right]$$

If we put :

$$JR \cos (3\rho - r_0 + \eta) = A \qquad (4b)$$
$$-JR \sin (3\rho - r_0 + \eta) = B \qquad (4c)$$

we have :

$$F_n = \sum_c \left[A \cos \rho (d - 3) + B \sin \rho (d - 3)\right] \qquad (4d)$$

Now, the matrix notation is most convenient to represent the seven values of F_n which may be considered as column vector $\{F_n\}$. In fact the pairs of constant values A and B corresponding to each constituent may be arranged as a row vector designated by (A, B), and a matrix may be drawn having the daily values of $\cos \rho (d - 3)$ and $\sin \rho (d - 3)$ as alternate rows. Thus we arrive at the following expression where are seen the symbols of the constituent corresponding to each pair of rows, as well as the values of $(d - 3)$ with which $\cos \rho (d - 3)$ and $\sin \rho (d - 3)$ were computed :

$\{F_n\}^T = (A, B)$ (4e)

d-3=-3	d-3=-2	d-3=-1	d-3=0	d-3=1	d-3=2	d-3=3	
1, 000	1, 000	1, 000	1, 000	1, 000	1, 000	1, 000	(S_0)
0, 000	0, 000	0, 000	0, 000	0, 000	0, 000	0, 000	
0, 999	0, 999	0, 999	1, 000	0, 999	0, 999	0, 999	(K_1)
-0, 051	-0, 034	-0, 017	0, 000	0, 017	0, 034	0, 051	
0, 240	0, 663	0, 904	1, 000	0, 904	0, 663	0, 240	(O_1)
0, 970	0, 775	0, 428	0, 000	-0, 428	-0, 775	-0, 970	
1, 000	1, 000	1, 000	1, 000	1, 000	1, 000	1, 000	(S_2)
0, 000	0, 000	0, 000	0, 000	0, 000	0, 000	0, 000	
0, 290	0, 659	0, 911	1, 000	0, 911	0, 659	0, 290	(M_2)
0, 957	0, 752	0, 413	0, 000	-0, 413	-0, 752	-0, 957	
0, 290	0, 659	0, 911	1, 000	0, 911	0, 659	0, 290	(MS_4)
0, 957	0, 752	0, 413	0, 000	-0, 413	-0, 752	-0, 957	
-0, 832	-0, 131	0, 660	1, 000	0, 660	-0, 131	-0, 832	(M_4)
0, 555	0, 991	0, 752	0, 000	-0, 752	-0, 991	-0, 555	
-0, 427	0, 228	0, 784	1, 000	0, 784	0, 228	-0, 427	(Q_1)
0, 905	0, 974	0, 621	0, 000	-0, 621	-0, 974	-0, 905	
-0, 380	0, 261	0, 794	1, 000	0, 794	0, 261	-0, 380	(N_2)
0, 925	0, 965	0, 608	0, 000	-0, 608	-0, 965	-0, 925	

Note. — The comma in the above matrix should be read as a decimal point.

If we designate by G the matrix in (4*e*), this expression can be written as follows :

$$\{F_n\}^T = (A,B) \cdot G \qquad (4f)$$

It is now necessary to find sets of multipliers ± 1 which allow the formulation of equations in which only A or only B appears. These multipliers can be expressed in the form of column vectors, designated by $\{D_d\}$ and the combinations of the monthly process may be expressed by :

$$\{F_n\}^T \cdot \{D_d\} = (A,B) \cdot G \cdot \{D_d\} \qquad (4g)$$

As A and B are, respectively, the odd and even elements of row vector (A, B), the product $G \cdot \{D_d\}$ will be a column vector with the even elements zero if B is to be eliminated, and with the odd elements zero if we wish to eliminate A. Remembering that the odd rows of G in (4*e*) are values of $\cos \rho\ (d - 3)$, and the even rows those of $\sin \rho\,(d - 3)$, it is seen that for symmetrical values of $(d - 3)$ the cosines are equal in value and in sign, whereas the sines have equal values but opposite signs. Thus it is possible to find $\{D_d\}$ according to the signs of $\cos \rho\,(d - 3)$ and $\sin \rho\,(d - 3)$. In fact, expression (4*e*) shows that the values of $\cos \rho\ (d - 3)$ for S_2 are all equal to 1; hence if we take these values as vector $\{0\}$ of table 4-II, it is seen that $G \cdot \{0\}$ will be a column vector having as elements the algebraical sums of the G rows. Hence

TABLE 4-II

Weekly multipliers

$d - 3$	Combinations		
	$\{0\}$	$\{b\}$	$\{4\}$
-3	1	1	2
-2	1	1	2
-1	1	1	0
0	1	0	0
1	1	-1	0
2	1	-1	2
3	1	-1	2

all the even elements of column vector $G \cdot \{0\}$ will be zero, which is the condition necessary to eliminate B from row vector (A, B). Thus by replacing $\{D_d\}$ by $\{0\}$ in equation (4*g*), we can write the following expression, where neither the zero values of column vector $G \cdot \{0\}$ nor B appear :

$$\{F_n\}^T \cdot \{0\} = (A) \left\{ \begin{array}{c} 7.000 \\ 6.984 \\ 4.614 \\ 7.000 \\ 4.720 \\ 4.720 \\ 0.394 \\ \hline 2.170 \\ 1.350 \end{array} \right\} \begin{array}{l} (S_0) \\ (K_1) \\ (O_1) \\ (S_2) \\ (M_2) \\ (MS_4) \\ (M_4) \\ \\ (Q_1) \\ (N_2) \end{array} \qquad (4h)$$

It is seen in the above expression that the largest coefficients correspond to S_0 and to constituents K_1 and S_2, which, according to table 4-I, do not accomplish any cycle per month. This fact justifies the choice of symbol 0 to designate the combination based on the daily values of cos ρ $(d - 3)$ for S_2. In addition, we see from (4*h*) that only constituent M_4 is conveniently reduced by the combination, which means that another combination will be necessary to isolate approximately, let us say, the constituents with two cycles per month (M_2, MS_4 and O_1).

If a column vector with ± 1 elements is chosen so that the signs are the same as those of sin ρ $(d - 3)$ for M_2, MS_4 and O_1, we obtain vector $\{b\}$ of table 4-II. The product of $G \cdot \{b\}$ will be a column vector where all the odd elements will be zero, which is the condition to eliminate A from (A, B). Thus by replacing $\{D_d\}$ by $\{b\}$ in (4*g*), we obtain the following expressions where neither the zero values of product $G \cdot \{b\}$ nor A appear :

$$\{F_n\}^T \cdot \{b\} = (B) \left\{ \begin{array}{rl} 0.000 & (S_0) \\ -0.204 & (K_1) \\ 4.346 & (O_1) \\ 0.000 & (S_2) \\ 4.244 & (M_2) \\ 4.244 & (MS_4) \\ 4.596 & (M_4) \\ \hline 4.200 & (Q_1) \\ 4.996 & (N_2) \end{array} \right. \qquad (4i)$$

We have adopted a letter to designate the vector formed by the new multipliers, to specify that this vector was obtained from a sine row instead of a cosine row in matrix (4*e*). The *second* letter of the alphabet was chosen to indicate that the sine row signs which gave $\{b\}$ were those corresponding to the constituents having two cycles per month.

To isolate the fourth-diurnal constituents it is necessary to consider the values of cos ρ $(d - 3)$ for M_4, which has four cycles per month (table 4-I). After some trials it was concluded that the best results would be found by using only the negative values of cos ρ $(d - 3)$, and making positive multipliers 2 correspond to these values. In this way vector $\{4\}$ of table 4-II was obtained and the following expression was found :

$$\{F_n\}^T \cdot \{4\} = (A) \left\{ \begin{array}{rl} 6.000 & (S_0) \\ 7.992 & (K_1) \\ 3.492 & (O_1) \\ 8.000 & (S_2) \\ 3.796 & (M_2) \\ 3.796 & (MS_4) \\ -3.852 & (M_4) \\ \hline -0.418 & (Q_1) \\ -0.238 & (N_2) \end{array} \right. \qquad (4j)$$

Though there are strong contributions from all the constituents other than the fourth-diurnal in (4*j*), these will not be troublesome, because a good isolation of the species has already been obtained with the daily process.

It is now necessary to expand A and B according to the values of J and η. Table 3-III shows that :

$$\eta = 11.5\,q \qquad \text{for functions } X_n$$

and

$$\eta = 11.5\,q + 90° \quad \text{for functions } Y_n$$

Consequently, expressions (4*b*) and (4*c*) will give :

$$J_x R \cos\,(3\rho - r_o + 11.5\,q) = A$$

and

$$-J_x R \sin\,(3\rho - r_o + 11.5\,q) = B$$

for X_n, and :

$$-J_y R \sin\,(3\rho - r_o + 11.5\,q) = A$$

and

$$-J_y R \cos\,(3\rho - r_o + 11.5\,q) = B$$

for Y_n. Hence if we put :

$$3\rho - r_o + 11.5\,q = -r \tag{4k}$$

the last four expressions may be transformed into :

$$A = J_x R \cos r \tag{4l}$$

and

$$B = J_x R \sin r \tag{4m}$$

for X_n, and :

$$A = J_y R \sin r \tag{4n}$$

and

$$B = -J_y R \cos r \tag{4o}$$

for Y_n.

It is obvious from (4*k*) that $-r$ is the phase at 11.5 hours of the central day, as will be found by the analysis. Hence R cos r and R sin r are the unknowns of the problem. Expressions (4*h*) to (4*j*) and (4*l*) to (4*o*) will allow the establishment of two independent systems, one for R cos r and the other for R sin r. Let us give an example of the formation of two equations resulting from two combinations of the daily values of X_1 and Y_1. If $\{F_n\}$ is replaced by $\{X_1\}$ in expression (4*h*), row vector (A), according to (4*l*), will be :

$$(A) = (J_x R \cos r)$$

J_x being the numerical coefficients of X_1 in table 3-IV. But it is possible to represent the above row vector as the product of the row vector (R cos r) by a diagonal matrix constructed with the values of J_x. Thus we have :

$$(A) = (R\cos r)\;\begin{array}{c} \begin{array}{ccccccccc} S_0 & K_1 & O_1 & S_2 & M_2 & MS_4 & M_4 & Q_1 & N_2 \end{array} \\ \left\Vert \begin{array}{ccccccccc} 0.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 15.347 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14.568 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.692 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.020 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.083 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.631 \end{array} \right\Vert \end{array} \tag{4p}$$

If we replace in (4*h*) row vector (A) by the right-hand side of the above expression, then we obtain after numerical computation :

$$\{X_1\}^T \cdot \{0\} = (R\ \cos\ r) \left\{ \begin{array}{r} 107.365 \\ 66.329 \\ 0.000 \\ 7.989 \\ -0.011 \\ -0.008 \\ \hline 30.553 \\ 6.180 \end{array} \right\} \begin{array}{l} (K_1) \\ (O_1) \\ (S_2) \\ (M_2) \\ (MS_4) \\ (M_4) \\ \\ (Q_1) \\ (N_2) \end{array} \qquad (4q)$$

By transposing the vectors appearing in (4*q*) we obtain the equation X_{10} of the second row of table 4-III. The second subscript of X indicates the multiplication of $\{X_1\}^T$ by vector $\{0\}$ and also the number of monthly cycles of the constituent whose contribution is strongest in X_{10}.

If we wish to obtain the function Y_{1b} of table 4-III it is necessary to start from (4*i*) where $\{F_n\}$ must be replaced by $\{Y_1\}$ and (B) must be replaced by its expression ($J_y R$ cos *r*), given by (4*o*). Then a matrix analogous to that of (4*p*) will be constructed with the J_y values taken from table 3-IV for Y_1. Afterwards a development similar to that used to obtain (4*q*) will be made and the following expression will result :

$$\{Y_1\}^T \cdot \{b\} = -(R\ \cos\ r) \left\{ \begin{array}{r} -\ 3.148 \\ 70.740 \\ 0.000 \\ 0.386 \\ 0.194 \\ 0.845 \\ \hline 83.311 \\ 1.076 \end{array} \right\} \begin{array}{l} (K_1) \\ (O_1) \\ (S_2) \\ (M_2) \\ (MS_4) \\ (M_4) \\ \\ (Q_1) \\ (N_2) \end{array} \qquad (4r)$$

All the second subscripts of X and Y in table 4-III indicate very clearly what multipliers of table 4-II have to be used to obtain the new functions, as well as the number of cycles of the most important constituents in each equation.

To compute the numerical values of the functions X_{00}, X_{10}, etc., we only need to effect the operations as indicated on the left-hand sides of (4*h*), (4*i*) and (4*j*), where $\{F_n\}$ must be replaced by $\{X_n\}$ or $\{Y_n\}$, taken from form 3-2 and $\{0\}$, $\{b\}$ and $\{4\}$ taken from the multipliers in table 4-II. The results will be inscribed in form 4-1. This is the last step in the monthly process.

FORM 4-1

Functions of R cos *r*				Functions of R sin *r*			
X		Y		Y		X	
00	22929						
10	-1297	1*b*	210	10	-485	1*b*	- 544
20	-8261	2*b*	4460	20	-767	2*b*	-1396
44	- 98	4*b*	258	44	- 40	4*b*	142

TABLE 4-III

(*a*) Equations with R cos *r*

$$\begin{Bmatrix} X_{00} \\ X_{10} \\ -Y_{1b} \\ X_{20} \\ -Y_{2b} \\ X_{44} \\ -Y_{4b} \end{Bmatrix} = \left|\begin{array}{rrrrrrr:rr} +168.000 & +0.462 & +8.243 & 0.000 & -3.983 & -2.025 & -0.334 & +6.120 & -3.077 \\ 0.000 & +107.365 & +66.329 & 0.000 & +7.989 & -0.011 & -0.008 & +30.553 & +6.180 \\ 0.000 & -3.148 & +70.740 & 0.000 & +0.386 & +0.194 & +0.845 & +83.311 & +1.076 \\ 0.000 & +0.192 & -2.817 & +108.184 & +70.931 & -4.052 & -0.670 & -1.891 & +34.542 \\ 0.000 & -0.014 & -7.037 & 0.000 & +67.232 & -0.097 & -0.418 & -11.908 & +79.752 \\ 0.000 & +0.038 & -0.396 & 0.000 & +1.071 & +51.641 & -50.713 & +0.129 & -0.199 \\ 0.000 & -0.006 & -3.004 & 0.000 & +3.394 & +68.398 & +73.507 & -5.156 & +6.017 \end{array}\right| \begin{Bmatrix} S_0 \\ K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4 \\ \hdashline Q_1 \\ N_2 \end{Bmatrix}$$

(*b*) Equations with R sin *r*

$$\begin{Bmatrix} Y_{10} \\ X_{1b} \\ Y_{20} \\ X_{2b} \\ Y_{44} \\ X_{4b} \end{Bmatrix} = \left|\begin{array}{rrrrrr:rr} +106.902 & +74.117 & 0.000 & +0.425 & +0.216 & +0.072 & +36.155 & +0.506 \\ -3.160 & +63.323 & 0.000 & +7.183 & -0.010 & -0.091 & +70.404 & +13.148 \\ +0.462 & -7.373 & +108.184 & +74.801 & -0.108 & -0.036 & -5.168 & +37.486 \\ -0.006 & -2.689 & 0.000 & +63.776 & -3.643 & -7.854 & -4.358 & +73.489 \\ +0.218 & -2.414 & 0.000 & +3.037 & +61.207 & -61.237 & +0.820 & -0.576 \\ -0.001 & -0.492 & 0.000 & +1.197 & +57.733 & +60.511 & -0.811 & +2.078 \end{array}\right| \begin{Bmatrix} K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4 \\ \hdashline Q_1 \\ N_2 \end{Bmatrix}$$

TABLE 5-I

(*a*) Final system with R cos *r*

$$
\begin{Bmatrix} X_{00} \\ X_{10} + Y_{1b} \\ - Y_{1b} \\ X_{20} + Y_{2b} \\ - Y_{2b} \\ X_{44} - Y_{4b} \\ -X_{44} - Y_{4b} \end{Bmatrix}
=
\left|\begin{array}{rrrrrrr|rr}
+168.00 & -0.46 & +8.24 & 0.00 & -3.98 & +2.03 & -0.33 & +6.12 & -3.08 \\
0.00 & +110.51 & -4.41 & 0.00 & +7.61 & -0.21 & -0.86 & -52.78 & +5.10 \\
0.00 & -3.15 & +70.74 & 0.00 & +0.39 & +0.19 & +0.85 & +83.31 & +1.08 \\
0.00 & +0.21 & +4.22 & +108.18 & +3.70 & -3.96 & -0.25 & +10.02 & -45.21 \\
0.00 & -0.01 & -7.04 & 0.00 & +67.23 & -0.10 & -0.42 & -11.91 & +79.75 \\
0.00 & +0.03 & -3.40 & 0.00 & +4.47 & +120.04 & +22.79 & -5.03 & +5.82 \\
0.00 & -0.04 & -2.61 & 0.00 & +2.32 & +16.76 & +124.22 & -5.29 & +6.22
\end{array}\right|
\begin{Bmatrix} S_0 \\ K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4 \\ \hline Q_1 \\ N_2 \end{Bmatrix}
$$

(*b*) Final system with R sin *r*

$$
\begin{Bmatrix} Y_{10} - X_{1b} \\ X_{1b} \\ Y_{20} - X_{2b} \\ X_{2b} \\ Y_{44} + X_{4b} \\ -Y_{44} + X_{4b} \end{Bmatrix}
=
\left|\begin{array}{rrrrrr|rr}
+110.06 & +10.79 & 0.00 & -6.76 & +0.23 & +0.16 & -34.25 & -12.64 \\
-3.16 & +63.32 & 0.00 & +7.18 & -0.01 & -0.09 & +70.40 & +13.15 \\
+0.49 & -4.68 & +108.18 & +11.03 & +3.54 & +7.82 & -0.81 & -36.00 \\
-0.01 & -2.69 & 0.00 & +63.78 & -3.64 & -7.85 & -4.36 & +73.49 \\
+0.22 & -2.91 & 0.00 & +4.23 & +118.94 & -1.13 & +0.01 & +1.50 \\
-0.22 & +1.92 & 0.00 & -1.84 & -3.47 & +122.15 & -1.63 & +2.65
\end{array}\right|
\begin{Bmatrix} K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4 \\ \hline Q_1 \\ N_2 \end{Bmatrix}
$$

5. — DETERMINATION OF THE UNKNOWNS

Inspection of table 4-III, (*a*) and (*b*), shows that it is now very easy to obtain final equations in such a way that only a large contribution of one constituent exists in each of these equations. Table 5-I, (*a*) and (*b*), shows these final equations where the left-hand side indicates the operations to be carried out with the left-hand sides of equations shown in table 4-III, (*a*) and (*b*).

By neglecting the disturbing constituents Q_1 and N_2 there remains only one equation for each unknown, and both systems shown in table 5-I, (*a*) and (*b*), may be solved by means of the respective inverse matrices, to give the provisional values of the unknowns $\overline{R \cos r}$ and $\overline{R \sin r}$. These inverse matrices are seen in form 5-1, (*a*) and (*b*).

In both matricial formulas (*a*) and (*b*) of form 5-1, the numerical values of the known terms are computed as indicated on the right-hand side of these formulas, by using the numerical values taken from form 4-1. Provisional values $\overline{R \cos r}$ and $\overline{R \sin r}$ are then computed by multiplying the matrices by the corresponding vectors of the known terms and the results are inscribed on the left-hand side of (*c*) in form 5-1.

As the constituents Q_1 and N_2 were neglected in the solution of systems (*a*) and (*b*) of table 5-I, let us now explain how their disturbing effect can be attenuated.

Let us take the system in table 5-I (*a*) and represent it by a symbolic expression putting :

$\{L\}$ for the known terms;
E for the square matrix to the left of the dashed line;
E' for the matrix to the right of the dashed line;
$\{I\}$ for the unknowns' vector corresponding to the square matrix (to M_4 inclusive);
$\{I'\}$ for the vector corresponding to matrix E' and to the disturbing constituents (Q_1 and N_2).

Then we may write this system in the symbolic form :

$$\{L\} = E\ \{I\} + E'\ \{I'\} \tag{5a}$$

If E^{-1} is the inverse matrix under (*a*) in form 5-1, we obtain from (5*a*) :

$$E^{-1}\{L\} = \{I\} + E^{-1}E'\{I'\}$$

hence :

$$\{I\} = E^{-1}\{L\} - E^{-1}E'\{I'\} \tag{5b}$$

The matrix $-E^{-1}E'$ is that immediately following the provisional values $\overline{R \cos r}$ under (*c*) in form 5-1 (*).

A similar reasoning would allow us to obtain the matrix immediately following the values of R sin *r* by taking the system from table 5-I (*b*) and the inverse matrix from form 5-1 (*b*).

As $\{I'\}$ represents either the vector $\{R' \cos r'\}$ or $\{R' \sin r'\}$ corresponding to the disturbing constituents Q_1 and N_2, it is now necessary to obtain an approximate value of these vectors. The only way to do so

(*) In the results the fact that all the values of matrix (*a*) in form 5-1 are multiplied by 10^6 must be taken into account.

is to compute R′ and r' for Q_1 and N_2 in function of the provisional values for $\overline{R}$ and $\overline{r}$ for O_1 and M_2 respectively. The values of $\overline{R}$ and $\overline{r}$ are computed by using the provisional values $\overline{R \cos r}$ and $\overline{R \sin r}$ (see form 5-1 (*d*)). As for O_1 or M_2 we have $W = w = 0$, and thus we have from (2*c*) and (2*d*) :

$$\overline{R} = fH \tag{5c}$$

$$-\overline{r} = V + u - g \tag{5d}$$

For the disturbing constituents Q_1 or N_2 we must write :

$$R' = f' (1 + W) H' \tag{5e}$$

$$-r' = V' + u' + w - g' \tag{5f}$$

where w and W are the same for both constituents and take into account the disturbing effect of ρ_1 on Q_1 or that of ν_2 on N_2. Since $f' = f$, with (5*c*) and (5*e*) we obtain :

$$R' = \overline{R} \frac{H'}{H} (1 + W) \tag{5g}$$

With (5*d*) and (5*f*) we obtain :

$$r' = \overline{r} + (V + u) - (V' + u') - w + g' - g \tag{5h}$$

However, as u, w and the difference $(V - V')$ are exactly the same for O_1 and Q_1 as for M_2 and N_2, the expression (5*h*) may be transformed into :

$$r' = \overline{r} + V(M_2) - V(N_2) - w(N_2) + g' - g \tag{5i}$$

The values of V must be taken for 11.5 hours of the central day, to which correspond the values of the unknowns $\overline{R \cos r}$ and $\overline{R \sin r}$. Thus if we compute V for 0 hour of this day by using the tables 1(*a*) to 1(*c*) given at the end of the article, then a correction must be added in order to take into account the difference Δq between the speeds of M_2 and N_2. If the observations started at zero hour of the first day, this correction will be $11.5\,\Delta q$, whereas if they started t hours later the correction is given by $(11.5 - t)\,\Delta q$. This correction can be immediately found from table 2 by subtracting the Δ value for N_2 from that corresponding to M_2. In addition $w(N_2)$ can be taken from table 5. Hence (5*i*) may be transformed into :

$$r' = \overline{r} + V(M_2) - V(N_2) + \Delta(M_2) - \Delta(N_2) - w + g' - g \tag{5j}$$

In this expression all the terms are known except $(g' - g)$. If the harmonic constants for O_1, Q_1, M_2 and N_2 are known for any nearby station the difference $(g' - g)$ at this station for Q_1 and O_1 or for N_2 and M_2 may be used in (5*j*) to obtain r' for respectively Q_1 or N_2. If no information exists, we must then put $g' - g = 0$.

To compute R′, expression (5*g*) where $\overline{R}$ is known must be used and $(1+W)$ is taken from table 5 for N_2, and where the relationship H′/H is made equal to $H(Q_1) / H(O_1)$ or to $H(N_2) / H(M_2)$ for any nearby place, or, if no information exists, equal to the equilibrium relationship $H'/H = 0.191$ which is valid in both cases. In form 5-1, under (*d*), we may see all the details of the computation of R′ and r' for both Q_1 and N_2.

The construction of table 5, at the end of the article, giving the values of W and w, will now be explained.

FORM 5-1

(*a*) to find $10^6 \overline{R \cos r}$

$$\begin{Bmatrix} S_0 \\ K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4 \end{Bmatrix} = \begin{vmatrix} +5952 & +6 & -653 & 0 & +348 & +102 & +3 \\ 0 & +9066 & +464 & 0 & -1030 & +6 & +55 \\ 0 & +400 & +14145 & 0 & -121 & -10 & -92 \\ 0 & -34 & -591 & +9243 & -516 & +310 & -36 \\ 0 & +43 & +1482 & 0 & +14852 & +4 & +40 \\ 0 & +6 & +302 & 0 & -516 & +8546 & -1574 \\ 0 & +10 & +228 & 0 & -212 & -1154 & +8258 \end{vmatrix} \begin{Bmatrix} X_{00} = +22929 \\ X_{10} + Y_{1b} = -1087 \\ -Y_{1b} = -210 \\ X_{20} + Y_{2b} = -3801 \\ -Y_{2b} = -4460 \\ X_{44} - Y_{4b} = -356 \\ -X_{44} - Y_{4b} = -160 \end{Bmatrix}$$

(*b*) to find $10^6 \overline{R \sin r}$

$$\begin{Bmatrix} K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4 \end{Bmatrix} = \begin{vmatrix} +9048 & -1498 & 0 & +1129 & +19 & +60 \\ +445 & +15647 & 0 & -1714 & -55 & -100 \\ -23 & +622 & +9243 & -1670 & -346 & -701 \\ +21 & +652 & 0 & +15605 & +507 & +1008 \\ -7 & +359 & 0 & -598 & +8389 & +39 \\ +9 & -229 & 0 & +247 & +245 & +8205 \end{vmatrix} \begin{Bmatrix} Y_{10} - X_{1b} = +59 \\ X_{1b} = -544 \\ Y_{20} - X_{2b} = +629 \\ X_{2b} = -1396 \\ Y_{44} + X_{4b} = +102 \\ -Y_{44} + X_{4b} = +182 \end{Bmatrix}$$

(c)

$$
\begin{array}{l}
\\
S_0 \\ K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4
\end{array}
\begin{array}{c}
\overline{R \cos r} \\
\left\{\begin{array}{r}
+135.015 \\ -\ 5.369 \\ -\ 2.847 \\ -\ 32.775 \\ -\ 66.606 \\ -\ 0.559 \\ -\ 0.023
\end{array}\right\}
\end{array}
+
\begin{array}{c}
\\
\left|\begin{array}{rr}
+0.019 & -0.009 \\
+0.427 & +0.035 \\
-1.158 & -0.006 \\
-0.050 & +0.458 \\
+0.056 & -1.186 \\
+0.004 & +0.001 \\
+0.017 & -0.027
\end{array}\right|
\end{array}
\begin{array}{c}
R' \cos r' \\ Q_1,\ N_2 \\
\left\{\begin{array}{r}
-0.812 \\ +3.591
\end{array}\right\}
\end{array}
=
\left\{\begin{array}{r}
+135.0 \\ -\ 5.590 \\ -\ 1.928 \\ -\ 31.090 \\ -\ 70.910 \\ -\ 0.559 \\ -\ 0.134
\end{array}\right\}
$$

$$
\begin{array}{l}
\\
K_1 \\ O_1 \\ S_2 \\ M_2 \\ MS_4 \\ M_4
\end{array}
\begin{array}{c}
\overline{R \sin r} \\
\left\{\begin{array}{r}
-\ 0.214 \\ -\ 6.117 \\ +\ 7.643 \\ -\ 21.903 \\ +\ 1.502 \\ +\ 1.298
\end{array}\right\}
\end{array}
=
\begin{array}{c}
\\
\left|\begin{array}{rr}
+0.420 & +0.051 \\
-1.094 & -0.074 \\
-0.046 & +0.451 \\
+0.025 & -1.160 \\
-0.028 & +0.026 \\
+0.030 & -0.037
\end{array}\right|
\end{array}
\begin{array}{c}
R' \sin r' \\ Q_1,\ N_2 \\
\left\{\begin{array}{r}
+\ 1.242 \\ +14.854
\end{array}\right\}
\end{array}
=
\left\{\begin{array}{r}
+\ 1.065 \\ -\ 8.575 \\ +14.285 \\ -39.103 \\ +\ 1.853 \\ +\ 0.786
\end{array}\right\}
$$

(d)

Take V, w and 1 + W from form 7-1

V_{M_2}	$= 287.1$	$g_{Q_1} - g^*_{O_1}$	$=$
$-V_{N_2}$	$= -48.7$	σ	$=$
$\Delta_{M_2} - \Delta_{N_2}$	$= 6.2$	θ_1	$= 238°.2$
$-w_{N_2}$	$= -6.4$		
σ	$= 238.2$	$g_{N_2} - g^*_{M_2}$	$=$
		σ	$=$
		θ_2	$= 238°.2$

$c_1 = k^*_1 \ (1 + W) = 0.218$

$c_2 = k^*_2 \ (1 + W) = 0.218$

	$\overline{R}^2$	$\overline{R}$	$c\overline{R} = R'$	$\tan \overline{r}$
O_1	45.52	6.8	1.482	2.149
M_2	4916.10	70.1	15.282	0.329

	$\overline{r}$	$r' = \overline{r} + \theta$	$\cos r'$	$\sin r'$
O_1	245°.0	123.2	-0.548	0.838
M_2	198.2	76.4	0.235	0.972

* If no information exists

$g_{Q_1} - g_{O_1} = g_{N_2} - g_{M_2} = 0$ and $k_1 = k_2 = 0.191$

6. — DETERMINATION OF W AND w

In sub-section 2 where the limitations of a short-period analysis were mentioned, we stated that the constituents having nearly equal speeds appeared in the form of a whole effect and that this whole effect could be represented by a small alteration in amplitude and phase of the principal constituent of the group by means of a corresponding change in f and u. In this case the groups of constituents are the following :

S_2, K_2 and T_2
K_1 and P_1
Q_1 and ρ_1
N_2 and ν_2
MS_4, MK_4 and MT_4

where N_2 and Q_1 are also taken into consideration so that their effects on the other constituents may later be eliminated.

We may now assume that for the constituents of a same group phase lags g are equal and that the amplitudes retain the same relationships to one another as in the equilibrium tide. Basing ourselves on this assumption, we can establish mathematically the expressions giving w and W which appear in (2c) and (2d).

Let us designate by E, E′ and E″ the astronomical arguments of the three constituents of a specific group in which g will be a phase lag common to all constituents whose mean amplitudes will be respectively H, H′ and H″. The whole effect of the group will then be expressed by :

$$fH \cos (E-g) + f'H' \cos (E'-g) + f''H'' \cos (E''-g) =$$

$$= fH\left[\cos (E-g) + \frac{f'H'}{f\,H} \cos (E'-g) + \frac{f''H''}{f\,H} \cos (E''-g)\right] =$$

$$= fH\ (\cos E \cos g + \sin E \sin g +$$

$$+ \frac{f'H'}{fH} \cos E' \cos g + \frac{f'H'}{fH} \sin E' \sin g +$$

$$+ \frac{f''H''}{fH} \cos E'' \cos g + \frac{f''H''}{fH} \sin E'' \sin g)$$

$$= fH \cos g\ (\cos E + \frac{f'H'}{fH} \cos E' + \frac{f''H''}{fH} \cos E'') +$$

$$+ fH \sin g\ (\sin E + \frac{f'H'}{fH} \sin E' + \frac{f''H''}{fH} \sin E'')$$

or putting :

$$\cos E + \frac{f'H'}{fH} \cos E' + \frac{f''H''}{fH} \cos E'' = (1+W) \cos (E+w) \qquad (6a)$$

$$\sin E + \frac{f'H'}{fH} \sin E' + \frac{f''H''}{fH} \sin E'' = (1+W) \sin (E+w) \qquad (6b)$$

the combined constituent will be expressed by :

$$f\,(1 + W)\ H \cos (E + w - g) \qquad (6c)$$

and since $fH = R$ and $E - g = -r$ for a dominating constituent, the factor (1+W) will represent the change in its amplitude, and w the change in its phase, under the influence of the least important constituents. As the difference between the constituents' speeds is very small, we may assume without appreciable error that the phase shift of the constituents does not vary during the period of analysis, and is consequently the same for W and w which will thus be determined for the instant of the central day of the analysed period. We shall therefore write the expressions of the astronomical arguments $V + u$ and the equilibrium amplitudes of the constituents of groups S_2, K_1 and N_2 :

	$V + u$	Coefficient
S_2 :	$E = 30°t$	0.423
K_2 :	$E' = 30°t + 2h + u_{K_2}$	0.115
T_2 :	$E'' = 30°t - h + 282°$	0.025
K_1 :	$E = 15°t + h + 90° + u_{K_1}$	0.531
P_1 :	$E' = 15°t - h - 90°$	0.176
N_2 :	$E = 30°t + 2h - 3s + p + u_{M_2}$	0.174
ν_2 :	$E' = 30°t + 4h - 3s - p + u_{M_2}$	0.033

Thus, choosing the instants satisfying the condition $E = 0$, we shall have :

Group S_2 : $E = 0$ for $t = 0$
" K_1 : $E = 0$ " $t = -(h + 90° + u_{K_1}) / 15°$
" N_2 : $E = 0$ " $t = -(2h - 3s + p + u_{M_2}) / 30°$

and consequently E′ and E″ will be obtained by substituting the values of t in their respective expressions, which gives :

	$V + u$
K_2	$E' = 2h + u_{K_2}$
T_2	$E'' = -h + 282°$
P_1	$E' = -2h - u_{K_1} - 180°$
ν_2	$E' = 2h - 2p$

Substituting these elements in (6a) and (6b), and replacing the ratios H′/H and H″/H by the ratios of the amplitudes of the equilibrium tide corresponding to the constituents of each group, we shall have :

$$S_2 \begin{cases} (1+W)\cos w = 1 + f_{K_2}\,0.272\cos(2h+u_{K_2}) + 0.059\cos(h-282°) \\ (1+W)\sin w = f_{K_2}\,0.272\sin(2h+u_{K_2}) - 0.059\sin(h-282°) \end{cases}$$

$$K_1 \begin{cases} (1+W)\cos w = 1 - \dfrac{1}{f_{K_1}}\,0.331\cos(2h+u_{K_1}) \\ (1+W)\sin w = \dfrac{1}{f_{K_1}}\,0.331\sin(2h+u_{K_1}) \end{cases} \qquad (6d)$$

$$N_2 \begin{cases} (1+W)\cos w = 1 + 0.189\cos(2h-2p) \\ (1+W)\sin w = 0.189\sin(2h-2p) \end{cases}$$

The ratio between the two expressions of each group will give tan w and in consequence w, and the square root of the sum of these squares will be equal to $(1+W)$. But it is easy to ascertain that w and W will always be small, because ratios H′/H and H″/H are also, and this will allow us to simplify the expressions of groups S_2 and K_1. In fact, if we take $\cos w = 1$ and $\sin w = w$ as an *approximation*, and if we neglect the third term of group S_2 and the products Ww, we can write :

$$\text{Group } S_2 \begin{cases} W \approx & f_{K_2}\ 0.272 \cos(2h + u_{K_2}) \\ w \approx & f_{K_2}\ 0.272 \sin(2h + u_{K_2}) \end{cases}$$

$$\text{Group } K_1 \begin{cases} W \approx - & \dfrac{1}{f_{K_1}} 0.331 \cos(2h + u_{K_1}) \\ w \approx & \dfrac{1}{f_{K_1}} 0.331 \sin(2h + u_{K_1}) \end{cases}$$

which shows us that W and w are approximately proportional to f_{K_2} for group S_2, and approximately proportional to $1/f_{K_1}$ for group K_1. We may therefore conclude that if we use the exact expressions for computing W and w putting $f = 1$ in the expressions, we shall obtain a satisfactory approximation of W/f_{K_2} and w/f_{K_2} for the S_2 group; $(f_{K_1} \cdot W)$ and $(f_{K_1} \cdot w)$ for the K_1 group. We may thus write :

Group S_2

$$\begin{cases} (1+W/f_{K_2}) \cos \dfrac{w}{f_{K_2}} = 1+0.272 \cos(2h+u_{K_2}) + 0.059 \cos(h-282^\circ) \\ (1+W/f_{K_2}) \sin \dfrac{w}{f_{K_2}} = 0.272 \sin(2h+u_{K_2}) - 0.059 \sin(h-282^\circ) \end{cases} \tag{6e}$$

Group K_1

$$\begin{cases} (1+f_{K_1} W) \cos f_{K_1} w = 1-0.331 \cos(2h+u_{K_1}) \\ (1+f_{K_1} W) \sin f_{K_1} w = 0.331 \sin(2h+u_{K_1}) \end{cases} \tag{6f}$$

Now, as :

$$\begin{aligned} 2h + u_{K_2} &\approx 2(V + u)_{K_1} \\ 2h + u_{K_1} &= (2V + u)_{K_1} \\ 2h - 2p &= 3V_{M_2} - 2V_{N_2} \end{aligned}$$

we have the possibility of tabulating the values of W/f_{K_2} and w/f_{K_2} in function of $(V + u)$ of K_1; those of $(f_{K_1} \cdot W)$ and of $(f_{K_1} \cdot w)$ for $(2V + u)$ of K_1, and those of $(1 + W)$ and of w for the N_2 group in function of $(3V_{M_2} - 2V_{N_2})$ (table 5 of the appendix).

Since the differences between the speeds of MS_4, MK_4 and MT_4 and the ratios between their theoretical amplitudes are the same as those of S_2, K_2 and T_2, the values of W and w in the tables will be used for the two groups. With the same justification we shall use the values of $(1+W)$ and w for groups (N_2 and ν_2) and (Q_1 and ρ_1).

7. — DETERMINATION OF CONSTANTS

As soon as we have computed the values of R^2 and of $\tan r$ (form 7-1), we shall determine the values of R and r to be entered on the appropriate

FORM 7-i

	S_0	K_1	O_1	S_2	M_2	MS_4	M_4	N_2
tan r	-	- 0.191	4.448	- 0.459	- 0.551	- 3.315	- 5.866	
R^2	-	32.38	77.25	1170.64	6557.27	3.75	0.64	
V 1/1 T 1 *(a)*	-	9.8	143.7		153.5			217.2
month : T 1 *(b)*	-	209.0	22.2		231.1			341.3
day : T 1 *(c)*	-	3.9	258.5		262.5			210.2
sum	-	222.7	64.4		287.1	287.1	214.2	48.7
u	-	- 7.1	8.2		- 1.8	- 1.8	- 3.6	
Δ	-	173.0	160.3	345.0	333.3	318.3	306.7	
w	-	- 15.8	-	- 17.4	-	- 17.4	-	
r	-	169.2	257.3	155.3	208.9	106.8	99.7	
g	-	182.0	130.2	122.9	107.5	333.0	257.0	
f	-	1.070	1.063	1.000	0.983	0.983	0.966	
f (1 + W)	-	1.186	-	0.874	-	0.859	-	
R	-	5.69	8.79	34.21	80.98	1.94	0.80	
H	135.0	4.8	8.3	39.2	82.4	2.3	0.8	

Entries in table 5

K_1 : $(2V + u)_{K_1}$ = 78°3

S_2, MS_4 : $(V + u)_{K_1}$ = 215.6

N_2 : $3V_{M_2} - 2V_{N_2}$ = 43.9

f_{K_2} = 1.171

f_{K_1} = 1.070

From table 5

K_1 : wf_{K_1} = - 16°9 wf_{K_1} = 0.115

S_2, MS_4 : w/f_{K_2} = - 14.8 w/f_{K_2} = - 0.108

N_2 : w = 6.4   1 + W = 1.139

K_1 : w = - 15°8 w = - 0.126

S_2, MS_4 : w = - 17.4 w = 0.108

K_1 : 1 + W = 1.108

S_2, MS_4 : 1 + W = 0.874

lines of form 7-1 where we find details of the computation of V, u, f, w and W for the various constituents. The V values will be taken from tables 1(a) to 1(c) at the end of the article for the central day of the series of observations. Table 2 gives Δ for the central hour of the analysis and from tables 3 and 4, respectively, we obtain u and f for the central day. We shall obtain corrections w and W from table 5 in function of the indicated arguments.

8. — CONCLUSION

To permit the comparison of the quality of the results, we have prepared a table of the g and H values for the six constituents K_1, O_1, S_2, M_2, MS_4 and M_4 for analyses of 32, 24, 15 and 7 days.

Number of days	Phase lags g°						
	K_1	O_1	S_2	M_2	MS_4	M_4	
32	198	123	127	111	3	286	Liverpool Tidal Institute
24	207	115	128	109	23	267	Semi-graphic
15	202	117	122	103	348	259	British Admiralty
7	182	130	123	108	333	257	Present method
	Amplitudes						
32	4	6	35	84	2	2	Liverpool Tidal Institute
24	4	7	36	83	1	2	Semi-graphic
15	4	7	36	84	2	2	British Admiralty
7	5	8	39	82	2	1	Present method

This table shows a fairly reasonable agreement for the principal constituents. The tide studied is that of Aratú (Brazil) which has a nearly regular semi-diurnal tide.

The results of analysis by the semi-graphic method will be found in the July 1963 issue of the Review, page 69.

TABLE 1 (*a*)

V_0 values at 0 h on January 1st from 1900 to 2000
(Years followed by (b) are leap-years)

Year	M_2	N_2	K_1	O_1	Year	M_2	N_2	K_1	O_1
	°	°	°	°		°	°	°	°
1900 (b)	006, 3	063, 7	10, 2	356, 1	1950	071, 4	215, 9	10, 1	061, 3
1	107, 1	075, 7	10, 0	097, 1	1	172, 2	227, 9	09, 8	162, 3
2	207, 8	087, 8	09, 7	198, 1	2 (b)	272, 9	239, 9	09, 6	263, 3
3	308, 6	099, 8	09, 5	299, 1	3	349, 3	214, 5	10, 4	338, 9
4 (b)	049, 3	111, 3	09, 2	040, 1	4	090, 0	226, 5	10, 1	079, 9
5	125, 7	086, 4	10, 0	115, 7	5	190, 8	238, 6	09, 9	180, 9
6	226, 5	098, 4	09, 7	216, 7	6 (b)	291, 5	250, 6	09, 6	281, 9
7	327, 2	110, 5	09, 5	317, 7	7	007, 9	225, 2	10, 4	357, 5
8 (b)	068, 0	122, 5	09, 3	058, 7	8	108, 7	237, 2	10, 1	098, 5
9	144, 3	097, 1	10, 0	134, 3	9	209, 4	249, 2	09, 9	199, 5
1910	245, 1	109, 1	09, 8	235, 3	1960 (b)	310, 2	261, 3	09, 7	300, 5
1	345, 8	121, 1	09, 5	336, 3	1	026, 5	235, 9	10, 4	016, 1
2 (b)	086, 6	133, 2	09, 3	077, 3	2	127, 3	247, 9	10, 2	117, 1
3	163, 0	107, 7	10, 0	152, 9	3	228, 0	259, 9	09, 9	218, 1
4	263, 7	119, 8	09, 8	253, 9	4 (b)	328, 8	272, 0	09, 7	319, 1
5	004, 5	131, 8	09, 6	354, 9	5	045, 2	246, 5	10, 4	034, 7
6 (b)	105, 2	143, 8	09, 3	095, 9	6	145, 9	258, 6	10, 2	135, 7
7	181, 6	118, 4	10, 1	171, 5	7	246, 7	270, 6	10, 0	236, 7
8	282, 4	130, 5	09, 8	272, 5	8 (b)	347, 4	282, 6	09, 7	337, 7
9	023, 1	142, 5	09, 6	013, 5	9	063, 8	257, 2	10, 5	053, 3
1920 (b)	123, 9	154, 5	09, 4	114, 5	1970	164, 6	269, 2	10, 2	154, 3
1	200, 2	129, 1	10, 1	190, 1	1	265, 3	281, 3	10, 0	255, 3
2	301, 0	141, 1	09, 9	291, 1	2 (b)	006, 1	293, 3	09, 8	356, 3
3	041, 7	153, 2	09, 6	032, 1	3	082, 4	267, 9	10, 5	071, 9
4 (b)	142, 5	165, 2	09, 4	133, 1	4	183, 2	279, 9	10, 3	172, 9
5	218, 9	139, 8	10, 1	208, 7	5	283, 9	291, 9	10, 0	273, 9
6	319, 6	151, 8	09, 9	309, 7	6 (b)	024, 7	304, 0	09, 8	014, 9
7	060, 4	163, 8	09, 7	050, 7	7	101, 1	278, 6	10, 5	090, 5
8 (b)	161, 1	175, 9	09, 4	151, 7	8	201, 8	290, 6	10, 3	191, 5
9	237, 5	150, 5	10, 2	227, 3	9	302, 6	302, 6	10, 1	292, 5
1930	338, 2	162, 5	09, 9	328, 3	1980 (b)	043, 3	314, 7	09, 8	033, 5
1	079, 0	174, 5	09, 7	069, 3	1	119, 7	289, 2	10, 6	109, 1
2 (b)	179, 8	186, 5	09, 5	170, 3	2	220, 4	301, 3	10, 3	210, 1
3	256, 1	161, 1	10, 2	245, 9	3	321, 2	313, 3	10, 1	311, 1
4	356, 9	173, 2	10, 0	346, 9	4 (b)	061, 9	325, 3	09, 9	052, 1
5	097, 6	185, 2	09, 7	087, 9	5	138, 3	299, 9	10, 6	127, 7
6 (b)	198, 4	197, 2	09, 5	188, 9	6	239, 1	311, 9	10, 4	228, 7
7	274, 8	171, 8	10, 2	264, 5	7	339, 8	324, 0	10, 1	329, 7
8	015, 5	183, 8	10, 0	005, 5	8 (b)	080, 6	336, 0	09, 9	070, 7
9	116, 3	195, 9	09, 8	106, 5	9	156, 9	310, 6	10, 6	146, 3
1940 (b)	217, 0	207, 9	09, 5	207, 5	1990	257, 7	322, 6	10, 4	247, 8
1	293, 4	182, 5	10, 3	283, 1	1	358, 5	334, 5	10, 2	348, 3
2	034, 1	194, 5	10, 0	024, 1	2 (b)	099, 2	346, 7	09, 9	089, 3
3	134, 9	206, 5	09, 8	125, 1	3	175, 6	321, 3	10, 7	164, 9
4 (b)	235, 6	218, 6	09, 5	226, 1	4	276, 3	333, 3	10, 4	265, 9
5	312, 0	193, 2	10, 3	301, 7	5	017, 1	354, 3	10, 2	006, 9
6	052, 8	205, 2	10, 1	042, 7	6 (b)	117, 8	357, 4	09, 9	107, 9
7	153, 5	217, 2	09, 8	143, 7	7	194, 2	331, 9	10, 7	183, 5
8 (b)	254, 3	229, 2	09, 6	244, 7	8	295, 0	344, 0	10, 5	284, 5
9	330, 6	203, 8	10, 3	320, 3	9	035, 7	356, 0	10, 2	025, 5
1950	071, 4	215, 9	10, 1	061, 3	2000 (b)	136, 5	008, 0	10, 0	126, 5

Note. — **The comma both in this table and those which follow should be read as a decimal point.**

TABLE 1 (*b*)

Changes in V_0 at 0 h for the first day of each month.

Month	M_2	N_2	K_1	O_1	Month	M_2	N_2	K_1	O_1
January	000, 0	000, 0	000, 0	000, 0	July	266, 9	062, 2	178, 4	088, 6
February	324, 2	279, 2	030, 6	293, 6	August	231, 1	341, 3	209, 0	022, 2
March	001, 5	310, 7	058, 2	303, 3	September	195, 3	260, 5	239, 5	315, 8
April	325, 7	229, 8	088, 7	237, 0	October	183, 9	217, 1	269, 1	274, 8
May	314, 2	186, 4	118, 3	195, 9	November	148, 0	136, 3	299, 6	208, 4
June	278, 4	105, 6	148, 8	129, 6	December	136, 6	092, 9	329, 2	167, 4

TABLE 1 (*c*)

Changes in V_0 at 0 h for every day of the month.
(For leap years, before reading the table add one day to dates before 29 February).

Day	M_2	N_2	K_1	O_1	Day	M_2	N_2	K_1	O_1
1	000, 0	000, 0	00, 0	000, 0	17	329, 9	120, 9	15, 8	314, 1
2	335, 8	322, 6	01, 0	334, 6	18	305, 5	083, 4	16, 8	288, 8
3	311, 2	285, 1	02, 0	309, 3	19	281, 1	046, 0	17, 7	263, 4
4	286, 9	247, 7	03, 0	283, 9	20	256, 8	008, 5	18, 7	238, 0
5	262, 5	210, 2	03, 9	258, 5	21	232, 4	331, 1	19, 7	212, 7
6	238, 1	172, 8	04, 9	233, 2	22	208, 0	293, 6	20, 7	187, 3
7	213, 7	135, 3	05, 9	207, 8	23	183, 6	256, 2	21, 7	161, 9
8	189, 3	097, 9	06, 9	182, 4	24	159, 2	218, 7	22, 7	136, 6
9	164, 9	060, 4	07, 9	157, 1	25	134, 8	181, 3	23, 7	111, 2
10	140, 6	023, 0	08, 9	131, 7	26	110, 5	143, 8	24, 6	085, 8
11	116, 2	345, 5	09, 9	106, 3	27	086, 1	106, 4	25, 6	060, 5
12	091, 8	308, 1	10, 8	081, 0	28	061, 7	068, 9	26, 6	035, 1
13	067, 4	270, 6	11, 8	055, 6	29	037, 3	031, 5	27, 6	009, 7
14	043, 0	233, 2	12, 8	030, 2	30	012, 9	354, 0	28, 6	344, 4
15	018, 7	195, 7	13, 8	004, 9	31	348, 6	316, 6	29, 6	319, 0
16	354, 3	158, 3	14, 8	339, 5	32	324, 2	279, 2	30, 6	293, 6

Table 2
Values of Δ

Time origin of observations	M_2	S_2	N_2	K_1	O_1	M_4	MS_4
	°	°	°	°	°	°	°
0	333, 3	345, 0	327, 1	173, 0	160, 3	306, 7	318, 3
1	304, 3	315, 0	298, 6	157, 9	146, 4	248, 7	259, 3
2	275, 3	285, 0	270, 2	142, 9	132, 4	190, 7	200, 3
3	246, 3	255, 0	241, 7	127, 8	118, 5	132, 7	141, 3
4	217, 4	225, 0	213, 3	112, 8	104, 6	74, 8	82, 4
5	188, 4	195, 0	184, 9	97, 8	90, 6	16, 8	23, 4
6	159, 4	165, 0	156, 4	82, 7	76, 7	318, 8	324, 4
7	130, 4	135, 0	128, 0	67, 7	62, 7	260, 9	265, 4
8	101, 4	105, 0	99, 5	52, 6	48, 8	202, 9	206, 4
9	72, 5	75, 0	71, 1	37, 6	34, 9	144, 9	147, 5
10	43, 5	45, 0	42, 7	22, 6	20, 9	87, 0	88, 5
11	14, 5	15, 0	14, 2	7, 5	7, 0	29, 0	29, 5
12	345, 5	345, 0	345, 8	352, 5	352, 0	331, 0	330, 5
13	316, 5	315, 0	317, 3	337, 4	339, 1	273, 0	271, 5
14	287, 6	285, 0	288, 9	322, 4	325, 2	215, 1	212, 6
15	258, 6	255, 0	260, 5	307, 4	311, 2	157, 1	153, 6
16	229, 6	225, 0	232, 0	292, 3	297, 3	99, 1	94, 6
17	200, 6	195, 0	203, 6	277, 3	282, 3	41, 2	35, 6
18	171, 6	165, 0	175, 1	262, 2	269, 4	343, 2	336, 6
19	142, 7	135, 0	146, 7	247, 2	255, 5	285, 2	277, 7
20	113, 7	105, 0	118, 3	232, 2	241, 5	227, 2	218, 7
21	84, 7	75, 0	89, 8	217, 1	227, 6	169, 3	159, 7
22	55, 7	45, 0	61, 4	202, 1	213, 6	111, 3	100, 7
23	26, 7	15, 0	32, 9	187, 0	199, 7	53, 3	41, 7

TABLE 3

Values of u at 0 h on January 1st from 1900 to 2000

Year	M_2	K_1	O_1	Year	M_2	K_1	O_1
	°	°	°		°	°	°
1900	2, 1	8, 9	-10, 9	1950	-0, 5	-1, 6	1, 8
1	1, 8	8, 2	-10, 5	1	0, 3	1, 0	- 1, 1
2	1, 4	6, 5	- 8, 5	2	1, 0	3, 5	- 3, 9
3	0, 8	3, 7	- 5, 0	3	1, 5	5, 7	- 6, 5
4	0, 1	0, 3	- 0, 5	4	1, 9	7, 5	- 8, 8
5	-0, 6	-3, 1	4, 2	5	2, 1	8, 6	-10, 3
6	-1, 3	-6, 0	8, 0	6	2, 1	8, 9	-11, 0
7	-1, 8	-8, 0	10, 2	7	1, 8	8, 0	-10, 3
8	-2, 1	-8, 9	11, 0	8	1, 3	6, 1	- 8, 1
9	-2, 1	-8, 6	10, 4	9	0, 7	3, 2	- 4, 3
1910	-2, 0	-7, 6	8, 8	1960	-0, 1	-0, 2	0, 3
1	-1, 6	-5, 8	6, 6	1	-0, 8	-3, 7	4, 9
2	-1, 0	-3, 5	4, 0	2	-1, 4	-6, 4	8, 5
3	-0, 3	-1, 0	1, 2	3	-1, 8	-8, 2	10, 5
4	0, 4	1, 6	- 1, 7	4	-2, 1	-8, 9	10, 9
5	1, 1	4, 0	- 4, 5	5	-2, 1	-8, 5	10, 2
6	1, 7	6, 2	- 7, 1	6	-1, 9	-7, 3	8, 5
7	2, 0	7, 8	- 9, 2	7	-1, 5	-5, 5	6, 2
8	2, 1	8, 8	-10, 6	8	-0, 9	-3, 2	3, 6
9	2, 0	8, 8	-10, 9	9	-0, 2	-0, 6	0, 7
1920	1, 7	7, 7	- 9, 9	1970	0, 6	2, 0	- 2, 2
1	1, 2	5, 5	- 7, 3	1	1, 2	4, 4	- 5, 0
2	0, 5	2, 5	- 3, 3	2	1, 7	6, 5	- 7, 5
3	-0, 2	-1, 0	1, 4	3	2, 1	8, 0	- 9, 5
4	-0, 9	-4, 3	5, 8	4	2, 1	8, 8	-10, 7
5	-1, 5	-6, 9	9, 0	5	2, 0	8, 7	-10, 9
6	-1, 9	-8, 5	10, 7	6	1, 6	7, 4	- 9, 6
7	-2, 1	-8, 9	10, 9	7	1, 1	7, 1	- 6, 8
8	-2, 1	-8, 3	9, 9	8	0, 4	1, 9	- 2, 6
9	-1, 8	-6, 9	8, 0	9	-0, 3	-1, 6	2, 2
1930	-1, 4	-5, 0	5, 6	1980	-1, 0	-4, 8	6, 4
1	-0, 7	-2, 6	2, 9	1	-1, 6	-7, 3	9, 4
2	0, 0	0, 0	0, 0	2	-2, 0	-8, 6	10, 8
3	0, 7	2, 5	- 2, 8	3	-2, 1	-8, 9	10, 8
4	1, 3	4, 9	- 5, 6	4	-2, 1	-8, 1	9, 6
5	1, 8	6, 9	- 8, 0	5	-1, 8	-6, 7	7, 7
6	2, 1	8, 3	- 9, 8	6	-1, 3	-4, 6	5, 2
7	2, 1	8, 9	-10, 8	7	-0, 6	-2, 2	2, 5
8	1, 9	8, 5	-10, 7	8	0, 1	0, 4	- 0, 4
9	1, 5	7, 0	- 9, 1	9	0, 8	2, 9	- 3, 3
1940	0, 9	4, 4	- 5, 9	1990	1, 4	5, 3	- 6, 0
1	0, 2	1, 1	- 1, 5	1	1, 9	7, 2	- 8, 3
2	-0, 5	-2, 4	3, 2	2	2, 1	8, 5	-10, 0
3	-1, 1	-5, 5	7, 2	3	2, 1	8, 9	-10, 9
4	-1, 7	-7, 7	9, 9	4	1, 9	8, 3	-10, 6
5	-2, 0	-8, 8	10, 9	5	1, 4	6, 6	- 8, 7
6	-2, 1	-8, 8	10, 6	6	0, 8	3, 9	- 5, 3
7	-2, 0	-7, 9	9, 2	7	0, 1	0, 6	- 0, 8
8	-1, 7	-6, 2	7, 2	8	-0, 6	-2, 9	3, 9
9	-1, 1	-4, 1	4, 8	9	-1, 2	-5, 9	7, 8
1950	-0, 5	-1, 6	1, 8	2000	-1, 7	-7, 9	10, 1

TABLE 4

Values of f at 0 h on January 1st from 1900 to 2000

Year	M_2	K_1	O_1	K_2	Year	M_2	K_1	O_1	K_2
1900	1, 007	0, 993	0, 987	0, 962	1950	0, 964	1, 111	1, 180	1, 310
1	1, 019	0, 954	0, 923	0, 874	1	0, 964	1, 112	1, 181	1, 314
2	1, 029	0, 918	0, 865	0, 807	2	0, 967	1, 103	1, 167	1, 284
3	1, 035	0, 892	0, 823	0, 764	3	0, 975	1, 085	1, 138	1, 223
4	1, 038	0, 882	0, 807	0, 748	4	0, 985	1, 059	1, 095	1, 140
5	1, 036	0, 889	0, 818	0, 759	5	0, 997	1, 025	1, 041	1, 044
6	1, 030	0, 912	0, 856	0, 797	6	1, 009	0, 987	0, 978	0, 948
7	1, 021	0, 946	0, 912	0, 860	7	1, 021	0, 947	0, 914	0, 862
8	1, 010	0, 986	0, 976	0, 945	8	1, 030	0, 913	0, 858	0, 798
9	0, 997	1, 024	1, 039	1, 041	9	1, 036	0, 890	0, 819	0, 760
1910	0, 985	1, 058	1, 094	1, 137	1960	1, 038	0, 882	0, 806	0, 748
1	0, 975	1, 085	1, 137	1, 222	1	1, 035	0, 892	0, 822	0, 763
2	0, 967	1, 103	1, 167	1, 283	2	1, 029	0, 917	0, 864	0, 805
3	0, 964	1, 112	1, 181	1, 314	3	1, 019	0, 953	0, 922	0, 873
4	0, 964	1, 111	1, 180	1, 311	4	1, 008	0, 992	0, 986	0, 960
5	0, 969	1, 100	1, 163	1, 273	5	0, 995	1, 030	1, 048	1, 058
6	0, 977	1, 081	1, 130	1, 207	6	0, 983	1, 063	1, 101	1, 152
7	0, 987	1, 053	1, 084	1, 119	7	0, 973	1, 089	1, 143	1, 233
8	1, 000	1, 017	1, 027	1, 021	8	0, 967	1, 106	1, 170	1, 290
9	1, 012	0, 977	0, 963	0, 926	9	0, 964	1, 113	1, 182	1, 315
1920	1, 023	0, 939	0, 900	0, 846	1970	0, 965	1, 110	1, 178	1, 306
1	1, 032	0, 907	0, 847	0, 788	1	0, 970	1, 097	1, 158	1, 264
2	1, 037	0, 887	0, 814	0, 755	2	0, 978	1, 076	1, 123	1, 193
3	1, 038	0, 883	0, 808	0, 749	3	0, 989	1, 047	1, 075	1, 103
4	1, 034	0, 896	0, 830	0, 771	4	1, 002	1, 011	1, 016	1, 006
5	1, 027	0, 924	0, 876	0, 819	5	1, 014	0, 971	0, 952	0, 913
6	1, 017	0, 961	0, 936	0, 891	6	1, 025	0, 933	0, 899	0, 835
7	1, 005	1, 001	1, 001	0, 981	7	1, 033	0, 902	0, 840	0, 780
8	0, 992	1, 038	1, 061	1, 079	8	1, 037	0, 884	0, 810	0, 752
9	0, 981	1, 069	1, 112	1, 172	9	1, 037	0, 884	0, 809	0, 751
1930	0, 972	1, 093	1, 151	1, 249	1980	1, 033	0, 900	0, 836	0, 777
1	0, 965	1, 108	1, 175	1, 300	1	1, 026	0, 930	0, 885	0, 829
2	0, 963	1, 113	1, 183	1, 317	2	1, 015	0, 968	0, 947	0, 905
3	0, 965	1, 108	1, 175	1, 300	3	1, 003	1, 007	1, 011	0, 997
4	0, 971	1, 093	1, 151	1, 251	4	0, 990	1, 044	1, 070	1, 095
5	0, 981	1, 070	1, 113	1, 174	5	0, 979	1, 074	1, 119	1, 186
6	0, 992	1, 039	1, 063	1, 082	6	0, 970	1, 096	1, 155	1, 258
7	1, 004	1, 002	1, 003	0, 984	7	0, 965	1, 109	1, 177	1, 304
8	1, 017	0, 962	0, 938	0, 894	8	0, 964	1, 113	1, 183	1, 316
9	1, 027	0, 925	0, 878	0, 821	9	0, 966	1, 106	1, 172	1, 294
1940	1, 034	0, 897	0, 831	0, 772	1990	0, 973	1, 090	1, 146	1, 240
1	1, 038	0, 883	0, 808	0, 750	1	0, 982	1, 066	1, 106	1, 161
2	1, 037	0, 886	0, 813	0, 754	2	0, 994	1, 034	1, 054	1, 066
3	1, 032	0, 906	0, 846	0, 786	3	1, 006	0, 996	0, 992	0, 968
4	1, 023	0, 938	0, 898	0, 844	4	1, 018	0, 956	0, 927	0, 879
5	1, 012	0, 976	0, 961	0, 924	5	1, 028	0, 920	0, 869	0, 810
6	1, 000	1, 016	1, 026	1, 019	6	1, 035	0, 894	0, 826	0, 766
7	0, 988	1, 052	1, 083	1, 116	7	1, 038	0, 883	0, 807	0, 749
8	0, 977	1, 080	1, 129	1, 204	8	1, 036	0, 888	0, 817	0, 758
9	0, 969	1, 100	1, 162	1, 272	9	1, 031	0, 910	0, 854	0, 794
1950	0, 964	1, 111	1, 180	1, 310	2000	1, 022	0, 022	0, 944	0, 856

TABLE 5

For computation of w and $1 + W$

S_2, MS_4, $2MS_6$			K_1, MK_3			N_2, MN_4, $2MN_6$		
Angle	w/f_{K_2}	W/f_{K_2}	Angle	wf_{K_1}	Wf_{K_1}	Angle	w	1 + W
°	°		°	°		°	°	
0	0, 7	-0, 214	0	0, 0	0, 331	0	0, 0	1, 184
10	- 6, 6	-0, 192	10	- 2, 5	0, 327	10	1, 6	1, 182
20	-12, 3	-0, 131	20	- 4, 9	0, 316	20	3, 1	1, 174
30	-15, 5	-0, 046	30	- 7, 3	0, 297	30	4, 6	1, 163
40	-16, 5	0, 047	40	- 9, 6	0, 271	40	5, 9	1, 147
50	-15, 6	0, 134	50	-11, 8	0, 239	50	7, 2	1, 127
60	-13, 4	0, 207	60	-13, 8	0, 201	60	8, 3	1, 104
70	-10, 3	0, 258	70	-15, 6	0, 157	70	9, 2	1, 077
80	- 6, 6	0, 284	80	-17, 1	0, 107	80	9, 9	1, 048
90	- 2, 6	0, 284	90	-18, 3	0, 053	90	10, 4	1, 017
100	1, 6	0, 256	100	-19, 1	-0, 003	100	10, 6	0, 984
110	5, 6	0, 204	110	-19, 3	-0, 060	110	10, 4	0, 953
120	9, 2	0, 131	120	-19, 0	-0, 118	120	10, 0	0, 922
130	12, 0	0, 041	130	-17, 8	-0, 173	130	9, 1	0, 893
140	13, 7	-0, 058	140	-15, 9	-0, 224	140	7, 8	0, 867
150	13, 6	-0, 157	150	-13, 1	-0, 268	150	6, 2	0, 846
160	11, 2	-0, 245	160	- 9, 3	-0, 302	160	4, 3	0, 830
170	6, 0	-0, 307	170	- 4, 9	-0, 323	170	2, 2	0, 819
180	- 0, 9	-0, 330	180	0, 0	-0, 331	180	0, 0	0, 816
190	- 7, 8	-0, 308	190	4, 9	-0, 323	190	- 2, 2	0, 819
200	-12, 6	-0, 247	200	9, 3	-0, 302	200	- 4, 3	0, 830
210	-14, 9	-0, 163	210	13, 1	-0, 268	210	- 6, 2	0, 846
220	-14, 8	-0, 067	220	15, 9	-0, 224	220	- 7, 8	0, 867
230	-13, 0	0, 029	230	17, 8	-0, 173	230	- 9, 1	0, 893
240	- 9, 8	0, 115	240	19, 0	-0, 118	240	-10, 0	0, 922
250	- 6, 0	0, 186	250	19, 3	-0, 060	250	-10, 4	0, 953
260	- 1, 8	0, 236	260	19, 1	-0, 003	260	-10, 6	0, 984
270	2, 6	0, 263	270	18, 3	0, 053	270	-10, 4	1, 017
280	6, 9	0, 265	280	17, 1	0, 107	280	- 9, 9	1, 048
290	10, 8	0, 241	290	15, 6	0, 157	290	- 9, 2	1, 077
300	14, 1	0, 192	300	13, 8	0, 201	300	- 8, 3	1, 104
310	16, 5	0, 124	310	11, 8	0, 239	310	- 7, 2	1, 127
320	17, 5	0, 039	320	9, 6	0, 271	320	- 5, 9	1, 147
330	16, 8	0, 051	330	7, 3	0, 297	330	- 4, 6	1, 163
340	13, 7	-0, 133	340	4, 9	0, 316	340	- 3, 1	1, 174
350	8, 0	-0, 193	350	2, 5	0, 327	350	- 1, 6	1, 182
360	0, 7	-0, 214	360	0, 0	0, 331	360	0, 0	1, 184
"Angle" = V + u for K_1 f = f for K_2			"Angle" = 2V + u for K_1 f = f for K_1			"Angle" = 3V for M_2 minus 2V for N_2		