THE ABSOLUTE PLUMB-LINE DEFLECTION II

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" It is depressing, if instructive, to find out how difficult it is even for informed people to take in a simple proposition, if it goes counter to what GALBRAITH calls 'the conventional Wisdom', ideas which have been accepted so long, that they are defended uncritically and even with passion from attack. " (Philip WILLS in an article " Air Traffic Control and Logic " in *Sailplane and Gliding,* February 1963).

Ref. 1 : " Radio Aids and Geodesy " in Vol. XXIV of *International Hydrographic Review,* 1947.

Ref. 2 : " The Absolute Plumb Line Deflection " in Vol. XL of *International Hydrographic Review,* 1963.

Since I have discovered that many people, at least in Denmark, did not understand, or completely misunderstood, ref. 2, I want to give a more complete, and I do hope more comprehensible, explanation of my thoughts.

Intentionally I try to evade long and tedious mathematical developments; but I shall try to explain the thoughts behind the principle proposed with appeal to simple logic and well known laws of nature.

The following notations are used :

Γ : (capital gamma) : The gravity in the spheric system.
γ : The gravity in the spheroidic system.
g : The gravity in the geoidic system.
r : Radius in the spheric system.
R : Radius vector in the spheroidic system.
φ' : Geocentric latitude, latitude in spheric system.
φ : Geographic or geodetic latitude, latitude in spheroidic system.
φ_G : Geoidic latitude, latitude in geoidic system.
a : Semi-major (equatorial) axis of ellipsoid of rotation.
b : Semi-minor (polar) axis of ellipsoid of rotation.
f : Flattening $\dfrac{a - b}{a}$ of ellipsoid of rotation.
M : Meridian radius of curvature of ellipsoid of rotation.
N : Grand normal, radius of curvature of ellipsoid of rotation at right angles to the meridian.

The notations are marked " o " when they are used to indicate the value on the surface of the globe in question.

First let me explain the strange word : anomaly. It simply means deviation from the normal, so it depends upon what is considered normal.

In ref. 1, page 35, I defined the gravity anomaly as : $g_0'' - \gamma_0$.

The two dots at g_0 indicate that the observed gravity g has been reduced to mean sea level by the free-air and the Bouguer correction, because this is the common practice in Denmark. γ_0 is the " normal " gravity computed from the International Ellipsoid having a flattening $f_0 = 1/297.0$.

As W. D. LAMBERT says in an article referred to later in great detail : " The question of the best method of reducing to sea level, whether by an isostatic reduction, by condensation, by inversion, or by no reduction at all except the free-air reduction for elevation, has not yet been settled to the entire satisfaction of all geodesists and is too large a question to be discussed here. "

Moreover this question is irrelevant as to principle; here we may abstract and imagine it possible to measure the gravity on the Geoid itself.

Net let us consider the different globes and their corresponding potential surfaces (ref. 1, page 15).

The sphere

The simplest potential surface approximating to the Earth is the Sphere. If we imagine a large fluid mass, with mean density the same as that of the Earth (about 5.6) and with volume the same as that of the Geoid, being alone in space, it would, solely under the influence of Newton's law of attraction, assume the form of a sphere.

Newton's law says in words that the attraction between two masses is proportional to the product of the masses and inversely proportional to the square of the distance between their centres of gravity, or :

$$k = \frac{F \cdot m_1 \cdot m_2}{r^2} \qquad (1)$$

where k is the force of attraction, F an attraction constant, m_1 and m_2 the two masses and r the distance between their centres of gravity.

The surface of this globe (being fluid) is not only a potential surface but also the visible boundary between empty space and the globe filled with matter. On and outside this boundary Newton's law of attraction is valid in its simple form.

On the surface of this globe a particle of unit mass would be attracted towards the centre with a force :

$$\Gamma_0 = \frac{F \cdot m}{r_0^2} \qquad (2)$$

where m is the total mass of the globe and r_0 the radius of the sphere (or distance to its centre of gravity). Γ_0 is independent of the position on the globe.

On the potential surface at double the distance from the centre ($r = 2r_0$), the force Γ would be only $\frac{1}{4}\,\Gamma_0$. All potential surfaces outside the globe would also be concentric spheres.

Inside the globe one cannot apply (2) directly, or the attraction would be infinite at the center around which all the mass is evenly distributed, so here the attraction from the matter in the globe must be zero. The force Γ is in fact the difference between the attraction of all masses below and the attraction of all the masses above the horizontal plane through the point in question. It is seen that all the potential surfaces inside the globe must also be concentric spheres and all the lines of force straight lines (radii) terminating at the centre, and that the force of attraction Γ would reach its maximum some distance below the surface of the globe (the isostatic layer).

The results of geophysics show that the Earth is not homogeneous but has an increasing density towards the centre. On the surface we have water with a density of about 1, sand, earth and rocks with a density of about 2-3, but in the centre the density is believed to be between 11 [1] and 18.5 [2].

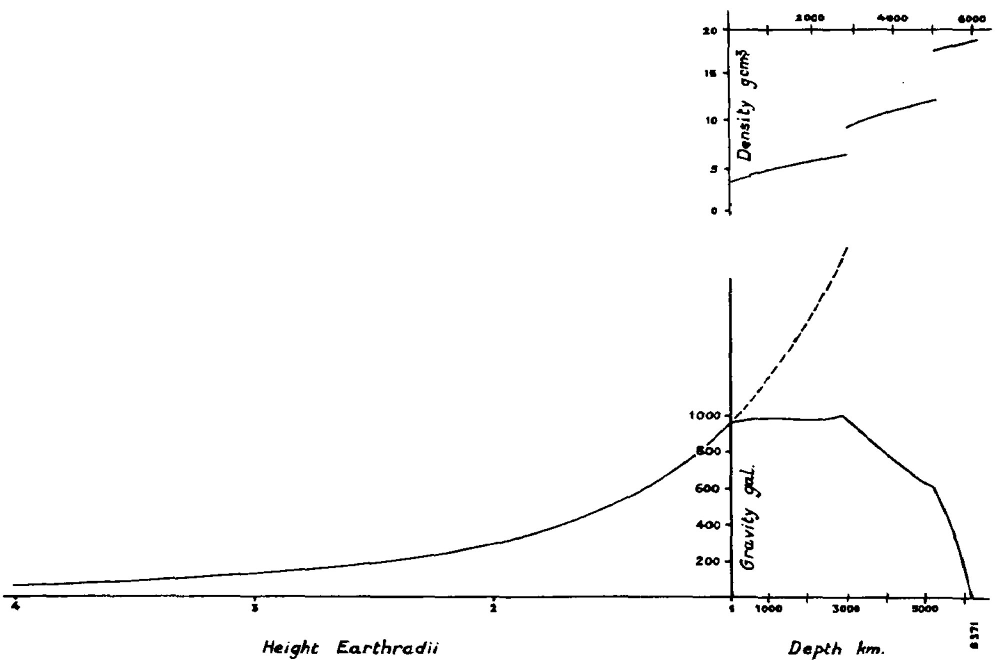

FIG. 1. — Density and the numerical value of gravity as function of the distance to the centre.

N.B. — The direction of gravity is of course in the direction of the abscissa.

We can imagine having the same variation of density in layers inside our idealized globe or sphere. This would not interfere with the variation of gravity on or outside the globe, but inside it would let the gravity rise slowly to a first (relative) maximum (isostatic layer) of about 999 gal at a depth of about 6-700 km below the surface [2] (see *fig.* 1).

If we pour some oil on our sphere it would be distributed over the whole surface; but it we pour some mercury on it, it would sink to the layer with the same density (13.6). It is seen that all the potential surfaces are still concentric spheres and the lines of force still straight lines.

We can also imagine our globe having an atmosphere like that of the Earth. Though the weight of a column of the atmosphere at sea level balances with that of a column of about 760 mm of mercury or about 10 m of water, the mass of the whole atmosphere is only about $\frac{1}{3\,000\,000}$ of the mass of the Earth [1].

As we don't know the mass of the Earth within this accuracy (1 in 10^6), nor can we observe the absolute gravity with this accuracy (ref. 1, page 35), so far no one has taken the atmosphere into account.

It is easily seen that the condition imposed on all potential functions is fulfilled, i.e. that the integral of the forces over the whole globe is zero, as the force acting on one random surface element of the potential surface is cancelled out by an equal but opposite directed force acting on the surface element on the opposite side of the potential surface. This is also valid for all the other surface elements, that is to say, there are no forces left to change the motion of the globe. The same applies to all the other spheres or potential surfaces.

If we consider ourselves observing Γ_0 (the absolute gravity or the numerical value of the gravity) on the surface of this idealized globe we would everywhere get the same result (apart from observational errors). We could also say that we have measured the distance r_0 to the centre of the globe, as r_0 is a function of Γ through Newton's law of attraction.

If we consider ourselves moving around above and below the surface of the idealized globe with an ordinary relative gravimeter measuring the gravity-difference from the gravity on the idealized globe, we could in fact use it as an altimeter measuring the distance to the surface of the globe. The only places where we could not apply this principle are where the variations of gravity in relation to the distance from the centre $\left(\frac{d\Gamma}{dr}\right)$ are zero, i.e. where the gravity has a minimum or maximum, at an infinite distance ($r = \infty$, $\Gamma = 0$) or at the isostatic layer 600-700 km below the surface of the globe. The principle is most accurate where the variation of Γ in relation to r is greatest, i.e. where :

$$\frac{d^2\,\Gamma}{dr^2} = 0$$

or the curve has an inflexion at the surface of the sphere.

The celestial sphere with $r = \infty$ and $\Gamma = 0$ belongs to this system.

To recapitulate :

In the spherical system the globe and all its other potential surfaces are concentric spheres.

On each potential surface the distance to the centre (r), the gravity (Γ), the curvature of any great circle $\left(\frac{1}{r}\right)$ and the surface curvature $\left(\frac{1}{r^2}\right)$ of any surface element are all constant.

The only force in action is the mass attraction, on and outside the globe according to Newton's simple law of attraction (2).

All the lines of force are straight lines terminating in the centre.

The Spheroid

If we imagine the sphere set in rotation around an axis through its centre, a new force, the centrifugal force, which is directly proportional to the rate of rotation and the distance to the axis of rotation, will appear in addition to the mutual mass attraction.

Around the Equator, where the centrifugal force operates at its greatest distance from the axis and consequently has its greatest value, it counteracts the mass attraction and tries to raise the surface of the globe.

As the globe, being a fluid, cannot be expanded or compressed (at least with an extremely high degree of approximation), and as the volume of the globe must thus remain constant, matter is taken from the areas around the poles (where the distance to the axis and consequently the centrifugal force are zero) and moved to a belt around the Equator. The surface of the globe around the poles is lowered and the mass attraction here is increased as the surface is brought nearer to the centre of gravity.

For a given rate of rotation there is a corresponding form of the globe : the greater the rate of rotation, or the greater the centrifugal forces are in relation to the mass attraction, the greater the flattening of the globe.

If the rate of rotation of the globe is slowed down, the flattening decreases, and if it is stopped the flattening again becomes zero, i.e. the globe is again a sphere.

In fact the globe behaves as an old fashioned centrifugal regulator.

Much energy is needed to overcome inertia and to move masses from the areas around the poles to the belt around the Equator when the globe is set in rotation. But when the globe has reached its fixed speed and a state of equilibrium has been reached, no energy is needed to keep it rotating.

When the globe is rotating at its fixed speed and has reached its equilibrium position, only the two forces, mass attraction and centrifugal force, are acting, and the gravity $\bar{\gamma}_0$ at every point of the surface of the rotating globe is at right angles to the surface element in question and is the resultant of these two components.

In this case a_0, b_0, the flattening f_0 of the globe, and the mass attraction, the centrifugal force and the resultant $\bar{\gamma}_0$ at any fixed point on the surface of the globe are constant.

Only when the rotation of the globe is accelerated or retarded do other forces intervene; a_0, b_0, f_0, the centrifugal force and the gravity $\bar{\gamma}_0$ are then variables, and only the volume and the mass of the globe and the mass attraction at any fixed point in space outside the surface of the globe remain constant.

As both the centrifugal forces (at right angles to the axis of rotation) and the mass attractions (through the centre) lie in the meridian planes, it follows that their resultants must also be in the meridian planes and that the Spheroid must thus be a body of rotation.

The surface of this rotating globe has not the same potential as the stationary sphere. This can be explained in the following way : on the sphere all the forces are directed towards the centre, contributing to the potential. On the rotating globe the centrifugal forces, directed away from the axis of rotation, also intervene and subtract from the potential created by the mass attraction. The potential surface belonging to the stationary system which has the same potential as the rotating spheroid is therefore a sphere raised above the surface of the stationary globe, becoming higher as the speed of rotation of the spheroid increases.

When the rate of rotation is retarded and stopped, the spheric potential surface lowers itself and at last coincides with the surface of the sphere with the same volume as the Geoid. The difference is so small that it cannot be shown on a figure.

We can define the Spheroid as the idealized, regular potential surface with the same distribution of masses in the core (not in the crust) as the Earth, rotating with the same speed and having the same potential as mean sea level; all the irregularities of the Geoid caused by islands and continents with mountains and valleys and all the other uneven distribution of masses in the crust of the Earth are eliminated.

Colonel N. P. JOHANSEN [3] defines the Spheroid as an ellipsoid of rotation concentric with the centre of gravity of the Earth, with the minor axis coinciding with the world axis, whose form and dimensions are such that, in approximating to the Geoide, the sum of the volumes cut off between the two surfaces outside and inside the ellipsoid are equal.

In ref. 1, page 16, I defined the Spheroid as the idealized globe, resulting from assuming that the Earth is alone in space, that it maintains its rate of rotation, and then assuming that all solids shall change into fluids while maintaining their densities.

This definition is not strictly correct. When the islands and continents change into fluids and run out into the ocean, the surface would rise (perhaps a couple of hundred metres) and not have the same potential as mean sea level on the Earth.

The error committed would perhaps be smaller than the uncertainty in our present knowledge of the Spheroid itself, but to be scientifically correct, I should prefer to alter this definition to :

The Spheroid is the idealized potential surface having the same potential as mean sea level and resulting from assuming that the Earth is alone in space, that it maintains its rate of rotation and that all solids change into fluids while maintaining their densities.

It is assumed that the Spheroid is an ellipsoid of rotation, and it is easily seen that such a body also satisfies the condition that the integral of γ_0 over the whole surface of the globe is zero. The force acting on one random surface element is cancelled out by an oppositely directed force acting on the surface element with opposite latitude and longitude. The moment of this couple is balanced out by the moment of the couple acting on the surface elements at the same latitude, opposite longitude, and opposite latitude, same longitude. This reasoning can be applied to all surface elements (*).

As we rise above the surface of the Spheroid the mass attraction decreases according to Newton's law, while the centrifugal force increases due to a longer arm of rotation (except at the poles). The potential surfaces in the lower atmosphere must therefore be concentric (not confocal) ellipsoids of rotation with increasing flattening with increasing height.

The surface elements of the different potential surfaces on the same line of force are not parallel but converge towards the poles. The gravity lines of force are not straight lines but curves with concavity towards the poles. Only at the poles, where there is no centrifugal force, and along the Equator, where the centrifugal force acts in a directly opposite direction to the mass attraction, are the lines of force still straight lines. As the latitude φ is the angle between the plumb line (tangent to the line of force) and the Equatorial plane, larger values (of the latitude φ) will be obtained when rising from the surface of the Spheroid through the lower atmosphere along a line of force.

In the upper atmosphere the rate of rotation is slowed down with increasing height and does not follow that of the Earth. The potential surfaces will be concentric ellipsoids of rotation with decreasing flattening with increasing height. The gravity lines of force will somewhere have an inflexion, and above this will be curved with the concavity towards the Equator.

Far outside the atmosphere no rotation takes place, only the mass attraction is acting; consequently all the potential surfaces are concentric spheres and the lines of force straight lines in the direction of the centre of the Earth.

Although the mass of the atmosphere is so small that it doesn't influence the gravity on the Earth to such an extent that it can at present be measured, it still plays a role conveying the rotation of the Earth some distance out in space. In this age of space geophysicists have a far wider knowledge of the atmosphere than in the past.

As geodesists work on the physical surface of the Earth (ref. 1, page 14) (from the level of the Dead Sea about 400 m below mean sea level, to the top of Mount Everest about 8 900 m above mean sea level), only the curvature of the lines of force in the lower atmosphere is of real interest.

Gauss, Helmert and others [4] have worked out formulae for this curvature (ref. 1, page 24) for the Bessel ellipsoid. I don't know if a similar formula has been worked out for the International Ellipsoid, but at least it can be done.

The first approximation to the Spheroid has only been known since 1924, when the International Union of Geodesy and Geophysics (IUGG) in

(*) According to this reasoning also, other closely resembling globes (ovaloids of rotation) might be potential surfaces (ref. 1, page 16).

Madrid defined the International Ellipsoid with the determining elements $a_0 = 6\,378\,388$ m and $f = 1/297.0$, and as a potential surface since 1930, when the same scientific union in Stockholm defined the variation of gravity γ_0 on this globe by the international gravity formula :

$$\gamma_0 = 978.0490\ (1 + 0.005\,2884\ \sin^2 \varphi_0 - 0.000\,0059\ \sin^2 2\varphi_0)$$

I prefer to call this International Ellipsoid the International Spheroid (ref. 1, page 18).

This idealized globe, the International Spheroid and its "normal" gravity are of course defined with much greater accuracy than that with which they are really known (ref. 1, page 43).

" Science advances rather by providing a succession of approximations to the truth, each more accurate than the last, but each capable of endless degrees of higher accuracy. " (Sir James JEANS : The Universe around us).

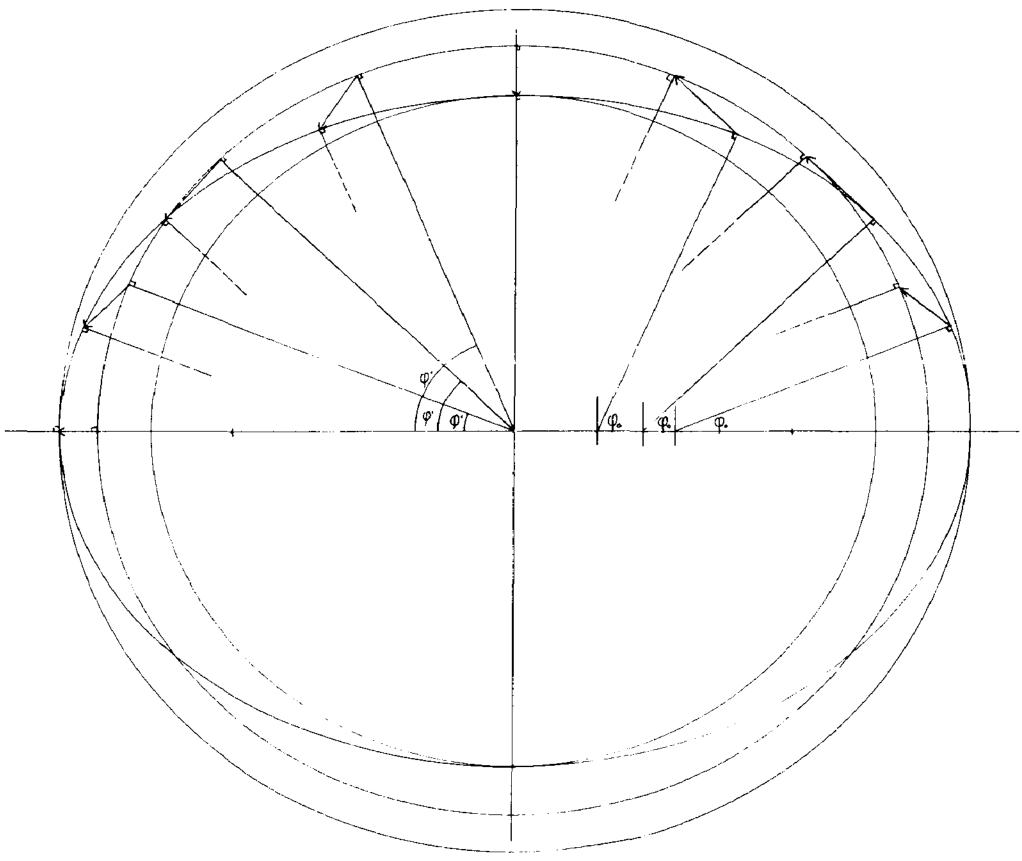

FIG. 2. — The Sphere and the Spheroid superimposed.

It is of course necessary to have an idealized globe of such regular form that you can develop a geometry and make your computations on it, define your chart planes and illustrate the undulations of the Geoid in relation to it.

The flattening of the International Spheroid is so small (1/297.0) that it cannot be distinguished from a sphere on normal graphic representations; on the Earth the centrifugal force is very small in relation to the mass attraction.

My figures are therefore drawn greatly exaggerated (flattening about 1/5) to illustrate clearly the effect of the flattening.

To investigate the relations between the Sphere and the Spheroid, we might superimpose them concentrically (fig. 2) and draw the curve representing gravity $|\gamma_0|$ as a function of φ_0 as given by the international gravity formula (fig. 3).

It is seen directly from the figure that the Spheroid has its greatest curvature for $\varphi_0 = 0$. It is greatest in a NS-direction $\frac{1}{M_0}$ corresponding to $1' = 1852.925$ m. In an EW-direction it is $\frac{1}{N_0\,(=a_0)}$ corresponding to $1' = 1855.398$ m.

The curvature decreases continuously towards the poles but the curvature is everywhere greater in a NS-direction than in an EW-direction. Only at the poles it has its smallest curvature $\frac{1}{M_0}$ corresponding to $1' = 1861.666$ m in all direction (which are all to the South) as $M_0 = N_0$.

The poles are so-called spherical points having the same surface curvature as a sphere with radius $r = M_0 = N_0$.

In fact a gravimeter might be used for latitude determination on the idealized globe except near the poles and the Equator, as gravity γ_0 and latitude φ_0 are functions of each other.

Moreover it is seen that what is vertical (the plumb line) on one globe is not vertical at the corresponding position on the other; and what is horizontal (the fluid surface) on one globe is not horizontal at the corresponding position on the other globe. But on both globes the vertical is always and everywhere at right angles to the horizontal (the surface element).

Since latitude is defined as the angle between the plumb line and the Equatorial plane then the latitude is not exactly the same for corresponding positions on the two globes. We can call the latitude on the Speroid φ_0, and on the Sphere (geocentric latitude) φ'. (It is not necessary to call it φ_0', since it is the same for all spheres (potential surfaces) as the gravity lines of force are straight lines).

The deflection of the plumb line $\varphi_0 - \varphi'$ or $\varphi' - \varphi_0$ (according to what globe is considered "normal") has been computed and tabulated both in *La Connaissance du Temps* and in IHB Special Publication No. 21, Table VII, together with the logarithm of the radius vector (R) of the International Spheroid (expressed in terms of the equatorial radius a as a unit), as they are geometrically connected by the formulae [5] :

$$\tan \varphi' = (1 - f_0)^2 \tan \varphi_0 \tag{3}$$

and

$$R = \sqrt{\frac{\cos \varphi_0}{\cos \varphi' \cos (\varphi_0 - \varphi')}} \tag{4}$$

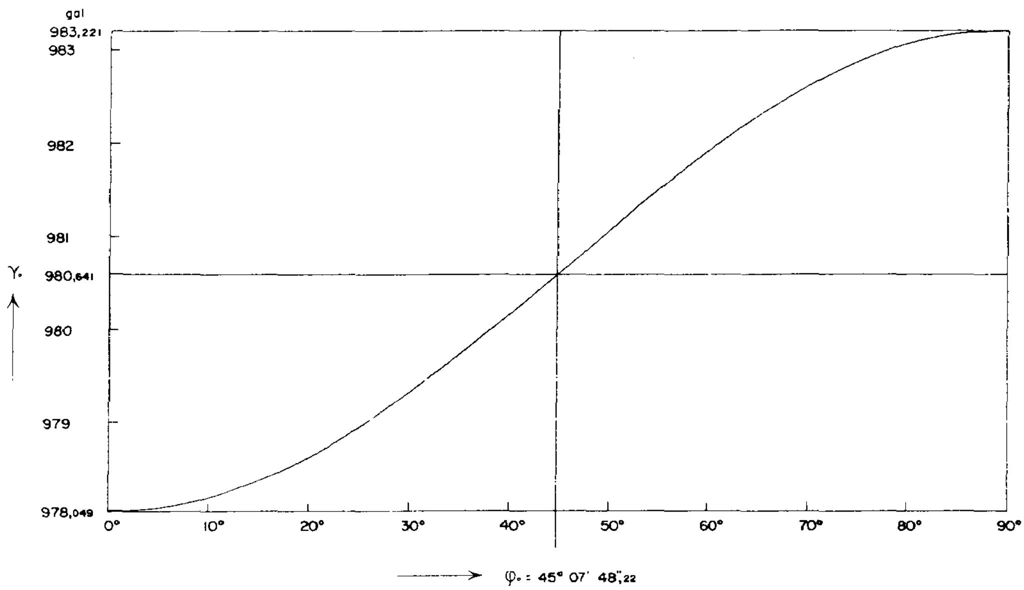

FIG. 3. — γ_0 as a function of latitude
$\gamma_0 = 978.0490\ (1 + 0.005\ 2884\ \sin^2 \varphi_0 - 0.000\ 0059\ \sin^2 2\ \varphi_0)$

It is seen from the tables that this deflection of the plumb line between these two globes has its maximum of 11′35″66 for φ_0 at about 45°, while it is zero at the poles and along the Equator, where $|\gamma_0|$ has maxima and minima (fig. 3). For the Bessel ellipsoid with a slightly smaller flattening (1/299.153) the greatest deflection is only 11′30″4.

We might visualize two spheres (potential surfaces) coincident with the Spheroid along the Equator and at the poles. The outer sphere coincident along the Equator has everywhere a smaller gravity and a smaller potential than the Spheroid. The inner sphere coincident at the poles has everywhere a greater gravity and a greater potential than the Spheroid. Between these two spheres we might imagine a multitude of concentric spheres intersecting the Spheroid at different latitudes. As the gravity on the Spheroid is continuously increasing with increasing φ_0, it follows that there must be one and only one sphere having the same potential as the spheroid, and this is what I call *the* Sphere. As mentioned previously it must have a greater radius than the sphere with the same volume and inner mass distribution as the Geoid.

If we look at the gravity formula :

$$\gamma_0 = 978.0490\ (1 + 0.005\,2884\ \sin^2 \varphi_0 - 0.000\,0059\ \sin^2 2\,\varphi_0)$$

it has the form :

$$\gamma_0 = A_0\ (1 + a_1\ \sin^2 \varphi_0 - a^2\ \sin^2 2\varphi_0)$$

A_0 is the " normal " gravity at the Equator (not to be confused with the Equatorial a_0).

If we had only this term, the formula would represent the outer sphere.

a_1, the coefficient of the spherical function or harmonic of the 2nd order, is dependent on the flattening and forces $|\gamma_0|$ to get the correct values around the poles.

a_2, the coefficient of the spherical function or harmonic of the 4th order, forces $|\gamma_0|$ to get the correct values about the middle of the quadrant; forces the globe to be an ellipsoid and not an ovaloid of rotation, so that the complete formula represents a spheroid in complete agreement with the International Ellipsoid.

It is seen that the gravity formula is only valid for one potential surface, the International Spheroid. If we want to have a formula for the potential surfaces, say 1 gal above the International Spheroid at the equator, it is not sufficient to subtract 1.000 from A_0, since the flattening of this potential surface, as mentioned before, is greater than that of the International Spheroid. Also the other coefficients a_1 and a_2 will change gradually.

By differentiation one derives the following formulae :

$$\gamma_0 = f(\varphi_0) = A_0 (1 + a_1 \sin^2 \varphi_0 - a_2 \sin^2 2\varphi_0) \tag{5}$$

$$\frac{d\gamma_0}{d\varphi_0} = f'(\varphi_0) = A_0 (2a_1 \sin \varphi_0 \cos \varphi_0 - 4a_2 \sin 2\varphi_0 \cos 2\varphi_0) \tag{6}$$

$$\frac{d^2\gamma_0}{d\varphi_0{}^2} = f''(\varphi_0) = A_0 (2a_1 \cos 2\varphi_0 - 8a_2 \; [2 \cos^2 2\varphi_0 - 1]) \tag{7}$$

The Sphere (with the same potential as the rotating Spheroid) must intersect the Spheroid where the numerical value of the gravity $|\gamma_0|$ has its inflexion (fig. 3), as it is here that the vertical component has its smallest value but greatest variation, while the horizontal component has its greatest value but smallest variation.

Setting $\frac{d^2\gamma_0}{d\varphi_0{}^2}$ (or $f''(\varphi_0)$) in formula (7) = 0 and solving the equation, I find that the Sphere intersects the Spheroid at latitude $\varphi_0 = 45°07'40''.22$ (geocentric latitude $\varphi' = 44°56'04''.60$) and has a gravity $|\Gamma_0| = |\gamma_0| =$ 980.64092 gal.

In fig. 2 I have drawn some vectors (from the Sphere to the Spheroid on the left side and from the Spheroid to the Sphere on the right) between positions having the same latitude (geocentric φ' and geographical φ_0 and vice versa).

They correspond to the deflecting vector $\vec{d}$ in ref. 2, page 77.

They must be visualized to represent a vector field transforming one globe to the other.

Each vector may be decomposed into a vertical and a horizontal component. It is seen that the vertical component raises or lowers the surface element so much that the gravity gets its correct value, while the horizontal component turns the surface element with its plumb line through the plumb line deflection between the two globes.

It is seen that the vertical component is the same as the gravity anomaly giving the vertical distance between the two globes.

If we consider the Sphere as the " normal ", idealized, imaginary globe while we measure the real numerical gravity $|\gamma_0|$ on the spheroid, the gravity anomaly will be $|\gamma_0| - |\Gamma_0|$ and will be positive from the pole down to the line of intersection. At the line of intersection the gravity anomaly is zero, and it is seen that all these positions have the same distance to the common axis of rotation and to the common centre of the two globes.

On the other side of the line of intersection the gravity anomaly $|\gamma_0| - |\Gamma_0|$ changes to negative and reaches its minimum as $\varphi_0 = 0$, and continues in the same way for the complete circumference.

It is seen that the gravity anomaly is a function of the position on the meridian $|\gamma_0| - |\Gamma_0| = F(\varphi_0)$. Where $F(\varphi_0) = 0$ the two globes intersect, but their normals (the direction of the two gravity vectors) are not the same.

Where $|\gamma_0| - |\Gamma_0| = F(\varphi_0)$ has a maximum or minimum, the two gravities γ_0 and Γ_0 do not have the same numerical value, but the 1st derivative of $F(\varphi_0) = |\gamma_0| - |\Gamma_0|$:

$$F'(\varphi_0) = \frac{d(|\gamma_0| - |\Gamma_0|)}{d\varphi_0}$$

is zero.

This only happens where the deflecting vector is acting in the same or in the opposite direction to the gravity $\bar{\gamma}_0$, i.e. where there is no horizontal component.

It is seen that the deflection of the plumb line $\varphi_0 - \varphi'$ is the horizontal component of the deflecting vector or $F'(\varphi_0)$ and is positive except for $\varphi_0 = 0$ or $\varphi_0 = 90°$, where it is zero.

As both the Sphere and the Spheroid are potential functions, and as the difference between two potential functions is again a potential function, and as the potential of the Sphere and the Spheroid are equal, it follows that the integral of the deflecting vector $\bar{d}$ over the whole surface of the globe is zero. This again will imply that the integral of all the vertical components (the gravity anomalies) and of all the horizontal components (the plumb line deflections) is zero.

If we imagine defining charts from our p.t. " normal " globe, the Sphere, and on them draw the iso-anomaly curves, they would all be latitude parallels. The zero iso-anomaly curve would depict where the two globes intersect. With the interval we choose, the other curves would depict where spheres (levels) with the same intervals in gravity intersect the Spheroid. The distance between the curves would be smallest around latitude 45° and depict the steepest slope to the North.

If we now consider a change from the Spheroid to the Sphere and allow the Spheroid to be our imaginary ideal, then all the deflecting vectors $\bar{d}$ change to act in the opposite direction; all the gravity anomalies are now $|\Gamma_0| - |\gamma_0|$ and change their sign as well as the plumb line deflections $\varphi' - \varphi_0$.

To recapitulate :

In the spheroidic system the globe and all its potential surfaces are (since the adoption of the International Reference Spheroid and the adoption of the international gravity formula to match) concentric ellipsoids of rotation with increasing flattening with increasing height, within the range of interest for geodesy.

Only two forces are in action : the mass attraction and the centrifugal force.

All the lines of force are, within the range of interest for geodesy, curved lines in the meridian plane with the concavity towards the poles. Only at the poles and around the Equator are the lines of force straight lines.

Where the Spheroid intersects the Sphere with the same potential, the two gravity vectors have the same numerical value but not the same direction.

The difference of the numerical value of the two gravity vectors $\bar{\gamma}_0$ and $\bar{\Gamma}_0$, the gravity anomaly, represents the vertical distance between the two equipotential globes. Where the gravity anomaly has its maxima or minima at the poles and around the Equator, the two gravity vectors have the same direction but not the same numerical value.

Where a potential surface is depressed by a new intervening force, the gravity is increased and the surface curvature is decreased. Where a potential surface is lifted by a new intervening force, the gravity is decreased and the surface curvature is increased.

The Geoid

Our Earth does not fulfill the ideal requirements put to our idealized globes.

It is not alone in space. The Sun and the Moon pull at it and create tidal waves running along its surface, being stopped or deflected by the continents and somewhere being dammed up to considerable heights, so that because of the short distance they can influence gravity much more than the Sun and the Moon themselves.

To eliminate these periodic disturbances we have chosen mean sea level and its imagined continuation under the continents as our reference. This surface we call the Geoid or the mathematical Earth surface.

Moreover our Earth is not fluid. Only about 8/11 of its surface is covered with water, while 3/11 rises above the water as continents and islands.

The Geoid has the advantage of having nearly the greatest potential of all accessible levels on the Earth, as only very few places on its physical surface are below mean sea level (e.g. the Dead Sea).

With an extremely high degree of approximation mean sea level can be considered to be a potential surface. Very quickly a fluid returns to its equilibrium at right angles to the resultant of all the forces acting on it, in contrast to solid matter.

All the visible irregularities in the distribution of masses on the Earth, continents and islands with mountains and valleys, etc., together with perhaps just as many irregularities in the varying density of the masses below the surface, give rise to undulations in the Geoid, elevations and depressions in relation to the Spheroid positioned concentric on the same axis of rotation.

Because of all these undulations (varying surface curvature), it is impossible to define charts from the Geoid and develop a geometry on it, even if we knew it in detail. We must use an idealized globe, the Reference Spheroid, for that purpose.

All these elevations and depressions are in the form of a multitude of spherical functions or harmonics of higher and higher order and with smaller and smaller coefficients.

If some men build a very large house, they perform a potential work by heaping up solid matter where before was only thin air. By this potential work they make a small impression on potential surfaces in the near neighbourhood of the house. Just under the house the potential surface will be pulled up and just over the house pulled down. To the side of the house the horizontal component of the attraction will give rise to deflections of the plumb line. As the mass of the house is so small in relation to the mass of the Earth as a whole, the influence will only be active for a rather short distance, say some hundred metres. But still the influence of the house may dominate over the influence of much greater masses, e.g. a range of high mountains hundreds of kilometres away. If the house is built in the middle of a continent perhaps a thousand metres above mean sea level, its influence will not reach down to the Geoid and make any elevation on it, but it will give a plumb line deflection for a geodesists orientating his instrument with reference to the spirit level close beside the house.

Trained observers should therefore try to avoid making observations in the close vicinity of irregular mass distributions, just as a navigator should ensure that the helmsman has no knife or keys or other magnetic material in his pockets.

Because of his close proximity to the magnetic compass it might have a dominating influence over the disturbing influence of the ship's magnetism and the correcting magnets, while it does no harm when it is down in his cabin.

Perhaps some day we might be able to compensate for the deflection of the plumb line by heavy weights close to the instrument, creating a little local minimum or maximum in the gravity anomaly. This might not be of interest to practical geodesy, but perhaps for astronomical observatories.

A fluid, the sea as well as the liquid in a spirit level, registers the direction of the resultant of *all* forces acting on it, with no exception. The dominating influence is the mass attraction of the Earth as a whole, but then comes a multitude of other influences: the centrifugal force, the attraction from distant continents, mountains and valleys, invisible subterranean irregularities in the densities of matter, nearby houses, etc., perhaps even the observer himself, if the spirit level is sufficiently sensitive.

All these forces, except the mass attraction from the Earth as a whole, and the centrifugal force caused by the rotation of the Earth, comprise what in ref. 2 I called the deflecting vector $\bar{d}$. It is of course quite impossible to try to compute its direction and numerical value; it is only used as a means to show that *all* the disturbing forces act as one force varying continuously in direction and numerical value from point to point in space.

The resultant of the mass attraction of the Earth as a whole, the centrifugal force created by the rotation of the Earth and the deflecting force is the real gravity vector g at any point in space.

Its direction is registered by the spirit level; its numerical value can be measured by an absolute gravity observation (ref. 1, page 33) with an uncertainty of some milligals.

When you have such a "gravity datum", you can easily "transport" numerical gravity (measure the difference in numerical gravity) to any other place with portable relative gravimeters.

Though modern gravimeters are so sensitive that they can register small gravity differences corresponding to the 2nd decimal of a milligal, they can of course not yield higher absolute accuracy than the datum. At present, I believe, three different gravity datum systems are in use (ref. 1, page 35), which might give rise to future discrepancies.

As the deflecting vector $\bar{d}_0$ can act in any direction, and not as the centrifugal force always in the meridian plane, it is evident that the real gravity vector $\bar{g}$ need not be contained in the meridian plane or its direction intersect the axis of rotation. The actual gravity g_0 is not a function of one variable (φ_0) as γ_0, but of two variables φ_0 and λ_0. The Geoid is *not* a body of rotation.

If we superimpose the Spheroid and the Geoid concentrically and coaxially, the Geoid will as mentioned show elevations and depressions in relation to the Spheroid.

As the idealized gravity vector $\bar{\gamma}_0$ is composed of the mass attraction force and the centrifugal force, and as the real gravity vector $\bar{g}_0$ is composed of the mass attraction force, the centrifugal force and the deflecting force, it follows that $\bar{d}_0$ is the vector field between the globes, transforming the Spheroid into the Geoid, i.e. $\bar{\gamma}_0 + \bar{d}_0 = \bar{g}_0$.

As both the Spheroid and the Geoid are potential functions and as the difference between two potential functions is again a potential function, and as the potentials of the Geoid and the Spheroid by definition are equal, it follows that the integral of deflecting vector $\bar{d}$ over the whole surface of the globe is zero. This again will imply that the integral of all the vertical components (the gravity anomalies) and of all the horizontal components (the plumb line deflections) is zero.

As the deflecting vector $\bar{d}$ is a function of the position (φ_0, λ_0) and is a potential function, it also follows that its vertical component, the gravity anomaly $|g_0| - |\gamma_0| = G(\varphi_0, \lambda_0)$, is a continuous and differentiable function of the two independent variables φ_0 and λ_0.

In all the positions where $\bar{g}_0$ and $\bar{\gamma}_0$ have the same numerical value $(|g_0| - |\gamma_0|) = G(\varphi_0, \lambda_0) = 0$, the two globes intersect, but their normals have not the same direction.

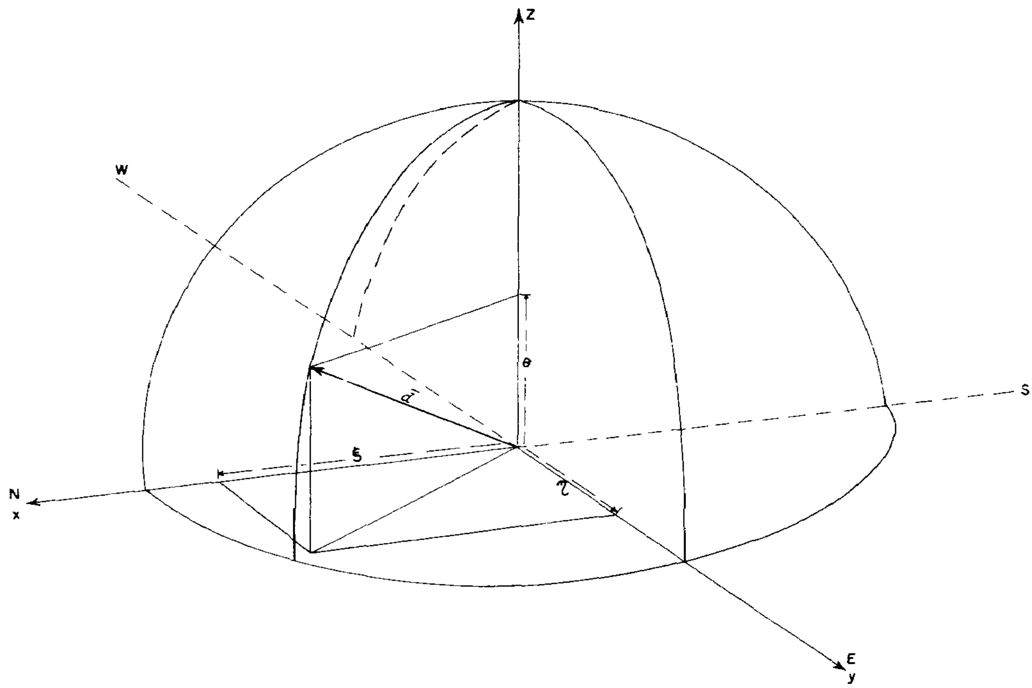

FIG. 4. — The deflecting vector $\bar{d}$ in a random point.
Its vertical component θ represents the gravity anomaly $|g_0| - |\gamma_0| = G(\varphi_0, \lambda_0)$.
Its two horizontal components : ξ in the direction N and η in the direction E cannot be observed directly but computed if the gravity-anomalies have been observed in a small neighbourhood of the points as
$G'_\varphi(\varphi_0, \lambda_0)$ and $G'_\lambda(\varphi_0, \lambda_0)$.

The horizontal component of $\bar{d}$ is in the same way a continuous and differentiable function of the position (φ_0, λ_0),

$$dG = \frac{\partial(|g_0| - |\gamma_0|)}{\partial \varphi_0} d\varphi_0 + \frac{\partial(|g_0| - |\gamma_0|)}{\partial \lambda_0} d\lambda_0 = G'_{\varphi_0}(\varphi_0, \lambda_0) + G'_{\lambda_0}(\varphi_0, \lambda_0)$$

the two parts representing the components ξ and η of the deflection of the plumb line in directions North and East.

At all the points where the gravity anomaly $|g_0| - |\gamma_0| = G(\varphi_0, \lambda_0)$ has a minimum or maximum, i.e. where both $G'_{\varphi_0}(\varphi_0, \lambda_0)$ and $G'_{\lambda_0}(\varphi_0, \lambda_0) = 0$, the two gravity vectors $\bar{g}_0$ and $\bar{\gamma}_0$ have the same direction, as the deflecting vector $\bar{d}$ is acting in the same or opposite direction to $\bar{\gamma}_0$.

This condition is sufficient as both the Geoid and the Spheroid always and everywhere have elliptic surface curvature (the centres of curvature on the same side).

Only at the very few positions where both the gravity anomaly and its two first derivatives in NS- and EW-direction are all zero, are the two gravity vectors identical (same value and direction), and the surface elements of the two globes coincide.

Our present knowledge of the shape of our Geoid is sparse.

HELMERT [1] computed the elevations under the continents and depressions under the sea to be of a magnitude of some hundred metres. The latest investigations utilizing artificial satellites, whose orbits have proved very sensitive to the Earth's gravitational field, show much smaller undulations (about 30 m) [8].

HELMERT [1] estimated the maximum deflection of the plumb line to be about 1.5 minute of arc and often to reach 10″ (seconds of arc) or more near the coasts.

To recapitulate :

The Geoid is influenced at each point by three forces : (1) the mass attraction from the Earth as a whole, (2) the centrifugal force created by the rotation of the Earth, (3) the deflecting force; in fact a multitude of forces from all the irregularities in the crust of the Earth, which at each point can be treated as one force.

It is in my opinion an impossible task to try to compute the undulations of the Geoid or irregularities in its lines of force, as it is impossible to find the necessary information. The reduction of $|g|$ to $|g_0|$ at mean sea level can only be an approximation as the densities are not known below the physical surface of the Earth. The form of the Geoid itself is of minor importance except where base-measurements are made. What really matters for the surveyor are the plumb line deflections, especially for all astronomical observations, where they are of dominating importance (ref. 1, page 28).

As the physical surface of the Earth is cut by a multitude of real potential surfaces (levels), shown by the spirit level or by hydrostatic levelling, and a multitude of idealized, imagined, potential surfaces of which the gravity of only one, the Spheroid, is known, I should prefer to have gravity formulae at suitable intervals, so I could interpolate the correct coefficients and find the variation of the idealized gravity γ in a NS-direction by differentiation in the level, where I am operating. Supplied with sufficient information about the densities in the upper layers of our idealized Spheroid, able mathematicians could provide this in the form of a critical table.

From a random point of observation in a higher level, the variation of the idealized gravity γ in relation to a parameter s in a N-direction could then be determined with great accuracy as $\frac{d\gamma}{d\varphi} \cdot \frac{d\varphi}{ds}$. The variation of φ_0 in relation to s_0 could be taken from the Meridian tables.

In Denmark quite a number of gravimetrical maps have been published. They may be of interest to prospectors and geophysicists, but they have had no influence on our geographical coordinates, and the latitudes φ_0 used for their reduction have been taken from our present charts defined from our old reference ellipsoid with a flattening of 1/300. They are only maps, not charts.

It is my firm conviction that when the Reference Spheroid is oriented with its centre coinciding with the centre of gravity of the Earth and with the same axis of rotation, then the zero iso-anomaly curves show where the Geoid and Reference Spheroid intersect, and that the plumb line deflection is then absolute and is zero at all the positions where the gravity anomaly has a minimum or maximum.

I am not alone in this conviction.

Since I wrote ref. 1 and 2 my attention has been drawn to a paper by Walter D. LAMBERT, USC & GS (retired), published in *Transactions American Geophysical Union,* Vol. 28, Number 2, April 1947, " Deflections of the Vertical from Gravity Anomalies ", of which I take the liberty of quoting a part (*) :

" With the increasing number of surveys based on astronomical coordinates and with Loran stations established on oceanic islands far from the mainland and its geodetic control, there has been an increasing interest in deflections of the vertical.

" Some of those who have a professional interest in deflections of the vertical but who have not analyzed the underlying conceptions seem to have the vague idea that there is something in nature that might properly be called *the* deflection of the vertical. *In the present state of geodesy there is nothing that may properly be called the* deflection of the vertical at a given point. All existing numerical values of the deflection depend on the assumed geodetic datum, and the geodetic datum might conceivably be anything whatever, though in practice the choice is confined within rather narrow limits.

" *The usual geodetic datum is essentially arbitrary.* Two geodesists with the same general background and the same mass of geodetic material before them might hit upon approximately the same choice of a datum and then again they might not. And since the choice of a datum is arbitrary, the deflections, dependent as they are on the datum, are arbitrary also.

" We diminish the arbitrariness of choice and, what is more important, we get an intrinsically better geodetic datum, *if we require that the center of the ellipsoid of reference shall coincide with the center of gravity of the Earth and that the axis of the ellipsoid shall coincide with the axis of rotation of the Earth.*

" As a matter of fact, our United States Datum, the predecessor of the North American Datum, was not adopted as the result of any special study. At first it just grew. Geodetic positions based on a supposedly temporary datum used in the eastern part of the country were found to agree satisfactorily with the astronomical positions further west, so the geodetic positions based on the eastern datum were accepted and the United States Datum was redefined in terms of a geodetic station in Kansas. Indeed, two of the items mentioned above were not explicitly considered when the decision was made; the position of the center of the ellipsoid of reference, and the direction of its axis. These are undetermined to this day. It is precisely these two items that will be emphasized in this paper.

" The word 'ellipsoid' is used throughout this note in the sense of an ellipsoid of revolution. The more general term 'spheroid' is avoided. It is not difficult to adapt the formula for theoretical gravity to a predetermined

(*) The italics are mine.

spheroid that is not an exact ellipsoid of revolution but the computation of triangulation on anything except an ellipsoid of revolution requires intricate numerical calculations; whereas this note implies throughout a comparison between astronomico-geodetic deflections and gravimetric deflections. Hence, for convenience, we are obliged to stick to the ellipsoid and base our gravity anomalies on an appropriate formula for theoretical gravity.

" About 100 years ago STOKES (...) did some pioneer theoretical work that may eventually enable us to refer our geodetic measurements to an ellipsoid thus ideally situated; its center at the center of gravity of the Earth and its axis coincident with the Earth's axis of rotation. If we insist that our geodetic datum shall be based on an ideally situated ellipsoid of this sort, the only elements of our datum left undetermined are the size and shape of the ellipsoid. If we specify these, then everything is determinate and we may properly speak of *the* deflection of the vertical at a given point. The problem then becomes one of computation.

" STOKES' mathematical developments in their original form do not give the deflections directly. The formula that he gave enables us to express the elevation of the geoid at a given point above the ellipsoid of reference in terms of the gravity anomalies. *A gravity anomaly and a deflection may seem rather different in kind, but the gravity anomaly is an anomaly in the vertical component of the gravitational attraction; a deflection is an anomaly in the horizontal component.* The nature of the force of which both are manifestations and the ingenuity of the mathematician make the connection between them. The ellipsoid is the particular ellipsoid already referred to, properly centered and properly oriented. The formula for the elevation of the geoid above this ellipsoid looks complicated, not to say perverse. No term of it has any obvious physical meaning. STOKES would probably have said that he was summing a series of spherical harmonics by an ingenious device. The more modern mathematician might say that he was solving an integral equation. (...).

" *In practical application we need a contour map of gravity anomalies. From it we can deduce the elevation of the geoid at various points. From enough such elevations we could draw contour lines of the geoid elevations referred to our chosen ellipsoid. These would be height contours, exactly similar to ordinary topographic contours. If we have enough such height contours, we can estimate the slope from the distance between neighboring contour lines. The slope of the geoid is simply another term for the deflection of the vertical in a direction perpendicular to the contours. STOKES himself suggested this.*

....

....

" It is interesting to consider how we might use gravity data to improve current processes for the computation of ordinary geodetic triangulation. If we knew the elevations of the geoid above the ellipsoid of reference, we could reduce the length of our measured bases to the level of that ellipsoid. At present, for lack of adequate information, they are reduced to the level of the geoid, which, for the various bases, has various positions with respect to the ellipsoid. Again, if we knew the deflections of the vertical, we could apply the proper corrections to our observed horizontal directions. These corrections may be quite appreciable, especially for inclined lines of sight at stations where the deflection is large, but for the present these corrections are almost universally ignored because the necessary data is lacking.

....

" The method of gravimetric deflections can be applied anywhere in the world, provided the gravity observations are available. We could determine from gravity observations a proper datum for South America and it would be on the same basis as a datum similarly determined for North America, even before those two continents are connected by triangulation. Gravimetric data could also be used to great advantage in the proposed readjustment of the triangulation of Europe. As mentioned in a preceding paragraph, the bases could be reduced to the ellipsoid, the observed directions could be corrected for deflection and perhaps the dimensions of the adopted ellipsoid could be improved. "

It is seen that in principle I am fully in agreement with Walter D. LAMBERT and with STOKES.

Only in respect of details concerning the application of the principle are we not fully in agreement.

Walter D. LAMBERT proposes using STOKES' complicated " not to say perverse" formula on a much greater area (several hundred km), and with much greater spacing of the gravity observations, to find a correct geodetic datum. This has been tried out in the U.S.A. and is of course a considerable step forward.

I propose using gravimetric observations spaced much closer, say with an interval of one second of arc ($\sim$ about 30 metres), in a much smaller area to position "bridge piers" (see ref. 1, page 42), where astronomical observations (both in latitude, longitude and azimuth) have been carried out to correct these observations for absolute plumb line deflection (deviation) from the actual potential surface on the Earth through the point of observation to the corresponding position on the surface of the idealized International Spheroid.

In this connection it must be borne in mind that at STOKES' time gravimetric observations were very labour- and time-consuming (major operations), while today gravimetric observations are carried out easily and quickly in a few minutes. Also the technique of numerical differentiation has made considerable progress.

This is only a very short and rough explanation of the principle and its application. Much work is needed to carry it out in practical form. This needs team-work and a lot of discussion by very skilled mathematicians and surveyors.

Someone may wonder why I waste so much of the precious space in this august journal on geodesy.

It is quite right that geodesy does not come within the tasks of most hydrographic offices; generally they get their data in the form of coordinates from the geodetic institutes. The United States Coast and Geodetic Survey is an exception.

But it is the purpose of the International Hydrographic Bureau to obtain uniformity as far as possible in hydrographic documents, so that mariners may use publications issued by other countries.

This uniformity is lacking in the wide field of nautical charts today. Every country is at present basing its surveys on its own globe as there are so many different geodetic datums. The geographic coordinates, which ought to be universal and absolute, are all relative (in relation to the country issuing the charts or the geodetic datum used).

In my opinion a chart can never be made more accurate than the accuracy with which you can determine the position of a single point (latitude φ_0, longitude λ_0 and azimuth α) on the globe from which you define your charts.

The position can be determined on the Geoid to within an uncertainty of the second decimal of a second of arc ($1''$ about 30 m, $0''.01$ about 0.3 m), which is sufficient for large scale charts. But when the position is changed from one globe (Geoid) to the other (International Spheroid), from which the charts are defined without respect to the deflection or deviation of the plumb line between the two globes, one-sided errors of several seconds of arc (hundreds of metres) are introduced.

So far geodesy has not taken into account that we are all situated on the same globe, containing the same masses and revolving with the same speed on the same axis.

As the Danish Geodetic Institute considers it absurd to discuss the problem, I hope the IHB can persuade the IUGG, the USC&GS or the Hydrographic Department of the U.K. to investigate the problem.

IHB Note

The Bureau will be pleased to publish in future editions of the Review any comments of the IUGG, the USC&GS, the Hydrographic Department of the U.K., or any other geodesists on the two articles by Commodore (Ret.) SCHMIDT of Denmark.

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