HARMONIC ANALYSIS OF TIDES THROUGH LINEAR COMBINATIONS OF ORDINATES

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1. — Introduction

The best known methods of harmonic analysis of tides using linear combinations of hourly heights are due to Dr. A. T. DOODSON, D. Sc., F.R.S. The latter has perfected a simple method for the use of the British Navy, and another more accurate and also more involved method which is used at the Tidal Institute of Liverpool. The author has set up the combinations in an almost empirical way. He has divided the analysis procedure into several stages, known as the daily, monthly and annual processes.

By successive filterings, we can thus isolate first the "species", i.e. long-period, diurnal, semi-diurnal or other groups of constituents, and then the constituents themselves.

Long after these methods had been perfected, Mr. ARAGNOL introduced symbolic polynomials [1], making it possible to look for the most desirable combinations systematically. At the end of his article, he shows the possibility of arriving at the isolation of the constituents themselves, without the need for successive filterings.

At the "Colloquy on the use of automatic computations for solving problems on tides", which took place in Paris in November, 1961, Mrs. COULMY, of the French Hydrographic Service, submitted a method of analysis [2] by a series of linear combinations of ordinates. This series of combinations may be replaced by a single overall combination worked out by symbolic polynomials. The author of this method does not apply the entire range of polynomials, stating that it would be tedious to look for hourly height combinations and that it would be preferable successively to combine pairs of ordinates and combinations of these ordinates. It may well be that if the analysis is made with electronic computers, the program will be more complicated by applying the polynomials integrally. However, if classical computers are used for the analysis, the very simple use of stencils, prepared in advance for each constituent, will permit a quick calculation of the harmonic constants for any constituent.

Although Mrs. COULMY's method is a general one, its author has established combinations for computing the harmonic constants of the ten major constituents, deduced from analysis of observations taken over a period of one month, which is most interesting from the surveyor's viewpoint.

TABLE 2-I

		Q_1		O_1		K_1		μ_2		N_2		M_2		L_2		S_2	MS_4		M_4	
p	p.180°	θ	ε	θ	ε	θ	ε	θ	ε	θ	ε	θ	ε	θ	ε	θ	θ	ε	θ	ε
1	180	13	-5,8	13	1,3	12	0,5	6	-12,2	6	- 9,4	6	- 6,1	6	- 2,8	6	3	- 3,0	3	- 6,1
2	360	27	1,8	26	2,5	24	1,0	13	3,6	13	9,7	12	-12,2	12	- 5,7	12	6	- 6,1	6	-12,2
3	540	40	-4,0	39	3,8	36	1,5	19	- 8,6	19	0,4	19	10,7	18	- 8,5	18	9	- 9,1	9	-18,3
4	720	54	3,5	52	5,0	48	2,0	26	7,2	25	- 9,0	25	4,6	24	-11,3	24	12	-12,2	12	-24,4
5	900	67	-2,3	65	6,3	60	2,5	32	- 5,0	32	10,1	31	- 1,5	30	-14,1	30	15	-15,2	16	27,5
6	1080	81	5,3	77	-6,4	72	3,0	39	10,8	38	0,7	37	- 7,6	37	12,6	36	18	-18,3	19	21,4
7	1260	94	-0,5	90	-5,1	84	3,4	45	- 1,4	44	- 8,7	43	-13,7	43	9,7	42	21	-21,3	22	15,3
8	1440	107	-6,3	103	-3,9	96	3,9	51	-13,4	51	10,4	50	9,2	49	6,9	48	24	-24,4	25	9,2
9	1620	121	1,2	116	-2,6	108	4,4	58	2,2	57	1,1	56	3,1	55	4,1	54	27	-27,4	28	3,1
10	1800	134	-4,6	129	-1,4	120	4,9	64	-10,0	63	- 8,3	62	- 3,0	61	1,2	60	31	-30,5	31	- 3,0
11	1980	148	3,0	142	-0,1	132	5,4	71	5,7	70	10,8	68	- 9,1	67	- 1,6	66	34	25,5	34	- 9,1
12	2160	161	-2,8	155	1,2	144	5,9	77	- 6,4	76	1,4	75	13,8	73	- 4,4	72	37	22,4	37	-15,2
13	2340	175	4,8	168	2,4	156	6,4	84	9,3	82	- 7,9	81	7,7	79	- 7,2	78	40	19,4	40	-21,3
14	2520	188	-1,1	181	3,7	168	6,9	90	- 2,9	89	11,1	87	1,6	85	-10,1	84	43	16,3	43	-27,4
15	2700	202	6,5	194	5,0	180	7,4	97	12,9	95	1,8	93	- 4,5	91	-12,9	90	46	13,3	47	24,5
16	2880	215	0,7	207	6,2	191	-7,2	103	0,7	101	- 7,6	99	-10,6	98	13,8	96	49	10,2	50	18,4
17	3060	228	-5,1	219	-6,5	203	-6,7	109	-11,6	108	11,5	106	12,3	104	11,0	102	52	7,1	53	12,3
18	3240	242	2,5	232	-5,2	215	-6,2	116	4,3	114	2,1	112	6,2	110	8,1	108	55	4,1	56	6,2
19	3420	255	-3,3	245	-4,0	227	-5,7	122	- 7,9	120	- 7,2	118	0,1	116	5,3	114	58	1,1	59	0,1
20	3600	269	4,2	258	-2,7	239	-5,2	129	7,9	127	11,8	124	- 6,0	122	2,5	120	61	- 2,0	62	- 6,0
21	3780	282	-1,6	271	-1,4	251	-4,7	135	- 4,3	133	2,5	130	-12,1	128	- 0,4	126	64	- 5,0	65	-12,1
22	3960	296	6,0	284	-0,2	263	-4,2	142	11,5	139	- 6,9	137	10,8	134	- 3,2	132	67	- 8,1	68	-18,2
23	4140	309	-0,2	297	1,1	275	-3,7	148	- 0,7	146	12,2	143	4,7	140	- 6,0	138	70	-11,1	71	-24,3
24	4320	322	-6,6	310	2,3	287	-3,2	154	-12,9	152	2,8	149	11,7	146	- 8,8	144	73	-14,2	75	-30,4
25	4500	336	2,0	323	3,6	299	-2,7	161	2,9	158	- 6,5	155	- 7,5	152	-11,7	150	76	-17,2	78	21,5
26	4680	349	-3,9	336	4,8	311	-2,2	167	- 9,3	165	12,6	161	-13,6	158	-14,5	156	79	-20,3	81	15,4
27	4860	363	3,7	349	6,1	323	-1,7	174	6,5	171	3,2	168	9,3	165	12,2	162	82	-23,3	84	9,3
28	5040	376	-2,1	361	-6,6	335	-1,2	180	- 5,7	177	- 6,2	174	3,2	171	9,3	168	85	-26,4	87	3,2
29	5220	390	5,5	374	-5,3	347	-0,7	187	10,1	184	12,9	180	- 2,9	177	6,5	174	88	-30,4	90	- 2,9

		MS_4		M_4	
p	pπ	θ	ε	θ	ε
31	5580	95	23,5	96	-15,1
32	5760	98	20,4	99	-21,1
33	5940	101	17,4	102	-27,2
34	6120	104	14,3	106	24,6
35	6300	107	11,3	109	18,5
36	6480	110	8,3	112	12,4
37	6660	113	5,2	115	6,3
38	6840	116	2,2	118	0,2
39	7020	119	- 0,9	121	- 5,8
40	7200	122	- 3,9	124	-11,9
41	7380	125	- 7,0	127	-18,0
42	7560	128	-10,0	130	-24,1
43	7740	131	-13,1	134	-30,2
44	7920	134	-16,1	137	21,6
45	8100	137	-19,2	140	15,5
46	8280	140	-22,2	143	9,5
47	8460	143	-25,3	146	3,4
48	8640	146	-28,3	149	- 2,7
49	8820	150	-31,4	152	- 8,8
50	9000	153	24,6	155	-14,9
51	9180	156	21,5	158	-21,0
52	9360	159	18,5	161	-27,1
53	9540	162	15,4	165	24,8
54	9720	165	12,4	168	18,7
55	9900	168	9,3	171	12,6
56	10080	171	6,3	174	6,5
57	10260	174	3,2	177	0,4
58	10440	177	0,2	180	- 5,7
59	10620	180	- 2,9	183	-11,8

We shall now study a somewhat different version from that of Mrs. COULMY, which in practice does not present any particular difficulty.

2. — Theory and application of the method

In the previous issue of this Review, we have shown that it is possible to eliminate a constituent or a group of constituents by a simple difference in the ordinates at instants $t = 0$ and $t = \theta$, provided

$$\theta = n\, 360°/q$$

where q is the hourly phase variation of the constituent. This means that the phase of the constituent is the same at intervals of θ hours. However, it is also possible to eliminate a constituent by adding the ordinates corresponding to phase variations of $(2p + 1)\, 180°$. These conclusions can be summarized as follows :

If, in the expression

$$\theta = p \cdot 180°/q \qquad (2a)$$

p is an odd number, we shall proceed with the elimination by adding the ordinates separated by θ-hour intervals, whereas if p is an even number, the ordinates should be subtracted to eliminate the constituent. We shall thus have the two symbolic binomials :

$$1 + z^{\theta} = 0 \text{ for } p \text{ odd number} \qquad (2b)$$

$$1 - z^{\theta} = 0 \text{ for } p \text{ even number} \qquad (2c)$$

By means of expression $(2a)$, we can set up a table (Table 2-I) for each constituent under consideration, as follows : For each constituent, the value of $180°/q$ is computed; this value is multiplied by the round figure p and the rounded products are entered in the θ columns. The value of θ multiplied by q gives us the approximate value of $p\, 180°$ appearing in the second column of the table. The difference ε between the two values is the phase difference due to the approximation of the selected θ intervals. In Table 2-I, we have underlined those values of θ which are best suited for eliminating the constituents concerned, for which ε is quite small.

Taking into consideration the special θ values of Table 2-I, we have set up Table 2-II in the same way as Table 3-I of our article in the previous issue of the I.H. Review.

The left-hand side of $(2b)$ and $(2c)$ may be developed as follows :

$$1 + z^{\theta} = z^{\theta/2}\ (z^{\theta/2} + z^{-\theta/2}) \qquad (2d)$$

and

$$1 - z^{\theta} = -z^{\theta/2}\ (z^{\theta/2} - z^{-\theta/2}) \qquad (2e)$$

For $z = e^{iq}$, we thus have

$$1 + z^{\theta} = e^{iq\theta/2}\ (e^{iq\theta/2} + e^{-iq\theta/2}) \qquad (2f)$$

$$1 - z^{\theta} = -e^{iq\theta/2}\ (e^{iq\theta/2} - e^{-iq\theta/2}) \qquad (2g)$$

but since

$$e^{iq\theta/2} + e^{-iq\theta/2} = 2 \cos q\theta/2 \qquad (2h)$$

$$e^{iq\theta/2} - e^{-iq\theta/2} = 2\, i \sin q\theta/2 \qquad (2i)$$

TABLE 2-II

N°	Polyn.	k	ω	E	R(*)
(1)	$1 + z^3$	2 cos 1,5 q	1,5 q	Q.D.	
(2)	$1 - z^3$	2 cos (1,5 q + 90°)	1,5 q + 90°		Q.D.
(3)	$1 + z^6$	2 cos 3 q	3 q	S.D.	D.
(4)	$1 - z^6$	2 cos (3 q + 90°)	3 q + 90°	Q.D.	S.D.
(5)	$1 + z^{12}$	2 cos 6 q	6 q	D.	S.D.
(6)	$1 - z^{12}$	2 cos (6 q + 90°)	6 q + 90°	S.D.	D.
(7)	$1 + z^{13}$	2 cos 6,5 q	6,5 q	O_1, Q_1	
(8)	$1 - z^{13}$	2 cos (6,5 q + 90°)	6,5 q + 90°	N_2, μ_2	
(9)	$1 + z^{94}$	2 cos 47 q	47 q	Q_1	
(10)	$1 + z^{108}$	2 cos 54 q	54 q	K_1	
(11)	$1 + z^{142}$	2 cos 71 q	71 q	O_1	
(12)	$1 - z^{144}$	2 cos (72 q + 90°)	72 q + 90°	K_1	
(13)	$1 + z^{284}$	2 cos 142 q	142 q	O_1	
(14)	$1 + z^{309}$	2 cos 154,5 q	154,5 q	Q_1	
(15)	$1 - z^{31}$	2 cos (15,5 q + 90°)	15,5 q + 90°	M_2	L_2
(16)	$1 + z^{57}$	2 cos 28,5 q	28,5 q	N_2	
(17)	$1 - z^{61}$	2 cos (30,5 q + 90°)	30,5 q + 90°	L_2	
(18)	$1 - z^{76}$	2 cos (38 q + 90°)	38 q + 90°	N_2	
(19)	$1 + z^{90}$	2 cos 45 q	45 q	S_2	
(20)	$1 - z^{90}$	2 cos (45 q + 90°)	45 q + 90°	μ_2	S_2
(21)	$1 - z^{103}$	2 cos (51,5 q + 90°)	51,5 q + 90°	μ_2	L_2
(22)	$1 + z^{118}$	2 cos 59 q	59 q	M_2	
(23)	$1 + z^{128}$	2 cos 64 q	64 q	L_2	
(24)	$1 + z^{148}$	2 cos 74 q	74 q	μ_2	
(25)	$1 - z^{180}$	2 cos (90 q + 90°)	90 q + 90°	S_2	
(26)	$1 - z^{190}$	2 cos (95 q + 90°)	95 q + 90°	N_2	
(27)	$1 - z^{177}$	2 cos (88,5 q + 90°)	88,5 q + 90°	MS_4	M_4
(28)	$1 + z^{177}$	2 cos 88,5 q	88,5 q	M_4	MS_4
(29)	$1 - z^{77}$	2 cos (38,5 q + 90°)	38,5 q + 90°	SN_4	
(30)	$1 + z^{163}$	2 cos 81,5 q	81,5 q	MN_4	

and

$$i e^{iq\theta/2} = e^{i(q\theta/2 + 90°)} \quad (2j)$$

it follows from (2*f*) and (2*h*)

$$1 + z^{\theta} = e^{iq\theta/2} \cdot 2 \cos q\theta/2 \quad (2k)$$

and from (2*g*), (2*i*) and (2*j*)

$$1 - z^{\theta} = - e^{i(q\theta/2 + 90°)} \cdot 2 \sin q\theta/2$$

(*) Reinforced constituents.

or

$$1 - z^\theta = e^{i(q\theta/2+90°)} \cdot 2 \cos (q\theta/2 + 90°) \qquad (2l)$$

If a similar expansion is made for $z = e^{-iq}$, we shall see that expressions (2*k*) and (2*l*) may be written in a general form : —

$$1 + z^\theta = e^{\pm iq\theta/2} \cdot 2 \cos q\theta/2 \qquad (2m)$$

$$1 - z^\theta = e^{\pm i(q\theta/2\ +90°)} \cdot 2 \cos (q\theta + 90°) \qquad (2n)$$

Hence we have, still in Table 2-II

$$k = 2 \cos q\theta/2 \text{ and } \omega = q\theta/2 \qquad (2o)$$

for combinations expressed by (2*b*), and

$$k = 2 \cos (q\theta/2 + 90°) \text{ and } \omega = q\theta/2 + 90° \qquad (2p)$$

for combinations expressed by (2*c*).

In order to isolate the various constituents, a suitable product of binomials $1 \pm z^\theta$ of Table 2-II should be obtained, so as to have a complete polynomial for isolating one of the constituents.

TABLE 2-III

Combinaisons	J	η	t'	qt'	$\Delta = \eta + \frac{1}{2}$ qt'	$10^6/a$	$-10^6/b$
Q_1 (1) (8) $(6)^2$ (10) (13)	56,83	14°,12	7	93°,79	61°,02	12871	-12052
O_1 (1) (8) $(6)^2$ (12) (14)	- 57,03	196.95	6	83,66	238,78	-11767	13146
K_1 [(1) (8) $(6)^2$ (6) (9) (11)] × 2	223,12	5,91	6	90,25	51,04	3176	- 3163
μ_2 (1) (7) $(4)^2$ (5) (17) (19) (22) (26)	- 169,91	48,07	3	83,90	90,02	- 3955	4400
N_2 (1) (7) $(4)^2$ (5) (19) (22) (23) (24)	97,71	71,20	3	85,32	113,86	6959	- 7550
M_2 (1) (7) $(4)^2$ (5) (19) (21) (23) (26)	95,65	65,12	3	86,95	108,60	7202	- 7598
L_2 (1) (7) $(4)^2$ (5) (15) (18) (21) (22) (25)	169,23	170,81	3	88,59	215,10	4128	- 4230
S_2 (1) (7) $(4)^2$ (5) (16) (18) (20) (22) (23)	- 185,46	75,00	3	90,00	120,00	- 3813	3813
M_4 $(5)^2$ $(3)^4$ $(2)^2$ $(27)^2$	953,69	305,51	11	277,65	84,34	696	- 815
MS_4 $(5)^2$ $(3)^4$ $(2)^2$ $(28)^2$	1005,95	332,76	11	288,83	117,17	611	- 854

Note : a = 2J cos 1/2 qt' ; b = - 2J sin 1/2 qt'

Table 2-III shows the products chosen by Mrs. COULMY. A brief explanation in this regard is in order. In the group of diurnal constituents, we see that polynomials (1) and (8) are common to all constituents. Actually, Table 2-II shows that polynomial (1) eliminates the contribution of the fourth-diurnal constituents, that polynomial (8) specifically eliminates constituents N_2 and μ_2, and that (6) eliminates M_2, S_2 and L_2; it also approximately doubles the contribution of the diurnal constituents. Polynomial (6) is therefore used twice, as indicated by $(6)^2$. The formation of products of binomials common to all constituents with the same subscript can be similarly explained.

In order to isolate a constituent, the product of the binomials should be completed with the binomials isolating each constituent of the group, except the constituent to be isolated.

For groups of fourth-diurnal constituents, the author of the method has used higher powers further to improve the isolation of the constituents.

The values of J and η as seen in Table 2-III are obtained by the following expressions :

$$J = k_1 \cdot k_2 \cdot k_3 \cdot \ldots\ldots\ldots$$

$$\eta = \omega_1 + \omega_2 + \omega_3 + \ldots\ldots\ldots$$

by introducing speed q of the constituent to be isolated into the expressions of k and ω of Table 2-II, corresponding to the polynomials of the proposed combination. Column qt' gives us the phase variation of the constituent for interval t', which is equal to the rounded number of hours in which $qt' \approx (2m + 1)\ \pi/2$. In order to isolate the constituents, we suggest setting

TABLE 2-IV

Q_1						O_1					
D_t	t		D_t	t		D_t	t		D_t	t	
	X	Y		X	Y		X	Y		X	Y
1	0	7	-1	284	291	1	0	6	1	309	315
1	3	10	-1	287	294	1	3	9	1	312	318
-2	12	19	2	296	303	-2	12	18	-2	321	327
-1	13	20	1	297	304	-1	13	19	-1	322	328
-2	15	22	2	299	306	-2	15	21	-2	324	330
-1	16	23	1	300	307	-1	16	22	-1	325	331
1	24	31	-1	308	315	1	24	30	1	333	339
2	25	32	-2	309	316	2	25	31	2	334	340
1	27	34	-1	311	318	1	27	33	1	336	342
2	28	35	-2	312	319	2	28	34	2	337	343
-1	37	44	1	321	328	-1	37	42	-1	346	352
-1	40	47	1	324	331	-1	40	46	-1	349	355
1	108	115	-1	392	399	-1	144	150	-1	453	459
1	111	118	-1	395	402	-1	147	153	-1	456	462
-2	120	127	2	404	411	2	156	162	2	465	471
-1	121	128	1	405	412	1	157	163	1	466	472
-2	123	130	2	407	414	2	159	165	2	468	474
-1	124	131	1	408	415	1	160	166	1	469	475
1	132	139	-1	416	423	-1	168	174	-1	477	483
2	133	140	-2	417	424	-2	169	175	-2	478	484
1	135	142	-1	419	426	-1	171	177	-1	480	486
2	136	143	-2	420	427	-2	172	178	-2	481	487
-1	145	152	1	429	436	1	181	187	1	490	496
-1	148	155	1	432	439	1	184	190	1	493	499

K_1											
D_t	t		D_t	t		D_t	t		D_t	t	
	X	Y		X	Y		X	Y		X	Y
1	0	6	1	145	151	1	359	365	1	504	510
1	3	9	1	146	152	1	362	368	1	505	511
-3	12	18	-3	154	160	-3	371	377	-3	513	519
-1	13	19	-1	155	161	-1	372	378	-1	514	520
-3	15	21	-3	157	163	-3	374	380	-3	516	522
-1	16	22	-1	158	164	-1	375	381	-1	517	523
3	24	30	3	166	172	3	383	389	3	525	531
3	25	31	3	167	173	3	384	390	3	526	532
3	27	33	3	169	175	3	386	392	3	528	534
3	28	34	3	170	176	3	387	393	3	529	535
-1	36	42	-1	178	184	-1	395	401	-1	537	543
-3	37	43	-3	179	185	-3	396	402	-3	538	544
-1	39	45	-1	181	187	-1	398	404	-1	540	546
-3	40	46	-3	182	188	-3	399	405	-3	541	547
1	49	55	1	191	197	1	408	414	1	550	556
1	52	58	1	194	200	1	411	417	1	553	559
1	94	100	1	236	242	1	453	459	1	595	601
1	97	103	1	239	245	1	456	462	1	598	604
-3	106	112	-3	248	254	-3	465	471	-3	607	613
-1	107	113	-1	249	255	-1	466	472	-1	608	614
-3	109	115	-3	251	257	-3	468	474	-3	610	616
-1	110	116	-1	252	258	-1	469	475	-1	611	617
3	118	124	3	260	266	3	477	483	3	619	625
3	119	125	3	261	267	3	478	484	3	620	626
3	121	127	3	263	269	3	480	486	3	622	628
3	122	128	3	264	270	3	481	487	3	623	629
-1	130	136	-1	272	278	-1	489	495	-1	631	637
-3	131	137	-3	273	279	-3	490	496	-3	632	638
-1	133	139	-1	275	281	-1	492	498	-1	634	640
-3	134	140	-3	276	282	-3	493	499	-3	635	641
1	142	148	1	285	291	1	501	507	1	644	650
1	143	149	1	288	294	1	502	508	1	647	653

up two exactly equal combinations, one starting with the first ordinate of the series corresponding to $t = 0$, which we shall call X, and the other which we shall name Y, the first ordinate of which corresponds to instant t'.

In order to draw stencils, we must find those particular polynomials that are the results of the product of the various binomials indicated in parentheses by the figures of the second column of Table 2-III. These figures correspond to the binomials in Table 2-II. Although this operation is lengthy, it will only have to be carried out once so as to set up tables similar to Table 2-IV drawn for the diurnal constituents. In that table, columns D_t represent the coefficients of z in the final polynomial and the columns of t for X are the powers of z in this polynomial. As to the values of t corresponding to Y, they are equal to $t + t'$, t' being given by Table 2-III.

In the columns of Table 2-IV corresponding to constituent K_1, it shall be noticed that the same combination reoccurs from $t = 359$ hours, which is almost exactly the time needed for the constituent to accomplish 15 cycles $(p/2)$. Since we have 696 observation hours available for a 29-day observation period, it will be possible to have the contribution of K_1 twice during this period, giving us a double value of this contribution.

Figure 2A shows the stencil which has been set up with figures giving combination X corresponding to constituent Q_1. In practice, it is sufficient to add up the products of the visible ordinates by the multipliers appearing above these ordinates.

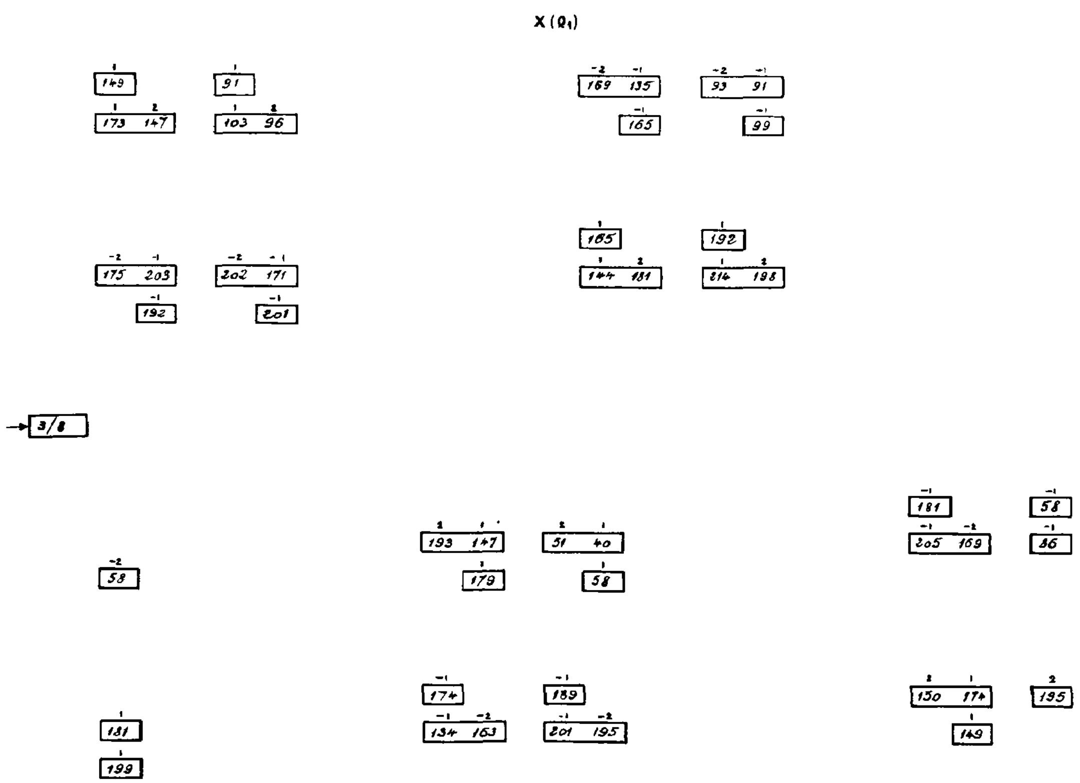

Fig. 2 A

Once the numerical values of X and Y for each constituent have been found, their phases $-r_o$ at zero hour of the first day and their respective amplitudes R should be computed. As both X and Y are obtained by means of combinations which are equal but shifted t' hours, we may write

$$X = JR \cos (\eta - r_o)$$

and

$$Y = JR \cos (\eta + qt' - r_o)$$

$-r_o$ being the phase of the constituent at zero hour of the first observation day. If we now put down

$$X + Y = A$$

and

$$X - Y = B$$

we shall have

$$A = J \cos R [\cos (\eta - r_o) + \cos (\eta + ql' - r_o)]$$

and

$$B = J \cos R [\cos (\eta - r_o) - \cos (\eta + qt' - r_o)]$$

Hence

$$A = 2JR \cos 1/2\ qt' \cos \left(\eta + \frac{1}{2}\ qt' - r_o\right)$$

and

$$B = 2JR \sin 1/2\ qt' \sin \left(\eta + \frac{1}{2}\ qt' - r_o\right)$$

If we put down

$$\left.\begin{array}{l} \eta + qt'/2 = \Delta \\ 2J \cos 1/2\ qt' = a \\ 2J \sin 1/2\ qt' = b \end{array}\right\} \quad (2p)$$

We shall have

$$\left.\begin{array}{l} A = a R \cos (\Delta - r_o) \\ \text{and} \\ B = b R \sin (\Delta - r_o) \end{array}\right\} \quad (2q)$$

However, we note that

$$-r_o = V_o + u - g$$

Hence

$$\Delta - r_o = V_o + u - g + \Delta \quad (2r)$$

If we put

$$\Delta - r_o = -r \quad (2s)$$

we obtain from (2r)

$$g = V_o + u + r + \Delta \quad (2t)$$

Expressions (2q) and (2s) will give

$$\left.\begin{array}{l} A = \ \ a R \cos r \\ B = -b R \sin r \end{array}\right\} \quad (2u)$$

Consequently

$$\left.\begin{array}{l} R \cos r = \ \ A/a \\ R \sin r = -B/b \end{array}\right\} \quad (2v)$$

The values of Δ, $10^6/a$ and $-10^6/b$, computed by means of expressions (2p), are given in Table 2-III.

3. — Various considerations on the practical use of the method

If a classical computer is used to carry out the operations some simplifications should be made. Firstly, it should be considered that when the effect of shallow water is negligible, constituent μ_2, which merges into $2SM_2$, will have an amplitude of similar size to that of $2N_2$. Since the hourly phase shift of these two constituents is equal to 0°.0728, they cannot be directly isolated by means of the 30-day observation analysis. Thus, in order to obtain μ_2, it will be necessary to correct the values of R cos *r* and

FORM 3-1

Date centrale : 3/8/1947 Port : Aratu (Brésil)

	S_0	Q_1	O_1	K_1	N_2	M_2	S_2	M_4	MS_4
X		- 153	33	1065	-1608	-7459	-2487	- 20	- 190
Y		13	416	207	-1235	2039	-2029	257	- 11
A = X + Y		- 140	449	1272	-2843	-5420	-4516	- 237	- 201
B = X - Y		- 166	- 383	858	- 373	-9498	- 458	277	- 179
$10^6/a$		12871	-11767	3176	6959	7202	6604	-6454	4656
$-10^6/b$		-12052	13145	-3163	-7567	-7596	-6604	-7381	6507
R cos r = A/a		-1,802	-5,283	4,040	-19,784	-39,035	-29,824	-1,530	-0,941
R sin r = B/b		2,001	-5,035	-2,714	2,816	72,166	3,025	2,045	-1,165
R^2		7,27	53,26	23,69	391,55	6731,66	898,62	6,52	2,24
tgr		1,110	0,954	-0,672	-0,143	-1,849	-0,101	-1,337	1,238

	S_0	Q_1	O_1	K_1	N_2	M_2	S_2	M_4	MS_4	
V 1/1/1947			143°,7	9°,8	217°,2	153°,5				K_1 : $(2V + u)_{K_1}$ = 74,6
ΔV (mois)			22,2	209,0	341,3	231,1				S_2, MS_4 : $(V + u)_{K_1}$ = 213,8
ΔV' (jour)			309,3	2,0	285,1	311,2				N_2 : $3V_{N_2} - 2V_{M_2}$ = 40,2
Somme		262,8	115,2	220,8	123,6	335,8		311,6	335,8	f_{K_2} = 1,170
Δ'		46,9	132,5	38,2	165,8	16,8	330°,0	227,3	158,7	f_{K_1} = 1,069
u		8,1	8,1	- 7,0	- 1,8	-1,8	0,0	-3,6	-1,8	
- w		5,9	-	-15,2	5,9		-17,4	...	-17,4	K_1 : wf_{K_1} = -16°,3
r		132,0	223.7	326,1	171,9	118,4	174,3	126,8	231,1	: Wf_{K_1} = 0,134
g		95,7	119,5	202,9	105,4	109,2	126,9	302,1	346,4	w = -15°,3
f		1,111	1,111	1,069	0,981	0,981	...	0,962	0,981	W = 0,125
1 + W		1,146	...	1,125	1,146	...	0,852	...	0,852	S_2, MS_4 : w/f_{K_2} = -14°,9
f (1 + W)		1,273	...	1,202	1,124	...	...	0,962	0,835	W/f_{K_2} = -0,127
R		2,69	7,30	4,87	19,79	82,05	29,98	2,55	1,50	w = -17,4
H		2,1	6,6	4,1	17,6	83,6	35,2	2,7	1,8	W = -0,148

$V_{Q_1} = V_{N_2} - V_{K_1}$ $V_{M_4} = 2V_{M_2}$ $V_{MS_4} = V_{M_2}$

$u_{Q_1} = u_{O_1}$; $f_{Q_1} = f_{O_1}$ $u_{M_4} = 2u_{M_2}$; $f_{M_4} = (f_{M_2})^2$ $u_{MS_4} = u_{M_2}$; $f_{MS_4} = f_{M_2}$

N_2 : w = 5,9

1 + W = 1,146

R sin r corresponding to this constituent, taking into account the effect of $2N_2$, computed according to M_2 and N_2, as indicated by DOODSON [3]. Even though the action of shallow water may be sufficient for $2SM_2$ to be more important than μ_2, it should not be neglected to correct the disturbance due to $2N_2$. Under these circumstances, we feel that in an abridged method, computation of the harmonic constants of μ_2, the amplitude of which is usually slight, may be deleted.

It will suffice to keep those combinations which eliminate the effect of group (μ_2, $2N_2$) among the combinations for isolating the other constituents.

In fact, the harmonic constants of μ_2, computed without taking $2N_2$ into account, may be completely wrong.

Constituent L_2 may also be ignored, provided the perturbations it causes on other constituents are eliminated by the combinations.

Anyway, L_2 is also a small constituent, the constants of which are usually ignored in simplified predictions.

There now remains to be studied a simplification for constituent S_2, which is perfectly possible. Actually, we see in Table 2-III that the influence of N_2 on S_2 is eliminated by means of binomials (16) and (18). The combination expressed by binomial (18) may be abandoned, and whatever precision is lost is largely made up by the resulting simplification. Indeed, the complexity and extension of the stencils increase according to the number of constituents to be isolated in each species. Thus even with the envisaged simplifications, the stencils for isolating the semi-diurnal constituents are more complicated than those for diurnal constituents, whose number of constituents under consideration is very small. In fact, even if μ_2 and L_2 are not computed, their effect on M_2, S_2 and N_2 are eliminated.

To get an idea of the effect of all these constituents on S_2, we have computed them leaving out binomial (18) and the results (multiplied by 10^4) appear on the following table :

S_2	Q_1	O_1	K_1	μ_2	N_2	M_2	L_2	M_4	MS_4
1	30	4	1	163	54	4	7	0	0

This table shows us that the effect of μ_2 is preponderant, but if $S_2 = \mu_2 = 100$ cm (which would be really exceptional), we have a perturbation of S_2 equal to 1.6 cm. For normal values of μ_2, elimination is therefore perfectly acceptable. The analysis of the Port of Aratù (Brazil) which was carried out by this method, has confirmed the theoretic conclusions, since the constants which were found for S_2 are practically the same as those obtained by the method of the Tidal Institute. In form 3-1, the values of $10^6/a$ and $-10^6/b$, which correspond to this modification, appear in the S_2 column.

In the case of fourth-diurnal constituents, the polynomial resulting from the combinations of Table 2-III is not practical for manual computing.

We therefore suggest the following combinations :

For M_4 : $(3)^3$ (5) (7) (2) (27) (29) (30)

For MS_4 : $(3)^3$ (5) (7) (2) (28) (29) (30)

(29) and (30) being the combinations which enable us to rid M_4 and MS_4 of the contributions of MN_4 and SN_4 (see Table 2-II). The values of $10^6/a$ and $-10^6/b$ are those appearing in form 3-1

The residual perturbations (multiplied by 10^4) of the other constituents considered are :

	Q_1	O_1	K_1	μ_2	N_2	M_2	L_2	M_4	MS_4	MN_4	SN_4
M_4	10	3	2	24	0	1	0	—	22	11	39
MS_4	30	2	2	13	1	3	0	0	—	12	35

All these values are perfectly acceptable; in our opinion, isolation should not be carried too far.

What the values of Δ' mean within the proposed form still remains to be explained.

In order to compute the values of w and W for constituents S_2, K_1 and N_2, the arguments which w and W are functions of should correspond to the central date of the analysis, the latter being the same for all the constituents. Thus the stencils should have an arrow (fig. 2A) pointing towards that date. Moreover, the phases $-r$ of the constituents are rendered by the combinations for zero hours on zero day. This corresponds to :

9	days	before	the	central	day	for	Q_1
10	"	"	"	"	"	"	O_1
13	"	"	"	"	"	"	K_1
11	"	"	"	"	"	"	N_2
11	"	"	"	"	"	"	M_2
9	"	"	"	"	"	"	M_4
9	"	"	"	"	"	"	MS_4

If ρ is the daily speed of any constituent whatever and d is the number of days comprised between the initial date and the central date, the astronomical arguments computed for that date shall be decreased by ρd, which can be subtracted from Δ to give Δ'. For S_2, $\rho\, d = 0$, since $\rho = 0$.

We are convinced that without the assistance of electronic computers, the use of stencils is much more economical than successive combinations carried out binomial by binomial. We are in a position to furnish interested parties with tables for constructing stencils for M_2, S_2, N_2, M_4 and MS_4. For constituents S_2, M_4 and MS_4, these tables have been constructed in view of the combinations which we have set up.

Finally, we may note that Mrs. COULMY has prepared a very interesting schedule for curve smoothing by means of electronic computers. Although it can also be applied with conventional computers, in practice it would be too time consuming.

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