OPTIMISATION PROCESSES IN TIDAL ANALYSIS

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Introduction

An analysis of tidal observations is normally carried out, not as an end in itself, but in order to obtain data which can be used to prepare future tidal predictions. The best form of analysis is that which yields the constants producing the best future predictions. Since the future is not known, this condition cannot be used as the basis of an analysis method; the condition which can be used, however, is that the best form of analysis yields those constants which can hindcast the year of analysis most accurately. It is this concept of a " best fit " between the tidal observations and the predictions obtained from the analysis results which deserves examination.

The examples considered in this paper will all refer to the analysis of hourly heights of tide; the principles concerned would, however, apply equally well to the analysis of other tidal characteristics.

Least squares optimisation

If it can be assumed that, for a certain set of constants, the set $h_o - h_p$ (h_o being an observed height, and h_p a predicted value) forms a Normal Distribution with time-independent variance, then the least squares method will yield the most probable set of constants for a given sample. This is simply because the probability of the occurrence of a particular value of $h_o - h_p$ is P, where

$$\mathrm{P} \propto \exp\left\{\frac{-(h_o - h_p)^2}{2\,\mathrm{S}^2}\right\} \qquad (1)$$

(where S^2 is the variance of the Normal Distribution). Similarly the probability of the occurrence of a particular set of $h_o - h_p$ is P′, where

$$\mathrm{P}' = \pi(\mathrm{P}) \propto \exp\left\{-\frac{1}{2\,\mathrm{S}^2}\sum_{\mathrm{Set}}(h_o - h_p)^2\right\} \qquad (2)$$

The most probable set of constants is that for which P′ is largest, and is therefore that for which

$$\sum_{\mathrm{Set}}(h_o - h_p)^2 = \text{minimum} \qquad (3)$$

The least-squares method (or some approximation to it) is invariably used in tidal analysis (references 1-6) but not much study seems to have been made of whether it is, in fact, entirely suitable.

Some illustrative examples

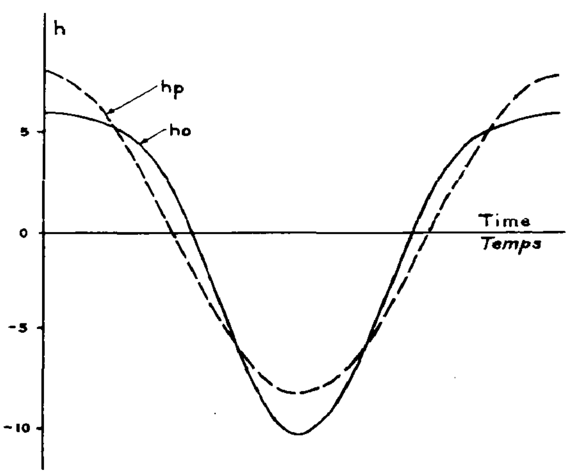

FIG. 1. — Effect on predictions of an inadequate theoretical model (using least-squares analysis).

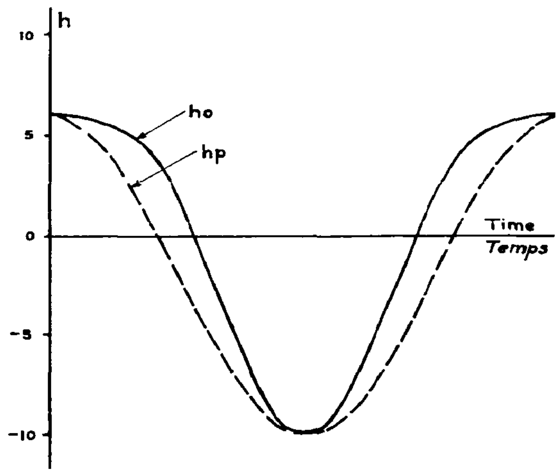

FIG. 2. — An alternative set of constants, for the same set of observations.

It is useful to consider a few simple and rather artificial examples, to see the effects of markedly non-Normal Distributions, and to consider in such cases what is really meant by " best fit ".

Consider a harbour where the tidal observations are (for a certain set of constants) usually perfectly represented by some theoretical function, but where on odd occasions meteorological conditions cause the sea-level to be raised by about two feet. From the point of view of tidal predictions, the best set of constants to use (assuming that the weather is unpredictable) is the set which provides perfect predictions for most of the time. A least-squares analysis of the whole of the observational data (including some of these surges) would not yield this set of constants. On the other hand, if the variations of wind-speed, wind-direction, barometric pressure and any other relevant meteorological parameters all had time-independent Normal Distributions, and if in each case the change in sea-level was proportional to the meteorological effect, then the variation of sea-level about the theoretical function would also be a time-independent Normal Distribution, and a least-squares analysis would be very suitable.

Another source of error can be inadequacy of the theoretical model to be used for the predictions, and in terms of which the analysis is to be carried out. If there is a port where the observations fit the expression :

$$h_o = 8 \cos \sigma t + 2 \cos 2 \sigma t \qquad (4)$$

but where the analysis is carried out in terms of the theoretical function

$$h_p = H_0 + H_1 \cos(-g_1 + \sigma t) \qquad (5)$$

and where a long sequence of observations is analysed, then figure 1 shows h_o (continuous line) and the h_p obtained from a least-squares analysis (broken line). Figure 2 also shows h_o, together with another h_p curve (broken line) which also satisfies equation 5, but which uses a different set of constants. The variance of h_o about h_p in figure 2 is four times what it is in figure 1; the profile of h_p shown in figure 2 could, however, be the more useful of the two for many purposes. Figure 3 shows the frequency distribution of the set $h_o - h_p$ in the case where h_p is deduced by least squares analysis : the curve does not resemble a Normal Distribution.

A third source of inaccuracy can be errors in the input data. A case might be considered where the observations fit the expression

$$h_o = 10 \cos \sigma t \quad (6)$$

and where the analysis is carried out in terms of the theoretical function

$$h_p = H_0 + H_1 \cos (-g_1 + \sigma t) \quad (7)$$

an exact number of periods being analysed. If the analysis is being carried out on a computer, and if the first of the observations is punched as 1010 instead of simply 10, and if this fault is missed by any checking process,

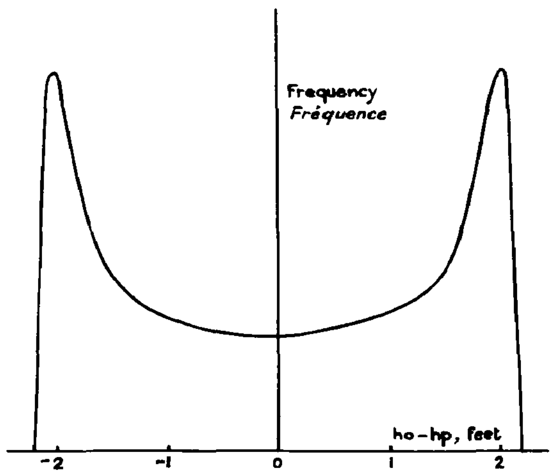

FIG. 3. — Frequency distribution of $h_o - h_p$, using data from figure 1.

then the h_p derived from the analysis of 360 hourly heights is given by

$$h_p = 2.78 + 15.55 \cos \sigma t \quad (8)$$

Figure 4 shows plotted the curves h_o (broken line) and h_p (continuous line).

The three examples given above are exaggerated, but they nevertheless show that if the difference between the set of observations and the theoretical function (for any set of constants) is sufficiently unlike the Normal Distribution, then the use of least squares optimisation is less than ideal. The examples also display three of the major sources of differences between h_o and h_p : meteorological effects, inadequacy of theoretical model, and data errors.

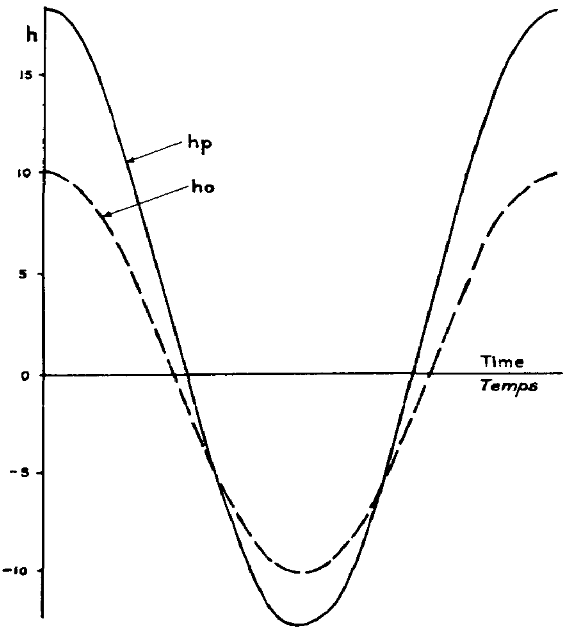

FIG. 4. — Effect on predictions of occasional severe data errors.

Sources of inaccuracy

It is necessary to consider whether, in practical cases, these sources of error will cause the set $h_o - h_p$ to be approximately a Normal Distribution, whose shape is independent of time.

It is felt that meteorological effects do not satisfy this; it seems quite possible that, for some ports at least, all the meteorological effects of a year, when examined together, might form a Normal Distribution, but the effects will not be randomly distributed in time. Normally there will be several months of calm weather during the late spring and the summer, and a number of short periods of storm during the autumn and winter (this argument applies to European ports, but similar arguments would apply to other areas). It is felt that since these effects are grouped in this way, they may affect the set of constants determined by the analysis. There is also the fact that for shallow-water ports, surges and tides may interact, so that the largest surges generally occur at about the same stage of the tide : this is another way in which the effects are not randomly distributed in time. The shorter the period of an analysis, the greater the effect on the final set of constants of any meteorological abnormalities.

As far as the question of the theoretical model's inadequacy is concerned, it is clear that the effects will vary considerably from port to port. If an analysis in terms of sixty constituents is carried out for a deep-water port having a small tidal range, then this will not be a source of trouble at all. When observations for a shallow-water port are being analysed, however, particularly if the tidal range is large, then even with sixty constituents in the tidal representation, the theoretical model will be found to be barely adequate. However, the very difficulties of this problem reduce its importance; the theoretical model cannot be easily improved since any improvement would mean introducing a large number of small amplitude constituents, all having different periods. This means that the errors are due to many small sources, and are well distributed in time, so that they will fit a Normal Distribution reasonably well, and will therefore not invalidate the use of a least-squares analysis.

The third source of error mentioned (data preparation errors) is very serious. Even if the data is checked for smoothness by a computer program, trouble can still occur; when the errors discovered by the smoothness program are being corrected, fresh errors can easily be introduced by mis-punching, and if data have to be repeatedly read into a computer for checking, then a great deal of computer time can be wasted, and this is particularly important on expensive high-speed computers. If the smoothness tests of the checking program are so sensitive that they will notice (for example) that the readings taken from a particular group of tide-gauge charts are all wrong by one hour, then it will be pointing out non-existent errors whenever there is the slightest meteorological disturbance. Even if all errors are removed by the most careful and extensive checking, a pack of data cards can be mishandled so that their order is altered, or a paper tape can be slightly torn so that a digit or two may be misread. There is no reason at all to expect any errors to form a time-independent Normal Distribution.

Tilbury and Portsmouth

It is of interest to discuss at this stage the form of the residuals, $h_0 - h_p$, after a least squares analysis has been carried out. Consider firstly

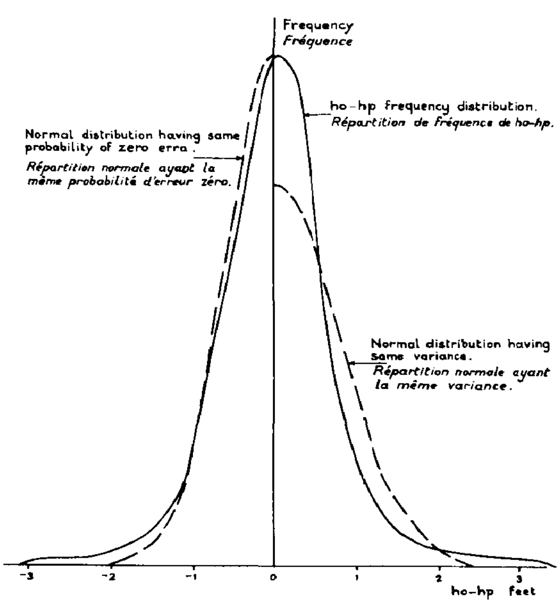

FIG. 5. — Tilbury $h_0 - h_p$ frequency distribution.

the ideal case where the observations, h_o, can (for a certain set of constants) equal a theoretical distribution plus a set of residuals which form a Normal Distribution. For a large sample, as is generally encountered in tidal analysis, the residuals obtained in this ideal case after a least squares analysis would very nearly form a Normal Distribution. The deviation from a Normal Distribution of the set of residuals in any actual analysis will therefore give some indication of how far the actual case deviates from the ideal case.

The first case examined is a least squares analysis of a year's observations at Tilbury; this port was chosen because it is a shallow-water port, it experiences a fair number of appreciable surges, and its tide gauge is extremely well-maintained so that there are no significant instrumental errors. The analysis was carried out in terms of the sixty constituents (plus z_0) listed in reference 1. The frequency distribution of the set $h_o - h_p$ is shown in figure 5 as a continuous line; the broken line to the left of the origin shows the Normal Distribution curve having the same probability of zero error, and the broken line to the right of the origin shows the Normal Distribution curve having the same variance as the $h_o - h_p$ set. The $h_o - h_p$ frequency distribution fits a Normal Distribution reasonably well for errors of up to about 1.2 feet, but the frequency of errors larger than this is much higher than is the case for the fitted Normal Distribution. Still considering the Normal Distribution drawn to the left of the origin, the following set of probabilities can be deduced.

	Normal Distribution	$h_o - h_p$ set
$1.0 > \lvert h_o - h_p \rvert$	90.7 %	83.5 %
$1.5 > \lvert h_o - h_p \rvert \geqslant 1.0$	8.1 %	9.6 %
$2.0 > \lvert h_o - h_p \rvert \geqslant 1.5$	1.2 %	3.5 %
$2.5 > \lvert h_o - h_p \rvert \geqslant 2.0$	0.0002 %	1.6 %
$\lvert h_o - h_p \rvert \geqslant 2.5$	0.0000003 %	1.8 %

It seemed likely that these large errors were due to meteorological effects; inadequacy of the theoretical model could produce a large number of small errors, but would hardly produce many large ones, and the data had been carefully checked so there would be few errors from this source. Meteorological disturbances would generally be of a few hours' or days' duration; values of $h_o - h_p$ dependent upon them would tend to be congregated together more than would be the case if the effects were randomly distributed in time. Of the 8 496 members of the set $h_o - h_p$, 486 had modulus greater than 1.5 feet, and these were grouped as follows :

41	occurred singly	(458)
48	" in groups of two	(26)
102	" " " " three	(2)
56	" " " " four	(0)
55	" " " " five	(0)
36	" " " " six	(0)

7	"	"	"	"	seven	....	(0)
24	"	"	"	"	eight	....	(0)
9	"	"	"	"	nine	....	(0)
30	"	"	"	"	ten	....	(0)
78	"	"	"	"	more than ten	....	(0)

The figures in brackets show the average way in which 486 values would be grouped if they were distributed randomly in time.

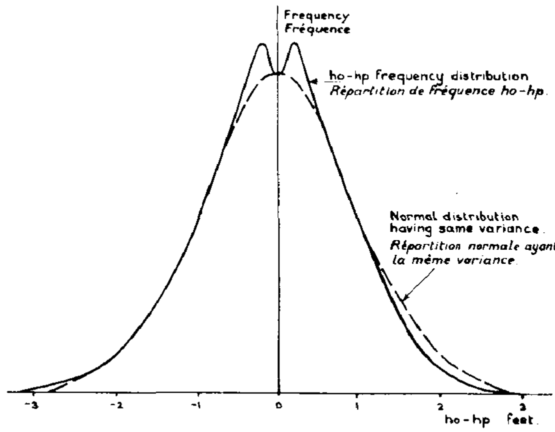

FIG. 6. — Portsmouth $h_o - h_p$ frequency distribution.

The second case examined was Portsmouth, where a least squares analysis was carried out, using a year's observations. This also was a shallow-water port, and in order to accentuate any effects due to inadequacy of the theoretical model, the analysis was only carried out in terms of z_0 and 22 constituents (S_a, S_{sa}, M_m, MS_f, M_f, Q_1, O_1, M_1, P_1, K_1, J_1, OO_1, $2N_2$, μ_2, N_2, ν_2, M_2, L_2, T_2, S_2, K_2, $2SM_2$). The continuous line in figure 6 shows the frequency distribution of $h_o - h_p$; the figure also contains (as a broken line) the form of the Normal Distribution having the same variance. The variance of these curves is nearly four times that of the left hand Normal Distribution in figure 5. To a slight extent, the same phenomenon exists with Portsmouth as with Tilbury, that there are more large errors than the Normal Distribution would suggest.

	Normal Distribution	$h_o - h_p$ set
$2.0 > \lvert h_o - h_p \rvert$	96.3 %	96.4 %
$3.0 > \lvert h_o - h_p \rvert \geqslant 2.0$	3.5 %	3.2 %
$4.0 > \lvert h_o - h_p \rvert \geqslant 3.0$	0.17 %	0.27 %
$\lvert h_o - h_p \rvert \geqslant 4.0$	0.003 %	0.18 %

Again these large errors are probably due to meteorological causes.

The most interesting feature of the Portsmouth curve is that the frequency distribution has a pair of maxima at about $\lvert h_o - h_p \rvert = 0.2$ feet. There is such symmetry in the curve, and the set of $h_o - h_p$ contains so many elements, that it would seem that this is a real phenomenon, having some physical explanation. It seemed possible that the absence of M_4 from the analysis list could account for these peaks; a set of predictions, h'_p, was prepared using a more complete set of constants for Portsmouth, and it was found that the set $h_p - h'_p$ formed an almost perfect Normal Distribution. It must therefore be assumed that the explanation lies in local meteorological conditions, or in the seiche characteristics of Portsmouth harbour.

Conclusions

The conclusions reached so far are that a poor theoretical model will not normally invalidate the use of a least squares analysis, but that meteorological effects may do so, and that data errors are also important, particularly if there are a number of large errors present. The question arises as to whether there is any improvement that can be made in the analysis process to mitigate these effects. It has been suggested earlier in this work that examination by eye, or by smoothness tests on the computer, can remove most data errors. There may remain the occasional severe error no matter how much checking and correcting is done; some errors, also, are very difficult to find by any mechanical process : if, for example, a few tide-gauge charts in the year are in British Summer Time, and the rest are in G.M.T., and if this goes unnoticed, then the data might pass any test of smoothness, and the analysis would be severely affected.

One way in which the effects of any data errors or meteorological abnormalities can be reduced is by performing the analysis twice. The idea is that the analysis is done once as usual, and the results are used to prepare a set of h_p. In any case where h_o and h_p differ substantially (by more than, say, two feet), the value of h_o is replaced by h_p. The analysis is then repeated, using this modified set of observations as data. This process was carried out for the case of the Tilbury data : values of h_o were replaced by h_p when $|h_o - h_p| \geqslant 2.0$ feet. The sets of $h_o - h_p$ (using the original h_o) obtained after the first and second analyses were examined, and the following results were found.

$\lvert h_o - h_p \rvert$	1st analysis	2nd analysis
$\geqslant$ 2.0 feet	3.4 %	3.5 %
1.0 to 1.9 feet	13.2 %	12.4 %
0.5 to 0.9 feet	29.9 %	29.9 %
$<$ 0.5 feet	53.5 %	54.2 %

The number of errors greater than two feet was scarcely affected, but there was a marginal improvement in the number of errors in the 1-2-foot range. This case did not make full use of the technique, in that it is believed that there were no data errors, and a long period of observations was being used; for a short-period analysis, using unchecked data, the results would have been somewhat different.

It is, of course, only on a powerful computer, where the double analysis would not take very much longer than a single analysis preceded by one or two data-checking runs, that this process could be considered. In such cases, however, it has a number of advantages, in that it allows for all data errors without operator intervention, or time wasted in checking the data, and it also allows for surges at the same time. It is imagined that the whole double analysis for a year's observations in terms of sixty constituents would take about 6-7 minutes on the KDF9 computer if the program was written in User Code, and perhaps 9-10 minutes if the program was fairly carefully written in KDF9 Algol.

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