RELATIVE ACCURACY OF SOME METHODS OF HARMONIC ANALYSIS OF TIDES

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1. — Introduction

In this study we seek to establish the relative accuracy of some methods of harmonic tidal analysis, involving observations of a month's duration. It is obvious that the developments which we shall make will be of an entirely general character; however we shall here limit ourselves to short-period analyses.

If hourly heights of a tide curve are analysed by the least squares method, the problem of determining the accuracy of the results is a standard one whose solution is more than 100 years old. In fact it will be necessary to compute first the residuals v which will be the differences between the actual tide curve and the curve predicted with the help of harmonic constants derived from the analysis, so as to be able then to find the mean error of unit weight, given by the formula :

$$m_y = \sqrt{[vv] / (N - 2Q)} \qquad (1a)$$

In this formula N is the number of observations and Q that of the constituents under consideration.

The errors in the unknowns R cos r and R sin r will be given by the general relations :

$$m_1 = m_y \sqrt{[\alpha\alpha]}, \qquad m_2 = m_y \sqrt{[\beta\beta]} \qquad \ldots\ldots \qquad (1b)$$

where $[\alpha\alpha]$, $[\beta\beta]$, etc. are the weight numbers.

If for the various methods under consideration we can determine m in terms of m_y, the relative accuracy of these methods can be inferred.

2. — Mean errors in phase and amplitude

Although it is possible to compare the methods by merely comparing the mean errors in R cos r and R sin r, it appears interesting to express the mean errors in R and r in terms of those in R cos r and R sin r. We now know that R and r are computed from expressions :

$$R = \sqrt{(R \cos r)^2 + (R \sin r)^2} \qquad (2a)$$

and

$$r = \tan^{-1} (R \sin r) / (R \cos r) \qquad (2b)$$

where, if we put :

$$R \cos r = a \qquad (2c)$$

and

$$R \sin r = b \qquad (2d)$$

we obtain :

$$R = \sqrt{a^2 + b^2} \qquad (2e)$$

$$r = \tan^{-1} b/a \qquad (2f)$$

In order to apply the general formula for propagation of errors to these expressions, the partial derivatives $\partial R/\partial a$, $\partial R/\partial b$, $\partial r/\partial a$ and $\partial r/\partial b$ must firstly be found. From (2e) and (2f) we then have :

$$\begin{aligned} \partial R/\partial a &= a/R \\ \partial R/\partial b &= b/R \\ \partial r/\partial a &= -a/R^2 \\ \partial r/\partial b &= b/R^2 \end{aligned}$$

The mean errors in R cos r and R sin r being designated by m_a and m_b, respectively, through the application of the general formula for the propagation of errors, we obtain :

$$m_R = \frac{1}{R} \sqrt{a^2 m_a^2 + b^2 m_b^2}$$

and

$$m_r = \frac{1}{R^2} \sqrt{a^2 m_a^2 + b^2 m_b^2}$$

Now, in the classical methods, we generally have $m_a \approx m_b$ for unknowns a and b corresponding to the same constituent. If we then put $m_a = m_b = m$, these last expressions will take the form :

$$\begin{aligned} m_R &= m \\ m_r &= m/R \end{aligned}$$

which shows that while the amplitude is determined with the same accuracy as the unknows R cos r and R sin r, the phase accuracy is inversely proportional to the constituent's amplitude. This is the reason it is always difficult to determine the small constituents with accuracy.

3. — Determination of m for the Doodson methods

To express m in terms of m_y, it will be necessary to study the propagation of the errors of the hourly heights through the daily and monthly processes.

We know that the daily process gives us functions X_n and Y_n. If we denote these functions in a general way by F_n, the daily process can be written in the form :

$$F_n = \sum_t D_t y_t \qquad (3a)$$

The propagation formula applied to this expression gives us :

$$m_F^2 = \sum_t D_t^2 m_y^2 \tag{3b}$$

D_t being the hourly multipliers in the daily process, $\sum_t D_t^2$ can be computed by using the well-known tables of these multipliers (1). In the Tidal Institute's method of analysis, we find :

TABLE 3-I

	X_0	X_1, Y_1	X_2, Y_2	X_3	X_4, Y_4	X_6
$\sum_t D_t^2$	42	52	84	54	28	92

For the so-called British Admiralty method, duly developed by Doodson to give the same constituents as the Tidal Institute method, we see, according to table IA (2), that using formula (3b), for all the functions except X_4 we shall have :

$$m_{F_n}^2 = 24\ m_y^2 \tag{3c}$$

and for X_4

$$m_{X_4}^2 = 16\ m_y^2 \tag{3d}$$

In the two methods under discussion, Doodson did not find it necessary to employ functions Y_n for $n = 0$, 3 and 6. Thus, for these species, the daily process immediately supplies the known terms of the set of equations in R cos r and R sin r. These known terms may then be represented by :

$$X_{np} = \sum_d D'_d X_n \tag{3e}$$

and

$$X_{np'} = \sum_d D''_d X_n \tag{3f}$$

the first expression representing the known terms of the set of equations in R cos r and the second in the R sin r equations. In these formulas D'_d represents the combinations designated by a numerical symbol (here expressed by p), and D''_d the combinations designated by a literal symbol called p' having the same numerical order in the alphabet as p.

As for diurnal, semi-diurnal and fourth-diurnal tides, the known terms in the equations are all of the general form :

$$(L_{np} \text{ or } L_{np'}) = \pm X_{np} \pm Y_{np'} \pm X_{np'} \pm Y_{np} \tag{3g}$$

for the Tidal Institute method, and :

$$L_{np} = X_{np} \pm Y_{np'} \tag{3h}$$

and

$$L_{np'} = X_{np'} \pm Y_{np} \tag{3i}$$

for the Admiralty method.

As will be seen, the signs have no importance as far as the propagation of the X_{np}, $X_{np'}$, Y_{np} and $Y_{np'}$ errors on L is concerned. Therefore relation (3g) may be expressed in the form :

$$L = \sum_d (\pm D'_d \pm D''_d)\ X_n + \sum_d (\pm D'_d \pm D''_d)\ Y_n \tag{3j}$$

(1) I.H.R., Vol. XXXI, No. 1, May 1954, page 68.
(2) *Ibid.*, page 84.

D'_d and D''_d always corresponding to combinations p and p' of the same order. By next applying the general formula for the propagation of errors to (3e), (3f) and (3j), we respectively obtain :

$$m^2_{X_{np}} = \sum_d D'^2_d \, m^2_{X_n} \tag{3k}$$

$$m^2_{X_{np'}} = \sum_d D''^2_d m^2_{X_n} \tag{3l}$$

and

$$m^2_{L_{np}} = m^2_{L_{np'}} = \sum_d (D'^2_d \pm 2D'_d D''_d + D''^2_d) \, m^2_{X_n} + \sum_d (D'^2_d \pm 2D'_d D''_d + D''^2_d) \, m^2_{Y_n}$$

However, in tables II [(3)] and IIA [(3)], we see that column vectors D'_d and D''_d are all orthogonal, giving us $\sum_d D_d \, D_d = 0$ and the last expression then becomes :

$$m^2_{L_{np}} = m^2_{L_{np'}} = \sum_d (D'^2_d + D''^2_d) \, (m^2_{X_n} + m^2_{Y_n}) \tag{3m}$$

This expression will also be obtained from (3h) and (3i).

We can now draw up a table of the values of $\sum_d D'^2_d$, $\sum_d D''^2_d$ and $\sum_d (D'^2_d + D''^2_d)$ for the two methods of analysis.

TABLE 3-II(a)

Tidal Institute Method

p	0	1	2	3	4	5	6	7
$\sum_d D'^2_d$	29	58	70	66	70	56	62	58
p'		a	b	c	d	e	f	g
$\sum_d D''^2_d$		62	64	62	60	66	60	62
$\sum_d (D'^2_d + D''^2_d)$	29	120	134	128	130	122	122	120

TABLE 3-II(b)

British Admiralty Method

p	0	1	2	3	4	5	6	7
$\sum_d D'^2_d$				always	29			
p'		a	b	c	d	e	f	g
$\sum_d D''^2_d$		28	24	28	24	28	28	28
$\sum_d (D'^2_d + D''^2_d)$	29	57	53	57	53	57	57	57

In the Tidal Institute method we always have the same values of $\sum_t D^2_t$ (table 3-I) for X_n and for Y_n (n being the same for both functions). In these conditions, from (3b), (3k), (3l) and (3m), we may write :

$$m^2_{X_{np}} = \sum_d D'^2_d \cdot \sum_t D^2_t m^2_y \tag{3n}$$

$$m^2_{X_{np'}} = \sum_d D''^2_d \cdot \sum_t D^2_t m^2_y \tag{3o}$$

$$m^2_{L_{np}} = m^2_{L_{np'}} = 2 \sum_d (D'^2_d + D''^2_d) \, \sum_t D^2_t \, m^2_y \tag{3p}$$

(3) *Ibid.*, pages 69 and 85.

With the exception of L_{4p} and $L_{4p'}$, these same expressions can be used in the case of the British Admiralty method. As expressions (3c) and (3d) show, for L_{4p} and $L_{4p'}$, expression (3j) in this case gives :

$$m^2_{L_{4p}} = m^2_{L_{4p'}} = 40 \sum_d (D'^2_d + D''^2_d)\, m^2_y \qquad (3q)$$

It is now quite easy to find the numerical coefficient of m^2_y for all the known terms. For the Tidal Institute method it is only necessary to multiply suitably the values contained in tables 3-I and 3-II(a). In order to find the coefficient corresponding to a function X_{np}, the value of $\sum_t D^2_t$ corresponding to X_n must be looked up in table 3-I and the value of $\sum_d D'^2_d$ corresponding to p in table 3-II(a). The product of these two quantities will be the numerical coefficient of m^2_y in expression (3n). To find the numerical coefficient of m^2_y in expression (3o) it will suffice to look up the value of $\sum_d D''^2_d$ corresponding to p' in table 3-II(a) and to multiply it by the same value of $\sum_t D^2_t$ already found for X_n. For functions L_{np} and $L_{np'}$, the value of $\sum_d (D'^2_d + D''^2_d)$ will be found in either the p or p' column of table 3-II(a) and should be multiplied by twice the value of $\sum_t D^2_t$ corresponding to X_n and Y_n in table 3-I. This will give the result of the computation of expression (3p). All the vector columns of table 3-III(a) have been computed according to the instructions just set forth.

In the case of the British Admiralty method, it is much easier to find the numerical coefficients in expressions (3n) to (3p). In fact, with the exception of the fourth-diurnal constituents for which expression (3q) must be applied, we shall always have $\sum_t D^2_t = 24$, which allows us to confine our search to table 3-II(b) where the values of $\sum_d D'^2_d$, $\sum_d D''^2_d$ and $\sum_d (D'^2_d + D''^2_d)$ according to the second subscripts p and p' will be found. The vector columns of table 3-III(b) have been computed from these instructions, noting that for the fourth-diurnal constituents these values must be multiplied by 40. Here there is only one vector column for the R cos r equations and another for R sin r since all the unknowns are derived from equations containing all six species of the tide under consideration.

For the application of the Tidal Institute method, on examining table IV [4] we shall see that it consists of the inverse matrices required for the computation of R cos r and R sin r. From the way in which the values are set out, the unknowns in each set of equations, designated by I, are expressed as a function of the known terms, designated by L, in the general formula :

$$I = \sum \alpha L \qquad (3r)$$

According to the previously employed propagation formula, we shall have :

$$m^2 = \sum \alpha^2 m^2_L \qquad (3s)$$

The values of $\sum \alpha^2$ have already been computed for the two methods in order to apply formula (3s), taking the values of m^2_L from tables 3-III(a)

(4) *Ibid.*, page 71.

and 3-III(b). Table 5-I already contains the values of *m* for all unknowns, for both the semi-graphic and the least squares method.

TABLE 3-III

Coefficients of m_y^2

(a) *Tidal Institute method*

X_{00}	1218	X_{0a}	2604
X_{01}	2436	X_{0b}	2688
X_{02}	2940		
C_{10}	3016	D_{10}	
C_{11}	12480	D_{11}	
C_{12}	13936	D_{12}	
C_{13}	13312	D_{13}	
C_{14}	13520	D_{14}	
D_{1a}	12480	C_{1a}	
D_{1b}	13936	C_{1b}	
C_{20}	4872	D_{20}	
C_{21}	20160	D_{21}	
C_{22}	22512	D_{22}	
C_{23}	21504	D_{23}	
C_{24}	21840	D_{24}	
D_{2b}	22512	C_{2b}	
C_{12}	13936	D_{12}	
X_{32}	3780	X_{3b}	3466
X_{33}	3564	X_{3c}	3348
X_{34}	3780	X_{3d}	3240
C_{42}	7504	D_{42}	
C_{43}	7168	D_{43}	
C_{44}	7280	D_{44}	
C_{45}	6832	D_{45}	
X_{62}	6440	X_{6b}	5888
X_{64}	6440	X_{6d}	5520
X_{65}	5152	X_{6e}	6072
X_{66}	5704	X_{6f}	5520
X_{67}	5336	X_{6g}	5704

(b) *British Admiralty method*

X_{00}	696		
X_{01}	696	X_{0a}	672
X_{02}	696	X_{0b}	576
C_{10}	1392	D_{10}	1392
C_{11}	2736	D_{11}	2736
C_{12}	2544	D_{12}	2544
C_{13}	2736	D_{13}	2736
C_{14}	2544	D_{14}	2544
D_{1a}	2736	C_{1a}	2736
D_{1b}	2544	C_{1b}	2544
C_{20}	1392	D_{20}	1392
C_{21}	2736	D_{21}	2736
C_{22}	2544	D_{22}	2544
C_{23}	2736	D_{23}	2736
C_{24}	2544	D_{24}	2544
D_{2b}	2544	C_{2b}	2544
X_{32}	696	X_{3b}	576
X_{33}	696	X_{3c}	672
X_{34}	696	X_{3d}	576
C_{42}	2120	D_{4b}	2120
C_{43}	2280	D_{4c}	2280
C_{44}	2120	D_{4d}	2120
C_{45}	2280	D_{4e}	2280
X_{62}	696	X_{6b}	576
X_{64}	696	X_{6d}	576
X_{65}	696	X_{6e}	672
X_{66}	696	X_{6f}	672
X_{67}	696	X_{6g}	672

4. — Determination of *m* in the semi-graphic method

We know that in the semi-graphic method large disturbances arising from meteorological phenomena may be reduced by means of smoothing the contours which represent the tidal surface. In these conditions we may assume that the analysed ordinates are only affected by random smoothing errors.

To be able to study the propagation of errors in ordinates for the tidal surface in known terms of the equations to be solved, we shall follow the steps set forth in my article published in the International Hydrographic Review (Vol. XL, No. 2, July 1963). Once the ordinates of the tidal surface have been obtained they will be combined according to the expressions :

$$X_n = \sum_t D_t^n y_t$$

and (4a)

$$X_{n'} = \sum_t D_t^{n'} y_t$$

where D_t^n represents the multipliers in table 4-II [5] which consist of the column vectors indicated by the figures $n = 1, 2, 3, 4, 6$, and $D_t^{n'}$ represents the multipliers in the same table whose vector columns are indicated by the letters $n' = a, b, c, e, f$. To isolate one species, the subscripts of functions X_n and $X_{n'}$ are of the same order, i.e. the numerical order in the alphabet of letter n' is equal to n. The two equations (4a) are sufficient for the isolation of the third-diurnal and sixth-diurnal species, but for the other species form 4-A [6] shows that it is necessary to find the functions :

$$U_1 = X_1 + X_3/3$$
$$U_a = X_a - X_c/3$$
$$U_2 = X_2 + X_6/3$$
$$U_b = X_b - X_f/3$$
$$U_d = 1.15 \ X_d$$

From the general relations (4a) we may write the above expressions as :

$$U_1 = \sum_t (D_t^1 + D_t^3 / 3) \, y_t \qquad (4b)$$

$$U_a = \sum_t (D_t^a - D_t^c / 3) \, y_t \qquad (4c)$$

$$U_2 = \sum_t (D_t^2 + D_t^6 / 3) \, y_t \qquad (4d)$$

$$U_b = \sum_t (D_t^b - D_t^f / 3) \, y_t \qquad (4e)$$

$$U_d = 1.15 \sum_t D_t^d \, y_t \qquad (4f)$$

Then expressions (4a) to (4f) are all of the form :

$$F_n = \sum_t \delta_t \, y_t \qquad (4g)$$

to which we can apply the general formula for propagation of errors. We can therefore write :

$$m_{F_n}^2 = \sum_t \delta_t^2 \, m_y^2 \qquad (4h)$$

The values of δ_t for expressions (4a), which should be applied in the case of the third- and sixth-diurnal constituents, are respectively multipliers ($n = 3$, $n' = c$) and ($n = 6$, $n' = f$) from table 4-II [7]. The sum of the

(5) I.H.R., Vol. XL, No. 2, July 1963, page 80.
(6) *Ibid.*, page 83.
(7) *Ibid.*, page 80.

squares of these multipliers gives the values of $\sum \delta_t^2$ for X_3, X_c, X_6 and X_f as seen in table 4-I. As for the value

TABLE 4-I

	X_0	U_1, U_a	U_2, U_b	X_3, X_c	X_4	U_d	X_6, X_f
$\sum_t \delta_t^2$	24	20	18.67	18	24	21.12	12

of $\sum_t \delta_t^2$ for U_d, expression (4f) shows that it will be sufficient to multiply the sum of the squares of the multipliers in column (d) table 4-I, by the square of 1.15. The result is entered in table 4-I under U_d.

To find multipliers $\sum_t \delta_t^2$ corresponding to equations (4b) to (4e) the values of D_t^n and $D_t^{n'}$ must be taken from the same line in table 4-II [7] and then combined according to the expressions of the coefficients of y_t in (4b) to (4e). We thus obtain a series of multipliers δ_t which will directly give U_1, U_a, U_2 and U_b. The sums of the squares of the multipliers so obtained are those to be seen in table 4-I for the same functions.

Examining expressions (5f) [8] and (5g) [9], we find that all the known terms of the sets of equations are obtained by the sum or the difference of pairs of values of F_n : F_n being any one of functions U_1, U_a, U_2, U_b, X_3, X_c, etc. The mean errors of the known terms are then equal to those of F_n multiplied by $\sqrt{2}$ which, according to (4h), can be written :

$$m_L^2 = 2 \sum_t \delta_t^2 m_y^2 \qquad (4i)$$

Let us now investigate the propagation of m_L on the unknowns. Table 5-III [10] shows that all the unknowns are computed from the general formula (3r). Therefore the propagation formula applied to this expression gives an expression similar to (3s), i.e. according to (4i) :

$$m^2 = 2 \sum_t \delta_t^2 \sum \alpha^2 m_y^2 \qquad (4j)$$

Table 4-I shows that all the values of $\sum_t \delta_t^2$ are equal when the number n is equal to the numerical order of the letter n' in the alphabet except for $n = 4$ and $n' = d$. Though even in this case the mean of the values 24 and 21.12, equal to 22.56, can be taken as an adequate approximation. Thus for all the constituents, with the exception of the fourth-diurnal, we can find the numerical coefficients of m_y^2 in expression (4j) by multiplying the double of the values of $\sum_t \delta_t^2$ in table 4-I by the values of $\sum \alpha^2$ computed from the values of α for each constituent, given in table 5-III [10]. In the case of the fourth-diurnal constituents, we write $2 \sum_t \delta_t^2 = 45.12$. The square roots of the values thus found are entered in the S.G. columns of table 5-I.

(7) *Ibid.*, page 80.
(8) *Ibid.*, page 85.
(9) *Ibid.*, page 86.
(10) *Ibid.*, page 88.

TABLE 5-I

Mean errors in the unknowns in units of 10^{-3} m_v

	R cos r				R sin r			
	L.S.	T.I.	B.A.	S.G.	L.S.	T.I.	B.A.	S.G.
S_0	27	40	38	66				
M_m		57	59	75		60	62	72
MS_f		57	62	79		59	55	68
Q_1		68	94	104		70	94	104
O_1		72	91	104		70	90	104
M_1		73	95	104		72	95	104
K_1		64	85	104		83	85	104
J_1		79	102	104		80	102	104
OO_1		92	105	104		88	106	104
μ_2		73	95	108		74	95	108
N_2		73	93	106		72	93	106
M_2		72	92	104		72	93	104
L_2		73	91	106		72	91	106
S_2		71	84	107		71	79	107
$2SM_2$		75	99	101		74	98	101
MO_3	54	62	95	104	54	63	93	104
M_3		63	94	104		63	91	104
MK_3		66	104	104		63	87	104
MN_4		67	90	106		68	90	106
M_4		69	90	107		67	89	107
SN_4		67	87	107		66	87	107
MS_4		66	86	107		65	87	107
$2MN_6$		81	97	103		80	93	103
M_6		78	96	104		81	90	104
MSN_6		76	91	104		79	88	104
$2MS_6$		75	91	104		76	86	104
$2SM_6$		73	95	104		72	85	104

5. — Conclusion

To draw the final conclusions, the sets of normal equations for the determination of R cos *r* and R sin *r* were established according to the Horn method [11], taking into account 28 constituents. The matrices of these systems both have very strong diagonal elements in comparison with others. Under these conditions we are of the opinion that for an accuracy study it will be sufficient to consider the inverse matrices as diagonal matrices whose values are the inverse of those of the equations' diagonals. Since the values of the data of the equations' diagonals are all close to 347.5, the inverse matrices will have the terms of their diagonals close to 1/347.5. We know, however, that the weight numbers $[\alpha\alpha]$, $[\beta\beta]$,

(11) I.H.R., Vol. XXXVII, No. 2, July 1960, pp. 65-84.

$[\gamma\gamma]$, ..., etc. are exactly the elements of the diagonals of the inverse matrices in the sets of normal equations, which allows us to write :

$$[\alpha\alpha] \approx [\beta\beta] \approx [\gamma\gamma] \approx ..... \approx 0.002878$$

This being so expression (1b) will give :

$$m = m_y \sqrt{0.002878}$$

or

$$m = 0.054 \; m_y$$

for all the unknowns.

By comparing the values entered in table 5-I, we note that the Tidal Institute method is the most accurate of the three and that there is little difference between the British Admiralty and semi-graphic methods.

The random errors are already reduced before the analysis since, generally speaking, all the analyses carried out by electronic computers are preceded by a smoothing of the curve and the observations themselves are very accurately carried out. This being so, we feel that unless the analysis has a strictly scientific purpose there will be no need to apply the least squares method. In hydrographic offices not possessing all the resources of more well-equipped organizations, perfectly valid results will be obtained by applying the Tidal Institute method. It is interesting to note that Mr. Lennon of the Tidal Institute has drawn up a programme for using the IBM 1620 computer in the analysis of a one year's series of observations, which includes smoothing the curve in less than fifteen minutes. This may be considered as a remarkable achievement.

Author's note. — The major part of this article's subject was discussed by Doodson and Lennon at the 3rd International Symposium on Earth Tides, Trieste, 1959. The author was unaware of this fact at the time this article was written.