1. — Introduction

In this study we seek to establish the relative accuracy of some methods of harmonic tidal analysis, involving observations of a month's duration. It is obvious that the developments which we shall make will be of an entirely general character; however we shall here limit ourselves to short-period analyses.

If hourly heights of a tide curve are analysed by the least squares method, the problem of determining the accuracy of the results is a standard one whose solution is more than 100 years old. In fact it will be necessary to compute first the residuals $v$ which will be the differences between the actual tide curve and the curve predicted with the help of harmonic constants derived from the analysis, so as to be able then to find the mean error of unit weight, given by the formula:

$$m_v = \frac{\sqrt{\langle vv \rangle}}{(N - 2Q)} \quad (1a)$$

In this formula $N$ is the number of observations and $Q$ that of the constituents under consideration.

The errors in the unknowns $R \cos r$ and $R \sin r$ will be given by the general relations:

$$m_1 = m_v \sqrt{\langle \alpha \alpha \rangle}, \quad m_2 = m_v \sqrt{\langle \beta \beta \rangle} \quad \ldots \ldots \quad (1b)$$

where $[\alpha \alpha]$, $[\beta \beta]$, etc. are the weight numbers.

If for the various methods under consideration we can determine $m$ in terms of $m_v$, the relative accuracy of these methods can be inferred.

2. — Mean errors in phase and amplitude

Although it is possible to compare the methods by merely comparing the mean errors in $R \cos r$ and $R \sin r$, it appears interesting to express the mean errors in $R$ and $r$ in terms of those in $R \cos r$ and $R \sin r$. We now know that $R$ and $r$ are computed from expressions:
\[ R = \sqrt{(R \cos r)^2 + (R \sin r)^2} \]  \hspace{1cm} (2a)

and

\[ r = \tan^{-1} \frac{(R \sin r)}{(R \cos r)} \]  \hspace{1cm} (2b)

where, if we put:

\[ R \cos r = a \]  \hspace{1cm} (2c)

and

\[ R \sin r = b \]  \hspace{1cm} (2d)

we obtain:

\[ R = \sqrt{a^2 + b^2} \]  \hspace{1cm} (2e)

\[ r = \tan^{-1} \frac{b}{a} \]  \hspace{1cm} (2f)

In order to apply the general formula for propagation of errors to these expressions, the partial derivatives \( \partial R/\partial a \), \( \partial R/\partial b \), \( \partial r/\partial a \) and \( \partial r/\partial b \) must firstly be found. From (2e) and (2f) we then have:

\[
\begin{align*}
\frac{\partial R}{\partial a} &= a/R \\
\frac{\partial R}{\partial b} &= b/R \\
\frac{\partial r}{\partial a} &= -a/R^2 \\
\frac{\partial r}{\partial b} &= b/R^2
\end{align*}
\]

The mean errors in \( R \cos r \) and \( R \sin r \) being designated by \( m_a \) and \( m_b \), respectively, through the application of the general formula for the propagation of errors, we obtain:

\[ m_R = \frac{1}{R} \sqrt{a^2 m_a^2 + b^2 m_b^2} \]

and

\[ m_r = \frac{1}{R^2} \sqrt{a^2 m_a^2 + b^2 m_b^2} \]

Now, in the classical methods, we generally have \( m_a = m_b \) for unknowns \( a \) and \( b \) corresponding to the same constituent. If we then put \( m_a = m_b = m \), these last expressions will take the form:

\[ m_R = m \]

\[ m_r = m/R \]

which shows that while the amplitude is determined with the same accuracy as the unknowns \( R \cos r \) and \( R \sin r \), the phase accuracy is inversely proportional to the constituent's amplitude. This is the reason it is always difficult to determine the small constituents with accuracy.

3. — Determination of \( m \) for the Doodson methods

To express \( m \) in terms of \( m_y \), it will be necessary to study the propagation of the errors of the hourly heights through the daily and monthly processes.

We know that the daily process gives us functions \( X_n \) and \( Y_n \). If we denote these functions in a general way by \( F_n \), the daily process can be written in the form:

\[ F_n = \sum_{t} D_t y_t \]  \hspace{1cm} (3a)
The propagation formula applied to this expression gives us:

\[ m_k^2 = \sum_i D_i^2 m_i^2 \]  

(3b)

\( D_i \) being the hourly multipliers in the daily process, \( \sum_i D_i^2 \) can be computed by using the well-known tables of these multipliers \(^{(1)}\). In the Tidal Institute's method of analysis, we find:

**Table 3-I**

<table>
<thead>
<tr>
<th>X₀</th>
<th>X₁, Y₁</th>
<th>X₂, Y₂</th>
<th>X₃</th>
<th>X₄, Y₄</th>
<th>X₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum D_i^2 )</td>
<td>42</td>
<td>52</td>
<td>84</td>
<td>54</td>
<td>28</td>
</tr>
</tbody>
</table>

For the so-called British Admiralty method, duly developed by Doodson to give the same constituents as the Tidal Institute method, we see, according to table IA \(^{(2)}\), that using formula (3b), for all the functions except \( X_4 \) we shall have:

\[ m_{k_n}^2 = 24 m_n^2 \]  

(3c)

and for \( X_4 \)

\[ m_{k_4}^2 = 16 m_4^2 \]  

(3d)

In the two methods under discussion, Doodson did not find it necessary to employ functions \( Y_n \) for \( n = 0, 3 \) and 6. Thus, for these species, the daily process immediately supplies the known terms of the set of equations in \( R \cos r \) and \( R \sin r \). These known terms may then be represented by:

\[ X_{np} = \sum_{d} D'_d X_n \]  

(3e)

and

\[ X_{np'} = \sum_{d} D''_d X_n \]  

(3f)

the first expression representing the known terms of the set of equations in \( R \cos r \) and the second in the \( R \sin r \) equations. In these formulas \( D'_d \) represents the combinations designated by a numerical symbol (here expressed by \( p \)), and \( D''_d \) the combinations designated by a literal symbol called \( p' \) having the same numerical order in the alphabet as \( p \).

As for diurnal, semi-diurnal and fourth-diurnal tides, the known terms in the equations are all of the general form:

\[ (L_{np} \text{ or } L_{np'}) = \pm X_{np} \pm Y_{np'} \pm X_{np'} \pm Y_{np} \]  

(3g)

for the Tidal Institute method, and:

\[ L_{np} = X_{np} \pm Y_{np'} \]  

(3h)

and

\[ L_{np'} = X_{np'} \pm Y_{np} \]  

(3i)

for the Admiralty method.

As will be seen, the signs have no importance as far as the propagation of the \( X_{np}, X_{np'}, Y_{np} \) and \( Y_{np'} \) errors on \( L \) is concerned. Therefore relation (3g) may be expressed in the form:

\[ L = \sum_{d} (\pm D'_d \pm D''_d) X_n + \sum_{d} (\pm D'_d \pm D''_d) Y_n \]  

(3j)

---


\(^{(2)}\) Ibid., page 84.
D_1 and D_2 always corresponding to combinations p and p' of the same order. By next applying the general formula for the propagation of errors to (3e), (3f) and (3j), we respectively obtain:

\[ m_{x_{np}}^2 = \sum_d D_d^2 m_{x_n}^2 \]  \hspace{1cm} (3k)  

\[ m_{x_{np'}}^2 = \sum_d D_d^2 m_{x_n}^2 \]  \hspace{1cm} (3l)  

and

\[ m_{f_{np}}^2 = m_{f_{np'}}^2 = \sum_d (D_d'^2 \pm 2D_d D_d'^2 + D_d''^2) m_{x_n}^2 + \sum_d (D_d'^2 \pm 2D_d D_d'^2 + D_d''^2) m_{x_n}^2 \]

However, in tables II (3) and IIA (3), we see that column vectors D_d and D_d' are all orthogonal, giving us \( \sum_d D_d D_d = 0 \) and the last expression then becomes:

\[ m_{f_{np}}^2 = m_{f_{np'}}^2 = \sum_d (D_d'^2 + D_d''^2) (m_{x_n}^2 + m_{x_n}^2) \]  \hspace{1cm} (3m)  

This expression will also be obtained from (3h) and (3i).

We can now draw up a table of the values of \( \sum_d D_d^2, \sum_d D_d'^2 \) and \( \sum_d (D_d'^2 + D_d''^2) \) for the two methods of analysis.

### Table 3-II(a)

**Tidal Institute Method**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_d D_d^2 )</td>
<td>29</td>
<td>58</td>
<td>70</td>
<td>66</td>
<td>70</td>
<td>56</td>
<td>62</td>
<td>58</td>
</tr>
<tr>
<td>( \sum_d D_d' )</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sum_d D_d'' )</td>
<td>62</td>
<td>64</td>
<td>62</td>
<td>60</td>
<td>66</td>
<td>60</td>
<td>62</td>
<td>62</td>
</tr>
<tr>
<td>( \sum_d (D_d'^2 + D_d''^2) )</td>
<td>29</td>
<td>120</td>
<td>134</td>
<td>128</td>
<td>130</td>
<td>122</td>
<td>122</td>
<td>120</td>
</tr>
</tbody>
</table>

### Table 3-II(b)

**British Admiralty Method**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_d D_d^2 )</td>
<td>always 29</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sum_d D_d' )</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sum_d D_d'' )</td>
<td>28</td>
<td>24</td>
<td>28</td>
<td>24</td>
<td>28</td>
<td>28</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td>( \sum_d (D_d'^2 + D_d''^2) )</td>
<td>29</td>
<td>57</td>
<td>53</td>
<td>57</td>
<td>53</td>
<td>57</td>
<td>57</td>
<td>57</td>
</tr>
</tbody>
</table>

In the Tidal Institute method we always have the same values of \( \sum_t D_t \) (table 3-I) for \( X_n \) and for \( Y_n \) (n being the same for both functions). In these conditions, from (3b), (3k), (3l) and (3m), we may write:

\[ m_{x_{np}}^2 = \sum_d D_d^2 \cdot \sum_t D_t m_{y_t}^2 \]  \hspace{1cm} (3n)  

\[ m_{x_{np'}}^2 = \sum_d D_d^2 \cdot \sum_t D_t m_{y_t}^2 \]  \hspace{1cm} (3o)  

\[ m_{f_{np}}^2 = m_{f_{np'}}^2 = 2 \sum_d (D_d'^2 + D_d''^2) \sum_t D_t m_{y_t}^2 \]  \hspace{1cm} (3p)  

With the exception of $L_{4p}$ and $L_{4p'}$, these same expressions can be used in the case of the British Admiralty method. As expressions (3c) and (3d) show, for $L_{4p}$ and $L_{4p'}$, expression (3j) in this case gives:

$$m_{L_{4p}}^2 = m_{L_{4p'}}^2 = 40 \sum_d \left( D_{d}^2 + D_{d}^{'2} \right) m_d^2$$  \hspace{1cm} (3q)

It is now quite easy to find the numerical coefficient of $m_d^2$ for all the known terms. For the Tidal Institute method it is only necessary to multiply suitably the values contained in tables 3-I and 3-II(a). In order to find the coefficient corresponding to a function $X_{np}$, the value of $\sum D_f^t$ corresponding to $X_n$ must be looked up in table 3-I and the value of $\sum D_f^d$ corresponding to $p$ in table 3-II(a). The product of these two quantities will be the numerical coefficient of $m_d^2$ in expression (3n). To find the numerical coefficient of $m_d^2$ in expression (3o) it will suffice to look up the value of $\sum D_f^d$ corresponding to $p'$ in table 3-II(a) and to multiply it by the same value of $\sum D_f^t$ already found for $X_n$. For functions $L_{np}$ and $L_{np'}$, the value of $\sum (D_f^t + D_f^{'2})$ will be found in either the $p$ or $p'$ column of table 3-II(a) and should be multiplied by twice the value of $\sum D_f^t$ corresponding to $X_n$ and $Y_n$ in table 3-I. This will give the result of the computation of expression (3p). All the vector columns of table 3-III(a) have been computed according to the instructions just set forth.

In the case of the British Admiralty method, it is much easier to find the numerical coefficients in expressions (3n) to (3p). In fact, with the exception of the fourth-diurnal constituents for which expression (3q) must be applied, we shall always have $\sum D_f^t = 24$, which allows us to confine our search to table 3-II(b) where the values of $\sum D_f^t$, $\sum D_f^{'2}$ and $\sum (D_f^t + D_f^{'2})$ according to the second subscripts $p$ and $p'$ will be found. The vector columns of table 3-III(b) have been computed from these instructions, noting that for the fourth-diurnal constituents these values must be multiplied by 40. Here there is only one vector column for the $R \cos r$ equations and another for $R \sin r$ since all the unknowns are derived from equations containing all six species of the tide under consideration.

For the application of the Tidal Institute method, on examining table IV (4) we shall see that it consists of the inverse matrices required for the computation of $R \cos r$ and $R \sin r$. From the way in which the values are set out, the unknowns in each set of equations, designated by $I$, are expressed as a function of the known terms, designated by $L$, in the general formula:

$$I = \sum \alpha L$$  \hspace{1cm} (3r)

According to the previously employed propagation formula, we shall have:

$$m^2 = \sum \alpha^2 m_d^2$$  \hspace{1cm} (3s)

The values of $\sum \alpha^2$ have already been computed for the two methods in order to apply formula (3s), taking the values of $m_d^2$ from tables 3-III(a)

---

(4) Ibid., page 71.
and 3-III(b). Table 5-I already contains the values of \( m \) for all unknowns, for both the semi-graphic and the least squares method.

**Table 3-III**

Coefficients of \( m^3 \)

<table>
<thead>
<tr>
<th>(a) Tidal Institute method</th>
<th>(b) British Admiralty method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{00} ) 1218</td>
<td>( X_{00} ) 696</td>
</tr>
<tr>
<td>( X_{01} ) 2436</td>
<td>( X_{01} ) 696</td>
</tr>
<tr>
<td>( X_{02} ) 2940</td>
<td>( X_{02} ) 696</td>
</tr>
<tr>
<td>( X_{03} )</td>
<td>( X_{03} ) 696</td>
</tr>
<tr>
<td>( X_{04} )</td>
<td>( X_{04} ) 696</td>
</tr>
<tr>
<td>( X_{05} )</td>
<td>( X_{05} ) 696</td>
</tr>
<tr>
<td>( C_{10} ) 3016 D_{10}</td>
<td>( C_{10} ) 1392 C_{10}</td>
</tr>
<tr>
<td>( C_{11} ) 12480 D_{11}</td>
<td>( C_{11} ) 2736 D_{11}</td>
</tr>
<tr>
<td>( C_{12} ) 13936 D_{12}</td>
<td>( C_{12} ) 2544 D_{12}</td>
</tr>
<tr>
<td>( C_{13} ) 13312 D_{13}</td>
<td>( C_{13} ) 2736 D_{13}</td>
</tr>
<tr>
<td>( C_{14} ) 13520 D_{14}</td>
<td>( C_{14} ) 2544 D_{14}</td>
</tr>
<tr>
<td>( D_{1a} ) 12480 C_{1a}</td>
<td>( D_{1a} ) 2736 C_{1a}</td>
</tr>
<tr>
<td>( D_{1b} ) 13936 C_{1b}</td>
<td>( D_{1b} ) 2544 C_{1b}</td>
</tr>
<tr>
<td>( C_{20} ) 4872 D_{20}</td>
<td>( C_{20} ) 1392 D_{20}</td>
</tr>
<tr>
<td>( C_{21} ) 20160 D_{21}</td>
<td>( C_{21} ) 2736 D_{21}</td>
</tr>
<tr>
<td>( C_{22} ) 22512 D_{22}</td>
<td>( C_{22} ) 2544 D_{22}</td>
</tr>
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<td>( C_{23} ) 21504 D_{23}</td>
<td>( C_{23} ) 2736 D_{23}</td>
</tr>
<tr>
<td>( C_{24} ) 21840 D_{24}</td>
<td>( C_{24} ) 2544 D_{24}</td>
</tr>
<tr>
<td>( D_{2a} ) 22512 C_{2a}</td>
<td>( D_{2a} ) 2544 C_{2a}</td>
</tr>
<tr>
<td>( D_{2b} ) 13936 D_{12}</td>
<td>( D_{2b} ) 2544 D_{12}</td>
</tr>
<tr>
<td>( X_{32} ) 3780</td>
<td>( X_{32} ) 696</td>
</tr>
<tr>
<td>( X_{33} ) 3564</td>
<td>( X_{33} ) 696</td>
</tr>
<tr>
<td>( X_{34} ) 3780</td>
<td>( X_{34} ) 696</td>
</tr>
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<td>( C_{42} ) 7504 D_{42}</td>
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<td>( C_{45} ) 6832 D_{45}</td>
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</tr>
<tr>
<td>( X_{42} )</td>
<td>( X_{42} ) 696</td>
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<td>( X_{43} )</td>
<td>( X_{43} ) 696</td>
</tr>
<tr>
<td>( X_{44} )</td>
<td>( X_{44} ) 696</td>
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<td>( X_{56} ) 5704</td>
<td>( X_{56} ) 5520</td>
</tr>
<tr>
<td>( X_{57} ) 5336</td>
<td>( X_{57} ) 5704</td>
</tr>
</tbody>
</table>

4. — Determination of \( m \) in the semi-graphic method

We know that in the semi-graphic method large disturbances arising from meteorological phenomena may be reduced by means of smoothing the contours which represent the tidal surface. In these conditions we
may assume that the analysed ordinates are only affected by random smoothing errors.

To be able to study the propagation of errors in ordinates for the tidal surface in known terms of the equations to be solved, we shall follow the steps set forth in my article published in the International Hydrographic Review (Vol. XL, No. 2, July 1963). Once the ordinates of the tidal surface have been obtained they will be combined according to the expressions:

\[ X_n = \sum \frac{D_n^i y_t}{t} \]

and

\[ X_n' = \sum \frac{D_n'^i y_t}{t} \]

where \( D_n^i \) represents the multipliers in table 4-II \(^5\) which consist of the column vectors indicated by the figures \( n = 1, 2, 3, 4, 6 \), and \( D_n'^i \) represents the multipliers in the same table whose vector columns are indicated by the letters \( n' = a, b, c, e, f \). To isolate one species, the subscripts of functions \( X_n \) and \( X_n' \) are of the same order, i.e. the numerical order in the alphabet of letter \( n' \) is equal to \( n \). The two equations (4a) are sufficient for the isolation of the third-diurnal and sixth-diurnal species, but for the other species form 4-A \(^6\) shows that it is necessary to find the functions:

\[
\begin{align*}
U_1 &= X_1 + X_3/3 \\
U_a &= X_a - X_c/3 \\
U_2 &= X_2 + X_6/3 \\
U_b &= X_b - X_f/3 \\
U_d &= 1.15 X_d
\end{align*}
\]

From the general relations (4a) we may write the above expressions as:

\[
\begin{align*}
U_1 &= \sum \left( \frac{D_1^i + D_3^i}{3} \right) y_t \\
U_a &= \sum \left( D_a - D_c \right) y_t \\
U_2 &= \sum \left( D_2^i + D_6^i \right) y_t \\
U_b &= \sum \left( D_b - D_f \right) y_t \\
U_d &= 1.15 \sum D_d^i y_t
\end{align*}
\]

Then expressions (4a) to (4f) are all of the form:

\[ F_n = \sum \varepsilon_i y_t \]

(4g)

to which we can apply the general formula for propagation of errors. We can therefore write:

\[ m_n^2 = \sum m_i^2 \]

(4h)

The values of \( \varepsilon_i \) for expressions (4a), which should be applied in the case of the third- and sixth-diurnal constituents, are respectively multipliers \((n = 3, n' = c)\) and \((n = 6, n' = f)\) from table 4-II \(^7\). The sum of the

---


\(^6\) Ibid., page 83.

\(^7\) Ibid., page 80.
squares of these multipliers gives the values of $\Sigma \delta_t^2$ for $X_0$, $X_a$, $X_b$ and $X_c$ as seen in table 4-I. As for the value

$$
\begin{array}{cccccccc}
\text{X}_0 & \text{X}_a & \text{X}_b & \text{X}_c & \text{X}_d & \text{X}_e & \text{X}_f \\
\Sigma \delta_t^2 & 24 & 20 & 18.67 & 18 & 24 & 21.12 & 12
\end{array}
$$

of $\Sigma \delta_t^2$ for $U_d$, expression (4f) shows that it will be sufficient to multiply

the sum of the squares of the multipliers in column (d) table 4-I, by the square of 1.15. The result is entered in table 4-I under $U_d$.

To find multipliers $\Sigma \delta_t^2$ corresponding to equations (4b) to (4e) the

values of $D_t^2$ and $D_t^2$ must be taken from the same line in table 4-II (7)

and then combined according to the expressions of the coefficients of $y_t$

in (4b) to (4e). We thus obtain a series of multipliers $\delta_t$ which will

directly give $U_1$, $U_2$, $U_3$ and $U_4$. The sums of the squares of the multipliers

so obtained are those to be seen in table 4-I for the same functions.

Examining expressions (5f) (8) and (5g) (9), we find that all the known

terms of the sets of equations are obtained by the sum or the difference

of pairs of values of $F_n$ : $F_n$ being any one of functions $U_1$, $U_2$, $U_3$, $U_4$, $X_0$, $X_1$, etc. The mean errors of the known terms are then equal to those

of $F_n$ multiplied by $\sqrt{2}$ which, according to (4h), can be written :

$$
m_t^2 = 2 \Sigma \delta_t^2 m_y^2
$$

(4i)

Let us now investigate the propagation of $m_t$ on the unknowns. Table

5-III (10) shows that all the unknowns are computed from the general

formula (3r). Therefore the propagation formula applied to this expression

gives an expression similar to (3s), i.e. according to (4i) :

$$
m_t^2 = 2 \Sigma \delta_t^2 \Sigma \alpha^2 m_y^2
$$

(4j)

Table 4-I shows that all the values of $\Sigma \delta_t^2$ are equal when the number $n$

is equal to the numerical order of the letter $n'$ in the alphabet except for $n = 4$ and $n' = d$. Though even in this case the mean of the values 24

and 21.12, equal to 22.56, can be taken as an adequate approximation.

Thus for all the constituents, with the exception of the fourth-diurnal, we

can find the numerical coefficients of $m_t^2$ in expression (4j) by multiplying

the double of the values of $\Sigma \delta_t^2$ in table 4-I by the values of $\Sigma \alpha^2$ computed

from the values of $\alpha$ for each constituent, given in table 5-III (10). In the

case of the fourth-diurnal constituents, we write $2 \Sigma \delta_t^2 = 45.12$. The square

roots of the values thus found are entered in the S.G. columns of table 5-I.

(7) Ibid., page 80.
(8) Ibid., page 85.
(9) Ibid., page 86.
(10) Ibid., page 88.
### Table 5-I

*Mean errors in the unknowns in units of $10^{-1} m_v*$

<table>
<thead>
<tr>
<th></th>
<th>R cos $r$</th>
<th></th>
<th>R sin $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>27</td>
<td>40</td>
<td>38</td>
</tr>
<tr>
<td>$M_n$</td>
<td>57</td>
<td>59</td>
<td>75</td>
</tr>
<tr>
<td>$M_S$</td>
<td>57</td>
<td>62</td>
<td>79</td>
</tr>
<tr>
<td>$O_1$</td>
<td>68</td>
<td>94</td>
<td>104</td>
</tr>
<tr>
<td>$M_1$</td>
<td>72</td>
<td>91</td>
<td>104</td>
</tr>
<tr>
<td>$K_1$</td>
<td>64</td>
<td>85</td>
<td>104</td>
</tr>
<tr>
<td>$J_1$</td>
<td>79</td>
<td>102</td>
<td>104</td>
</tr>
<tr>
<td>$O_{O_1}$</td>
<td>92</td>
<td>105</td>
<td>104</td>
</tr>
<tr>
<td>$N_2$</td>
<td>73</td>
<td>95</td>
<td>108</td>
</tr>
<tr>
<td>$M_2$</td>
<td>73</td>
<td>93</td>
<td>106</td>
</tr>
<tr>
<td>$L_2$</td>
<td>73</td>
<td>91</td>
<td>106</td>
</tr>
<tr>
<td>$S_2$</td>
<td>71</td>
<td>84</td>
<td>107</td>
</tr>
<tr>
<td>$2SM_2$</td>
<td>75</td>
<td>99</td>
<td>101</td>
</tr>
<tr>
<td>$MO_3$</td>
<td>62</td>
<td>95</td>
<td>104</td>
</tr>
<tr>
<td>$M_3$</td>
<td>63</td>
<td>94</td>
<td>104</td>
</tr>
<tr>
<td>$MK_3$</td>
<td>66</td>
<td>104</td>
<td>104</td>
</tr>
<tr>
<td>$MN_4$</td>
<td>67</td>
<td>90</td>
<td>106</td>
</tr>
<tr>
<td>$M_4$</td>
<td>69</td>
<td>90</td>
<td>107</td>
</tr>
<tr>
<td>$SN_4$</td>
<td>67</td>
<td>87</td>
<td>107</td>
</tr>
<tr>
<td>$MS_4$</td>
<td>66</td>
<td>86</td>
<td>107</td>
</tr>
<tr>
<td>$2MN_6$</td>
<td>81</td>
<td>97</td>
<td>103</td>
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<tr>
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<td>96</td>
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</tr>
<tr>
<td>$2SM_6$</td>
<td>73</td>
<td>95</td>
<td>104</td>
</tr>
</tbody>
</table>

### 5. Conclusion

To draw the final conclusions, the sets of normal equations for the determination of $R \cos r$ and $R \sin r$ were established according to the Horn method \(^{(11)}\), taking into account 28 constituents. The matrices of these systems both have very strong diagonal elements in comparison with others. Under these conditions we are of the opinion that for an accuracy study it will be sufficient to consider the inverse matrices as diagonal matrices whose values are the inverse of those of the equations' diagonals. Since the values of the data of the equations' diagonals are all close to 347.5, the inverse matrices will have the terms of their diagonals close to $1/347.5$. We know, however, that the weight numbers \([\alpha_{\chi}], [\alpha_{\beta}],\)

\[ \gamma \gamma \], ..., etc. are exactly the elements of the diagonals of the inverse matrices in the sets of normal equations, which allows us to write:

\[ [a a] = [b b] = [c c] = .... = 0.002878 \]

This being so expression (1b) will give:

\[ m = m_0 \sqrt{0.002878} \]

or

\[ m = 0.054 m_0 \]

for all the unknowns.

By comparing the values entered in table 5-I, we note that the Tidal Institute method is the most accurate of the three and that there is little difference between the British Admiralty and semi-graphic methods.

The random errors are already reduced before the analysis since, generally speaking, all the analyses carried out by electronic computers are preceded by a smoothing of the curve and the observations themselves are very accurately carried out. This being so, we feel that unless the analysis has a strictly scientific purpose there will be no need to apply the least squares method. In hydrographic offices not possessing all the resources of more well-equipped organizations, perfectly valid results will be obtained by applying the Tidal Institute method. It is interesting to note that Mr. Lennon of the Tidal Institute has drawn up a programme for using the IBM 1620 computer in the analysis of a one year's series of observations, which includes smoothing the curve in less than fifteen minutes. This may be considered as a remarkable achievement.

Author's note. — The major part of this article's subject was discussed by Doodson and Lennon at the 3rd International Symposium on Earth Tides, Trieste, 1959. The author was unaware of this fact at the time this article was written.