

WAVE PATTERN DIAGRAMS

by Captain H. DUFFO, French Navy (Ret.)

Wave pattern diagrams, or, to use a more general term, wave refraction diagrams, do not seem to have been exploited as they merit. In many cases when the plotting of orthogonals would allow a problem to be quickly roughed out, technicians prefer to have recourse to tests on a small-scale model, tests which are long, laborious and costly, and where it is difficult to eliminate with certainty some disturbing effects.

A refraction diagram allows the détermination of the direction taken by off-shore swell from the point where this swell begins to "feel" the bottom until it breaks in a spot which, thanks to this diagram, may be exactly determined. Within the same limits it also permits the computation of amplitude by the formula :

$$H_d = H_\infty D_\infty^d K_\infty^d$$

We shall define several notations.

Generally speaking, we shall use those encountered in American literature since these are universally known.

- d = depth level below the still water;
- L_∞ = wave length in infinite depth;
- L = wave length at a given point, in finite depth;
- H_∞ = trough (or amplitude) in infinite depth;
- H = trough at a given point, in finite depth;
- C_∞ = velocity (or rate of wave propagation) in infinite depth;
- C = velocity at a given point, in finite depth;
- T = period (time elapsed between the passage of two successive wave crests over a given point).

d , L , H , and C may be assigned a suffix whose significance is usually obvious. For example in the case where the contour lines are numbered, d_0 , d_1 , d_2 , d_3 , etc. represent the depths on contour lines numbered 0, 1, 2, 3, etc.

Certain authors give the suffix 0 to the elements H , C , and L in order to show that infinite depth is encountered. We have thought it preferable to keep the suffix zero for an *initial* situation of whatever type as this is the standard notation.

D , or D_∞^d , is the factor by which H_∞ must be multiplied in order to find H_d (the amplitude in a depth d), in a case where there is *no refraction*, (i.e. when the crests run parallel to the contour lines).

More generally, and always when refraction is absent, D_{dp}^d is the factor by which H_{dp} must be multiplied in order to find H_{dq} . Obviously we obtain

$$D_x^d = D_x^d D_{dp}^d.$$

K is the refraction coefficient. We may recall that this ceases to be valid when the wave enters a zone of breakers. We shall also give K a lower and a higher suffix in order to specify the limits within which it is computed. K_A^B will be the refraction coefficient between A and B , A and B being on the same orthogonal. If the contour lines have been numbered, K_p^q , still along the same orthogonal, will be the refraction coefficient between contour lines marked p and q . When there is no suffix K is related to a point in the zone being studied, for example to M , and it represents K_x^M . For a particular off-shore swell, there will be a value of K corresponding to each point of the sea area.

We shall use indifferently the terms "wave ray" (recommended by Professor LACOMBE) and "orthogonal" (as aforesaid) to designate the orthogonals to the lines of the wave crests; to the contrary, "normal" will be applied more particularly to the normal to the contour lines.

Finally i , the angle of incidence, will represent the angle of the wave-ray with the normal to the contour lines. r , the angle of refraction, will be the angle made by the same normal and the ray having crossed the contour line separating two areas of different velocity.

We shall now recall several standard formulas. We shall number them in Roman figures — on occasion giving them a ' or a '' when the same formula has already been expressed in a different form. The formulas we ourselves establish will be numbered in arabic type.

The first of the standard formulas will be :

$$L = C T \quad (I)$$

The second is the expression of the Law of Descartes, known as Snell's Law in the English literature :

$$\frac{\sin i}{C} = \frac{\sin r}{C'} \quad (II)$$

C being the velocity on the side of the incident ray, C' on the side of the refracted ray.

The following relations between velocity, period and depth have been adopted by all authors :

$$C = \frac{g T}{2 \pi} \tanh \frac{2 \pi d}{C T} \quad (III)$$

$$C_\infty = \frac{g T}{2 \pi} \quad (III')$$

$$C = C_\infty \tanh \frac{2 \pi d}{C T} \quad (III'')$$

Finally, we have already quoted the formula that is valid along an orthogonal :

$$H_d = H_\infty D_\infty^d K_\infty^d, \quad (IV)$$

or in a more general way :

$$H_{d_q} = H_{d_p} D_{d_p}^{d_q} K_{d_p}^{d_q}, \tag{IV'}$$

$D_{d_p}^{d_q}$ is completely defined for a wave of given period; tables and diagrams make its computation possible as a function of period and depth. $D_{d_p}^{d_q}$ is easily deduced therefrom since $D_{d_p}^{d_q} = D_{d_p}^{d_p} D_{d_p}^{d_q}$.

On the other hand K , which depends on the shape of the bottom, must be computed separately by means of refraction diagrams for each case.

Standard Computation of the Refraction Coefficient

Figure 1 represents two contour lines Z_0 and Z bounding the area to be studied. We have plotted 2 orthogonals A_0A and B_0B which, if Z_0 borders the zone of infinite depth — and this is what we shall here assume — are parallel straight lines up to their arrival at A_0 and at B_0 . Then let M_0 and M be the mid points of A_0B_0 and AB respectively.

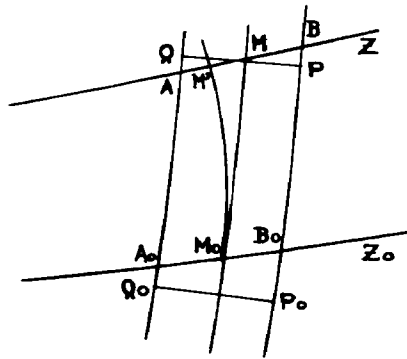


FIG. 1

Assuming that all the wave's energy lies between the two orthogonals, we may prove that the square of $K_{M_0}^M$ is equal to the ratio of intervals between adjacent orthogonals at M_0 and at M .

In the case of figure 1, we have :

$$(K_{M_0}^M)^2 = \frac{Q_0 P_0}{Q P} \tag{V}$$

The measurement of Q_0P_0 , the interval of two rectilinear and parallel wave rays, does not present any difficulties of principle. On the other hand, the interval QP at M can theoretically only be represented by a section of a straight line if the adjacent orthogonals are there both almost rectilinear and parallel. Generally speaking, M being approximately equidistant from the two orthogonals, the normals drawn from M onto each of them make a fairly noticeable angle.

Moreover, we assign the computed K value to point M , thus assuming that M_0M is on the same wave ray. However, this value of K is only a mean value which is only valid for M in the case of regular underwater topography

which generates regular orthogonals, i. e. in cases where there is little need for wave pattern diagrams. Let us imagine that between A_0A and B_0B the orthogonal coming from M_0 is plotted as M_0M' (figure 1). We may imagine what errors could be committed when the sea bottom is rough, unless a close-set network of orthogonals is available.

However, if the intervals are decreased, their measurement becomes inaccurate and the relative errors become rapidly prohibitive. There comes a moment when it is no longer desirable to increase the number of orthogonals.

These criticisms regarding the computation of K do not, in our opinion, justify the complaints made about the method which, after all, gives valuable, albeit approximate, information.

These criticisms have however prompted us to seek improvements, which finally seem to us to be satisfactory.

Computation of the refraction factor at any point on an orthogonal

Let us consider a wave of a given period propagating on the sea bottom between two contour lines Z_0 and Z .

Between Z_0 and Z we will plot $n-1$ closely placed contour lines, between two contour lines Z_0 and Z .

Let us now imagine an underwater bottom surface bounded by the same contour line $Z_0, Z_1, \dots, Z_{n-1}, Z$, which will be such that between two adjoining curves, Z_p and Z_{p+1} , the velocity remains constant and equal to C_{p+1} , corresponding to the depth of the contour line Z_{p+1} of the actual bottom. Between these two contour lines the bottom will therefore be horizontal and it will be at a depth equal to the depth of the contour line Z_{p+1} , the velocity for a wave of a given period, being solely a function of depth and conversely, (III). At Z_p the bottom will rise vertically until it reaches the level of Z_{p+1} , constituting a cylinder with Z_p as directrix and bounded by the Z_{p+1} horizontal level, (figure 2).

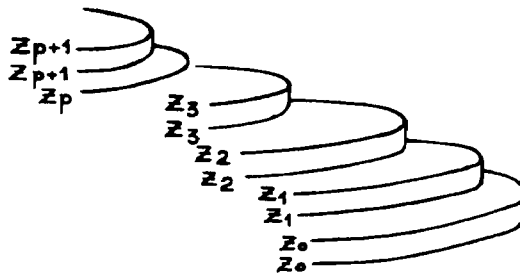


FIG. 2

The original underwater relief will therefore be replaced by a tiered relief, but which, if we increase the number of contour lines, will approximate the actual surface as closely as we wish. When n tends towards infinity, the two surfaces will coincide.

Let $X A_0$ be a wave ray making an angle of incidence i_0 with the normal $A_0 N$: this is refracted and makes an angle of refraction r_0 with this same normal when passing into the medium of velocity C_1 (figure 3). It will remain rectilinear between Z_0 and Z_1 and then between Z_1 and Z_2 , and so on, only changing angle when crossing the contour lines. Between A_0 and A the wave ray will appear in the form of a broken line made up of sections of straight lines which make angles of incidence and refraction i_p and r_p on any contour line Z_p .

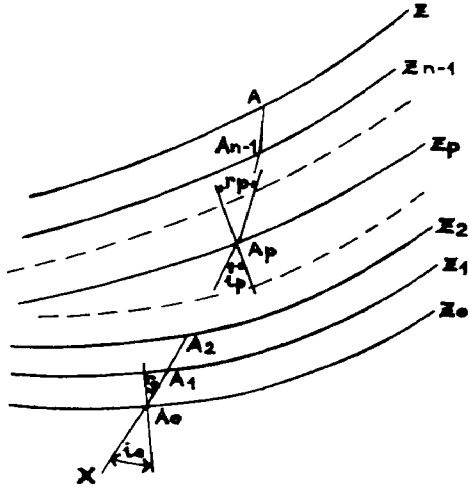


FIG. 3

Let us consider two adjacent contour lines, for example Z_0 and Z_1 (figure 4). The orthogonal being considered will be $X A_0 A_1$. Except in cases where $i_0 = 90^\circ$ and $i_1 = 90^\circ$ we may always plot a second orthogonal $X' A'_0 A'_1$ close enough to the first so that sections $A_0 A'_0$ and $A_1 A'_1$ may be taken as sections of the tangents at A_0 and at A_1 respectively, and so that the angle made by $X A_0$ and $X' A'_0$ is negligible compared with the difference between angles i_0 and r_0 . There refracted rays will likewise be parallel if terms of higher order are neglected.

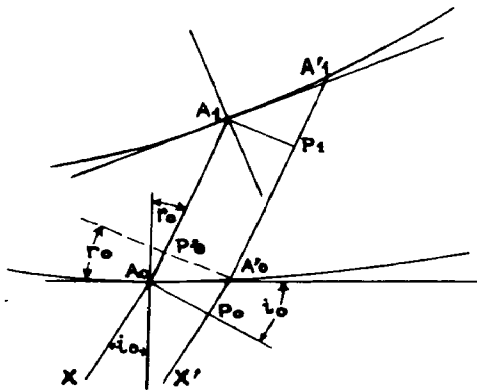


FIG. 4

Let us drop a perpendicular A_0P_0 onto $X'A'_0$ from A_0 (figure 4); and also $A'_0P'_0$ from A'_0 onto A_0A_1 . We then have $A'_0P'_0 = A_1P_1$.

According to the standard definition of the refraction coefficient between $A_0A'_0$ and $A_1A'_1$ the square of its value will be equal to

$$\frac{A_0P_0}{A_1P_1} \text{ or } \frac{A_0P_0}{A'_0P'_0} = \frac{A_0A'_0 \cos i_0}{A_0A'_0 \cos r_0} = \frac{\cos i_0}{\cos r_0}.$$

If A'_0 comes indefinitely closer to A_0 , then A'_1 will come indefinitely closer to A_1 and the refraction coefficient will no longer be defined as between two areas but as between two points A_0 and A_1 .

$$(K_{A_0}^{A_1})^2 = \frac{\cos i_0}{\cos r_0} \tag{1}$$

or :

$$\frac{1}{(K_{A_0}^{A_1})^2} = \frac{\cos r_0}{\cos i_0} \tag{1'}$$

Using the same reasoning we find :

$$\begin{aligned} \frac{1}{(K_{A_0}^{A_1})^2} &= \frac{\cos r_0}{\cos i_0} \\ \frac{1}{(K_{A_1}^{A_2})^2} &= \frac{\cos r_1}{\cos i_1} \\ &\dots\dots\dots \\ \frac{1}{(K_{A_{n-1}}^A)^2} &= \frac{\cos r_{n-1}}{\cos i_{n-1}} \end{aligned}$$

By multiplying member by member we obtain :

$$\frac{1}{(K_{A_0}^A)^2} = \frac{\cos r_0 \cos r_1 \cos r_2 \dots \cos r_{n-1}}{\cos i_0 \cos i_1 \cos i_2 \dots \cos i_{n-1}} \tag{2}$$

Tangents at A_0 and at A_1 make the angles ω_0 and ω_1 with a fixed direction (an arbitrary direction which could be the North). They intersect at O (figure 5), making an angle $\omega_1 - \omega_0 = \Delta\omega_0$.

Let us prolong the normals at A_0 and A_1 . These intersect at N with an angle $\Delta\omega_0$ and form triangle A_0A_1N .

Let us for the moment assume that $\Delta\omega$ *always remains positive*. For this purpose it is only necessary to choose an area where *the trend of the depth contour lines* always varies in the same direction, and also to make an appropriate choice for the *direction of the increasing ω* .

In figure 5 we see that $i_1 = r_0 + \Delta\omega_0$, and in figure 5' that $i_1 = r_0 - \Delta\omega_0$, if the angles of incidence and of refraction are reckoned as positive.

In either instance we shall always have :

$$i_1 = r_0 + \Delta\omega_0, \tag{3}$$

if we reckon i as *positive* when the *wave ray* is, in relation to the normal, *situated on the same side as point O* where the tangents intersect (figure 5) and as *negative in the opposite case* (figure 5').

r will have the same sign as i by virtue of (II).

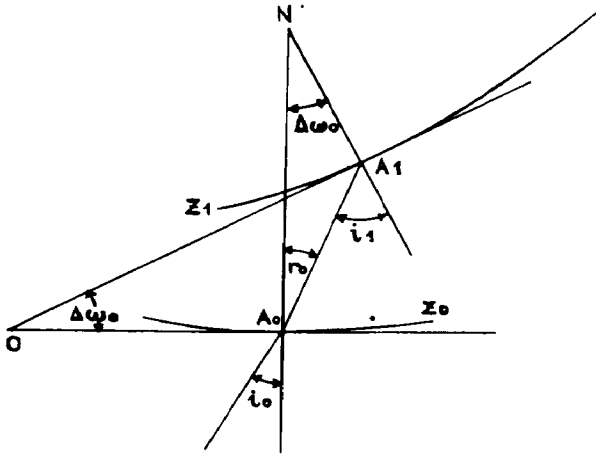


FIG. 5

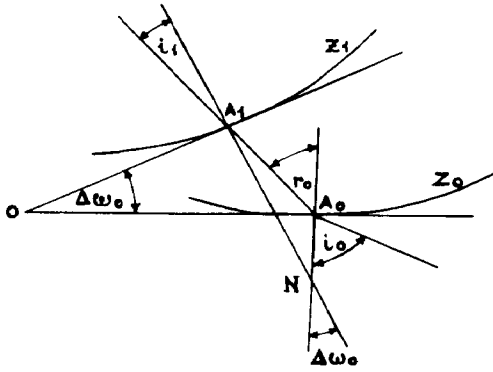


FIG. 5'

We may choose intermediate depth contour lines so that $\Delta\omega_0 = \Delta\omega_1 = \Delta\omega_2 = \dots = \Delta\omega_{n-1} = \epsilon$, ϵ being a small positive angle, equal to $\frac{\alpha}{n}$, when the tangents at A_0 and at A intersect with an angle α .

As a general rule we have :

$$i_{p+1} = r_p + \epsilon, \tag{3'}$$

and :

$$\cos r_p = \cos (i_{p+1} - \epsilon) = \cos \epsilon \cos i_{p+1} (1 + \tan \epsilon \tan i_{p+1})$$

The equality (2) becomes :

$$\frac{1}{(K_{A_0}^A)^2} = \frac{\cos^n \epsilon \cos r_{n-1} \times P}{\cos i_0} \tag{4}$$

by putting :

$$P = (1 + \tan \epsilon \tan i_1) (1 + \tan \epsilon \tan i_2) \dots (1 + \tan \epsilon \tan i).$$

When n tends towards infinity, i.e. when the number of tiers continues to increase and their height to be reduced, the fictitious surface being considered in figure 2 becomes increasingly similar to the real surface up to the moment when they merge together.

ϵ tends towards zero, $\cos \epsilon$ tends towards 1 and r_{n-1} tends towards i . It will only be necessary to determine the limit to which product P tends to obtain an accurate and precise value for the refraction coefficient.

Let us call M the absolute and maximum value which $\tan i$ may take. This, as we recall, remains finite ($i \neq \pm 90^\circ$).

Let us give n a sufficiently large value so that, since $\tan \epsilon$ is very small, the factors making up P will be not only positive but also very close to 1.

Using Napierian logarithms we obtain :

$$\log_e P = \log_e (1 + \tan \epsilon \tan i_1) + \log_e (1 + \tan \epsilon \tan i_2) + \dots + \log_e (1 + \tan \epsilon \tan i_{n-1}) + \log_e (1 + \tan \epsilon \tan i)$$

Expanding the various logarithms of the above sum in series, and putting them in columns, we shall have :

$$\begin{aligned} \log_e P = & \tan \epsilon \tan i_1 + \tan \epsilon \tan i_2 + \dots + \tan \epsilon \tan i_{n-1} + \tan \epsilon \tan i \\ & - \frac{1}{2} \tan^2 \epsilon \tan^2 i_1 - \frac{1}{2} \tan^2 \epsilon \tan^2 i_2 - \dots - \frac{1}{2} \tan^2 \epsilon \tan^2 i_{n-1} - \frac{1}{2} \tan^2 \epsilon \tan^2 i \\ & \dots \dots \dots \\ & \pm \frac{1}{m} \tan^m \epsilon \tan^m i_1 \pm \frac{1}{m} \tan^m \epsilon \tan^m i_2 \pm \dots \pm \frac{1}{m} \tan^m \epsilon \tan^m i_{n-1} \pm \frac{1}{m} \tan^m \epsilon \tan^m i \\ & \dots \dots \dots \end{aligned}$$

For the first line we may write :

$$\frac{\tan \epsilon}{\epsilon} \frac{\alpha}{n} (\tan i_1 + \tan i_2 + \tan i_3 + \dots + i_{n-1} + \tan i)$$

The sum within the brackets, divided by n , is in fact the mean value of $\tan i$ between ω_0 and ω , the difference $\omega - \omega_0$ being equal to α . When n tends towards infinity, $\frac{\tan \epsilon}{\epsilon}$ tends towards 1. Our first line tends towards $\int_{\omega_0}^{\omega} \tan i d\omega$.

Let us examine the q line, which may be written as :

$$\frac{(-1)^q}{q} \tan^q \epsilon (\tan^q i_1 + \tan^q i_2 + \dots + \tan^q i_{n-1} + \tan^q i)$$

The sum within the brackets is made up of n terms, each one having an absolute value smaller than M^q . The absolute value of this sum will be smaller than nM^q .

In order to simplify the notations we shall give the product $M \tan \epsilon$ the notation x . The absolute value of the entire q line will be smaller than :

$$\frac{n \tan^q \epsilon M^q}{q} = \frac{n x^q}{q}$$

The sum of lines giving the value of $\log_e P$, omitting the first, has a smaller absolute value than :

$$n \left(\frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \dots + \frac{1}{q} x^q + \dots + \frac{1}{m} x^m + \dots \right)$$

(*) The signs of each line are alternately + and —; if m is odd the sign will be +; if m is even the sign will be —.

i.e., by replacing the series within the brackets by its total, smaller than :

$$n [-\log_e (1 - x) - x].$$

This expression tends towards zero when n tends towards infinity. In effect :

$$x = M \tan \varepsilon = \frac{M \tan \varepsilon}{\varepsilon} \varepsilon = \frac{M \tan \varepsilon}{\varepsilon} \cdot \frac{\alpha}{n}$$

whence :

$$n = M \frac{\tan \varepsilon}{\varepsilon} \cdot \frac{\alpha}{x}$$

We have :

$$n [-\log_e (1 - x) - x] = -M \alpha \frac{\tan \varepsilon}{\varepsilon} \frac{\log_e (1 - x) + x}{x}$$

which tends towards zero, as we wished to demonstrate, when n tends towards infinity while x tends towards zero.

The limit of $\log_e P$ is thus : $\int_{A_0}^A \tan i \, d\omega$, when we are coming close to the actual under-water bottom surface, and :

$$(K_{A_0}^A)^2 = \frac{\cos i_0}{\cos i} e^{-\int_{A_0}^{A_1} \tan i \, d\omega}$$

The integral $\int_{A_0}^A \tan i \, d\omega$ is easy to compute. The curve representing $\tan i$ as a function of $d\omega$ is plotted taking a scale suited to the desired accuracy for the abscissae and the ordinates. Finally it will only be necessary to measure the area bounded by the curve, the ordinates ω_0 and ω the axis of the abscissae. The signs of these areas will be those of $\tan i$ since $d\omega$ has been assumed to be always positive. The integral in question could equally well be computed in an approximate way — and this would often be sufficient — either by the trapeze method or by Simpson's method.

We have assumed for the ease of computation that the ω angle would always vary in the same direction. It is easy to give this formula a more general value by eliminating this troublesome restriction.

Let us first note that when ω is constant the integral is zero (the area defined above is confined to a section of an ordinate) and that $e^0 = 1$. The formula then becomes :

$$(K_{A_0}^A)^2 = \frac{\cos i_0}{\cos i} \tag{V}$$

which is a standard formula directly established in the case where the contour lines are parallel straight lines. As for our formula, this is not valid when i or i_0 is equal to 90° , an eventuality which would merit special study.

Returning to the case where ω varies in some way or another. On the $B_0B_1B_2B$ curve (figure 6) representing tangent i versus ω , B_0 corresponds to ω_0 — the direction of the tangent at A_0 to the contour line — B corresponds to ω — the direction of the tangent at A . Points B_1 and B_2 , where the tangent to the curve in the figure is vertical, correspond to the points A_1 and A_2 of the orthogonal where ω changes to the opposite direction.

The formula is therefore applicable between A_0 and A_1 ; between A_1 and A_2 , and between A_2 and A . Since we have :

$$K_{A_0}^A = K_{A_0}^{A_1} \cdot K_{A_1}^{A_2} \cdot K_{A_2}^A,$$

then :

$$(K_{A_0}^A)^2 = \frac{\cos i_0}{\cos i} e^{-\int_{A_0}^{A_1} \tan i d\omega - \int_{A_1}^{A_2} \tan i d\omega - \int_{A_2}^A \tan i d\omega}$$

But beyond A_1 , at the same time as $d\omega$ changes sign, $\tan i$ must also change its sign because the tangents to the contour line intersect on the other side of the normal. The integral here computed with $d\omega$ constantly positive must likewise change sign. In the case of figure 6, the area swept

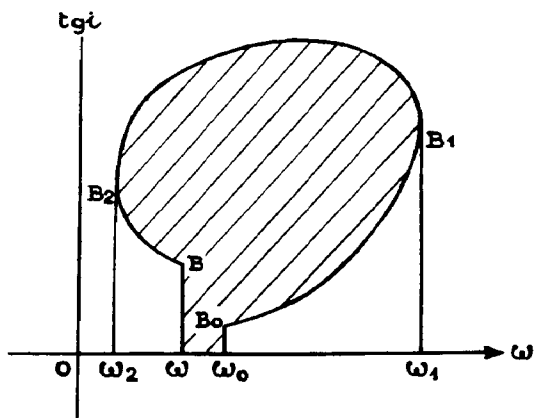


FIG. 6

by the ordinate is positive from ω_0 to ω_1 , negative from ω_1 to ω_2 and then again becomes positive after ω_2 . It is seen that the total area will be negative. This corresponds to the hachured area on figure 6. The sum of the 3 integrals which make up the exponent of e may be replaced by a single integral from A_0 to A . The formula :

$$(K_{A_0}^A)^2 = \frac{\cos i_0}{\cos i} e^{-\int_{A_0}^A \tan i d\omega} \quad (5)$$

is therefore valid for any shape of submarine relief, provided, however, the slope is sufficiently small for the reflected energy to remain negligible.

The possibility of having available accurate coefficients of refraction in a whole variety of cases — as are actually encountered — has manifold applications. For example, it would seem that this should facilitate the study of intersecting orthogonals.

However, our purpose is not to pursue our investigations in this direction, for we have a more urgent task.

The basic formula which we have established can only be used really successfully when we know how to plot accurate orthogonals, but it seems that the methods currently used are far from perfect. We shall therefore try to improve them.

Plotting of refraction diagrams

We are not the first to have had this idea : already much valuable progress has been made. Thus the wave crest method after being strongly criticized by Dr. PIERSON has been almost completely rejected by serious users. In the same way Professor LACOMBE has shown that grave errors could be avoided by using directly the relation deduced from Descartes Law (II) and not by employing a derivative formula.

However, whatever the precautions taken and the accuracy sought, truly satisfactory results cannot be expected as long as the orthogonals are plotted in too long sections along which the velocity is assumed constant. As Professor LACOMBE has remarked, this is the same as if the actual submarine topography — which is never known with perfect accuracy but which is usually thought to be different from a tiered formation — was replaced by a stepped relief with a number of depth contours in common with the actual topography, but constituted by vertical walls separated by horizontal areas.

When constructing a small-scale model, walls of suitable height are first established along contour lines plotted on the floor. These walls retain filling materials at each level. However hydraulic tests are never undertaken before the brick-layer has connected the tops of the various walls by fashioning with his trowel slopes which are arbitrary but which represent fairly closely what is actually the case.

Let us repeat that hydrographic surveys are never strictly accurate. They only offer an approximate representation of the bottom relief which is never absolutely stable. The template in which the vertical dimensions are too small for the full use of accurate soundings adds its own distortions to an already imperfect representation due to the limitations of the brick-layer's craft.

However, nobody can deny that valuable information is deduced from tests on a small-scale model which only represents a surface approximating the actual surface.

We now hope to be able to conceive a computational process which, with slopes which are arbitrary but close to reality, will allow us to fill the abnormal gaps still existing between the various tiers when the present procedures are used. To this end we shall increase to infinity the number of intermediate depth contours as was the case with the computation of the refraction coefficient. The height of the tiers will become smaller and smaller, but we will select a general lay-out without any restriction except the points of departure and of arrival. If, finally, we have removed artificial and excessive discontinuities which are due to the use of unperfected processes, we will only obtain a fictitious relief, invented by our computations. This will no doubt have the merit of being not too different from the actual relief, still not perfectly known.

Definition of the adopted fictitious surface

Let us assume that a sub-marine topography is reliably represented by a number of contour lines. This means that the lines must be sufficiently close so that between two neighbouring lines the slope varies, always in the same direction, either continuously increasing or decreasing, throughout almost all the area under consideration. This is in any case what we shall assume.

Between two adjacent depth contours Z and Z_0 , we propose to plot an orthogonal, corresponding to a particular wave, arriving at A_0 with an angle of incidence i_0 and after refraction ending on Z at A . Velocity is C_0 along Z_0 and C along Z . The tangents at A_0 and A to the contour lines intersect at O with an angle α . The angle made by $O A_0$ with the direction of origin coming from O is ω_0 . Each point of $A_0 A$ may be defined in a polar coordinate system, with O as pole, by the radius vector ρ and the angle ω (figure 7).

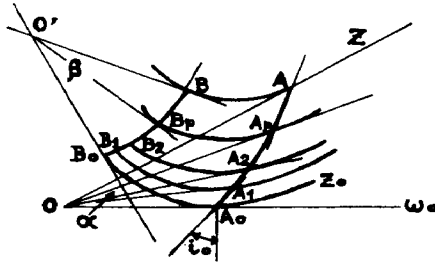


FIG. 7

The cylinder which is based vertically on $A_0 A$ cuts the actual sub-marine relief making a curve whose slope in relation to the horizontal will be either always increasing or always decreasing, as we have assumed. We shall now replace this slope by a curve F , which will be very close to it since it will have the same horizontal projection, the same point of departure and the same point of arrival. Between the two, the depth of its various points will vary continuously so that the corresponding velocity starting from C_0 at point A_0 will become progressively C at point A , according to a law which we shall choose arbitrarily. This law will be defined by :

$$\frac{d C}{C} = - k d \omega , \tag{6}$$

where k is a constant to be computed for each particular case.

Integration between A_0 and A gives :

$$\log_e \frac{C}{C_0} = - k (\omega - \omega_0) \tag{7}$$

Thus, giving the values they have at A to ω and C :

$$k = \frac{- \log_e \frac{C}{C_0}}{\alpha} \tag{7'}$$

and :

$$C = C_0 e^{-k(\omega - \omega_0)}. \tag{8}$$

k is positive when the bottom is rising, as is generally the case when going from deep sea towards the coast.

The surface generated by such curves as F corresponding to such orthogonals as A_0A for a particular wave is very close to the actual submarine relief since F is very close to this relief. It is this surface which we shall choose for determining the orthogonals, and we shall call it Σ .

Let us now imagine that by means of $n-1$ straight lines coming from O we divide the angle $\widehat{A_0OA}$ into n equal parts $\frac{\alpha}{n}$. These straight lines meet the orthogonal A_0A at points $A_1, A_2, \dots, A_p, \dots, A_{n-1}$.

Taking the straight line OA_p , the angle $\widehat{A_0OA_p}$ will be equal to p times $\frac{\alpha}{n}$, i.e. $\frac{p}{n} \alpha$. Let us carry out the same operation for all orthogonals such as A_0A , for example B_0B . The tangents at B_0 and B intersect at O' with an angle β which we shall divide into n equal parts. The straight line $O'B_p$ will be homologous to the straight line OA_p . The family of straight lines such as OA_p has an envelope whose A_p, B_p and the other similar points are limit points. We shall call this envelope Z_p and we shall say that it is a contour line of Σ .

In effect, the angle $\omega_p - \omega_0$ is equal to $\frac{p}{n} \alpha$. The angle $\omega'_p - \omega'_0$ is equal to $\frac{p}{n} \beta$. These two angles are different but the exponents of e in formulas (8) relating to A_p and to B_p are identical since both are equal to :

$$\frac{p}{n} \log_e \frac{C}{C_0}.$$

The velocity and thus the depth are therefore identical at A_p and B_p , i.e. over the whole Z_p curve. Clearly this is a contour line of the surface Σ as we wished to establish.

Between Z_0 and Z there are an infinity of Σ surfaces since Σ depends on the swell under consideration. However they are all very close to the actual surface and this is important.

For computing A_0A we shall take as an intermediate step the particular surface generated by the horizontals resting on the F curve and on the vertical of O . This surface, which resembles a hyperbolic paraboloid, is tangent to the Σ surface along F .

We shall deal with the particular case in which the contour lines are converging straight lines.

When the contour lines are converging straight lines

Let us plot $n-1$ straight lines between OZ_0 and OZ (figure 8), such as $OA_1, OA_2, \dots, OA_p \dots OA_{n-1}$, which are contour lines. As in the

case of the computation of the coefficient of refraction we should replace our surface by a tiered surface which resembles a corkscrew staircase, and we shall use the same rules for signs as for formula (3).

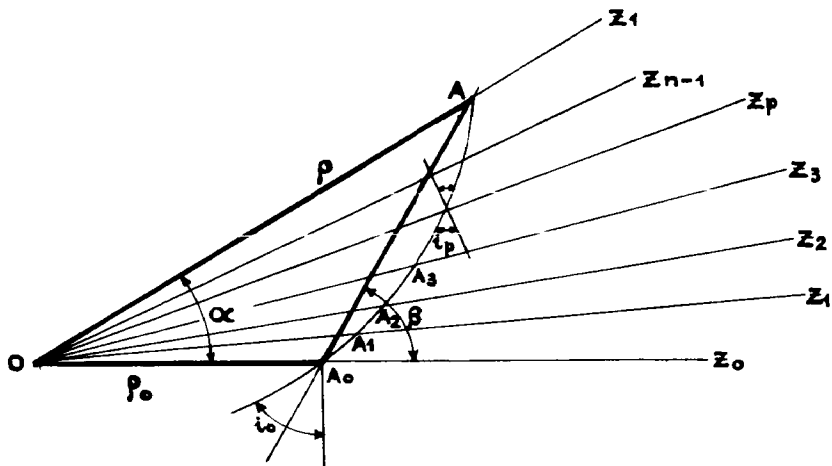


FIG. 8

In the case of figure 8, we see for example that i is positive. Snell's Law gives us :

$$\frac{\sin i_p}{C_p} = \frac{\sin r_p}{C_{p+1}} = \frac{\sin r_p - \sin i_p}{C_{p+1} - C_p} = \frac{2 \sin \frac{r_p - i_p}{2} \cos \frac{r_p + i_p}{2}}{C_{p+1} - C_p}$$

or :

$$\frac{C_{p+1} - C_p}{C_p} \frac{\sin i_p}{\cos \frac{i_p + r_p}{2}} = 2 \sin \frac{r_p - i_p}{2}$$

For any point on the orthogonal, if we cause n to increase indefinitely by neglecting the terms of higher order and by replacing i_p by i , i_{p+1} by $i + di$, r_p by r , C_p by C , C_{p+1} by $C + dC$, we shall find :

$$\tan i \frac{dC}{C} = r - i$$

Likewise the relation (3') becomes :

$$r - i = di - d\omega. \tag{3''}$$

whence :

$$di = d\omega + \frac{dC}{C} \tan i. \tag{9}$$

This formula is *general*. Now by assuming the variation of C along the orthogonal to be ruled by the arbitrary law defined by the formula

$$\frac{dC}{C} = -kd\omega, \text{ the particular formula is obtained :}$$

$$di = d\omega (1 - k \tan i) \tag{10}$$

It is easy to integrate by putting $k = \tan \varphi$.

Continuing to call α the angle (by definition positive) between the straight lines OZ_0 and OZ , we find :

$$\alpha = \cos^2 \varphi (i - i_0) - \sin \varphi \cos \varphi \log_e \frac{\cos (\varphi + i)}{\cos (\varphi + i_0)} \quad (11)$$

with :

$$\tan \varphi = \frac{-\log_e \frac{C}{C_0}}{\alpha} \quad (12)$$

However, when considering function $\rho = f(\omega)$, representing the A_0A curve in polar coordinates ($OA = \rho$ and $OA_0 = \rho_0$) we know that $d\rho = \rho \tan i d\omega$. An easy integration will give :

$$\log_e \frac{\rho_0}{\rho} = \sin \varphi \cos \varphi (i - i_0) + \cos^2 \varphi \log_e \frac{\cos (\varphi + i)}{\cos (\varphi + i_0)} \quad (13)$$

This last equation, after taking equation (11) into account, can be replaced by the following and more simple equation :

$$\log_e \frac{\rho_0}{\rho} = \frac{i - i_0 - \alpha}{\tan \varphi} \quad (14)$$

or again :

$$\rho = \rho_0 e^{-\frac{i - i_0 - \alpha}{\tan \varphi}} \quad (14')$$

However we shall also define and compute a new angle, β , which will be of prime importance when we pass from the case of straight lines to the general case.

The A_0A chord makes an angle which we shall call β with A_0Z_0 . In triangle A_0OA , we have :

$$\frac{\sin \beta}{\rho} = \frac{\sin (\beta - \alpha)}{\rho_0} = \frac{\sin \beta \cos \alpha - \sin \alpha \cos \beta}{\rho_0}$$

or :

$$\cotan \beta = \cotan \alpha - \frac{\rho_0}{\rho \sin \alpha} \quad (15)$$

or again :

$$\cotan \beta = \cotan \alpha - \frac{e^{-\frac{i - i_0 - \alpha}{\tan \varphi}}}{\sin \alpha} \quad (15')$$

By forming a system of 3 equations, for example (12), (11) and (14'), we could compute φ , then i and ρ in terms of ρ' , C , C_0 , i_0 and α , which are known.

We prefer however to compute φ , i and β by means of equations (12), (11) and (15'). The second method has many more advantages. Both arrive at the accurate determination of a point on an orthogonal and of its tangent. However, thanks to this second method, we shall be able to solve the problem quite straight forwardly in the general case where α is one of the unknowns.

The case of any kind of depth contour

Returning to the general case (figure 9), let us replace the submarine relief first of all by the surface Σ , which resembles it very closely, then by the surface which is tangent to Σ and generated by horizontals resting on both the F curve and the vertical from O.

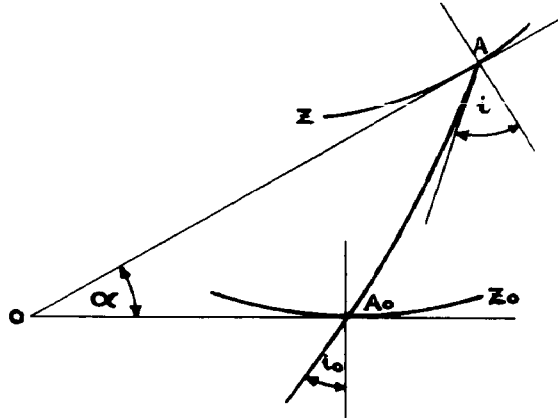


FIG. 9

The above equations, and in particular equations (12), (11) and (15'), are entirely valid for Σ , but we now have a further unknown α and there would seem to be a missing equation.

In fact, the shape of the Z contour line forms a fourth relation, and we will see that it will be possible to find a solution.

Our first task will be to work out tables or graphs giving i and β as functions of α and i_0 for several values of $\frac{C_0}{C}$. This is a most tiresome and lengthy task when a table of logarithms only, instead of an electronic computer, is available.

However it would certainly seem that we could profitably concentrate both our efforts and expenditure in this direction.

We have contented ourselves with gathering the data which allow i and β to be computed for a *single* value of $\frac{C_0}{C}$ as determined by $\log \frac{C_0}{C} = 0.01$.

For β we have drawn up a double graph, represented in plate I, where a family of α curves and a family of i_0 curves are shown. We expected to be able to take concentric circles for the α curves but since we could not anywhere find polar graph paper we have used ordinary rectangular grid paper, and we have adopted rectangular curves, the inner rectangle OABCDO (fig. 10) corresponding to $\alpha = 0$ and the outer rectangle to 90° . It would seem superfluous to go any further. Moreover we do not recommend choosing the Z_0 and Z contour lines whose tangents have turned more than about 50° - 60° .

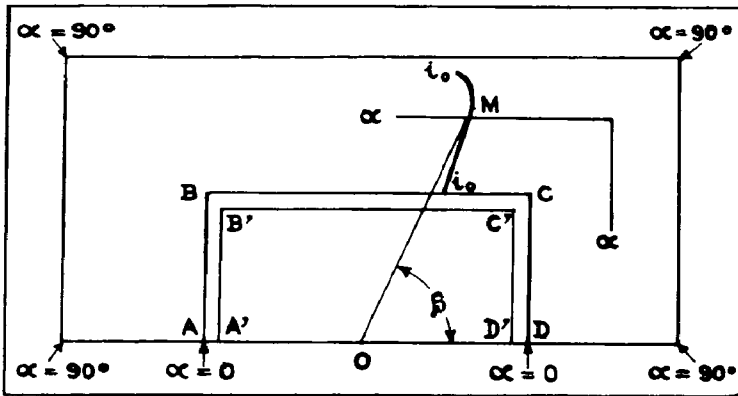


FIG. 10

Therefore the use of an intermediate contour line seems indicated. It will be seen that it has not been possible to graduate the $\alpha = 0$ curve in i_0 by means of our formulae for their terms are becoming indeterminate and the formulae were set out assuming that $\alpha \neq 0$. However if we make α tend toward zero, φ will tend toward $\frac{\pi}{2}$, (11) merely becomes Snell's Law and (15') will become :

$$\cotan \beta = \frac{i - i_0}{\log_e \frac{C_0}{C}} \tag{16}$$

This formula can easily be found by direct computation.

The i_0 curves have been determined in such a way that the angle \widehat{DOM} is equal to β if M is the point where an α curve encounters an i_0 curve.

The window A'B'C'D' has been constructed inside the rectangle ABCD and the borders A'B'C'D' have been provided with a scale of angles i_0 .

In order to make the plotting of orthogonals easier, it seems essential to have a specially prepared chart on which it will be easy to plot successive contour lines so that their velocities $C_0, C_1, C_2, C_3 \dots$ fulfill the condition $\frac{C_0}{C_1} = \frac{C_1}{C_2} = \frac{C_2}{C_3} = \frac{C_3}{C_4} = \dots = \gamma$ designating the common value of these ratios.

Thus, M. LAVAL, Ingénieur Général des Ponts et Chaussées, has recommended in his treatise on Maritime Engineering the plotting of "lines of equal values of $\frac{dC}{C}$ " which amounts to the same.

It would seem that it is at this point that the hydrographic surveyor should step in.

The hydraulics engineer must not be allowed to conceive these important curves, which will have both submarine relief and surface Σ in common. They should indeed be *smoothed*, that is, improved in a more or less arbitrary fashion.

Certainly the hydrographic surveyor, in order to allow the navigator to

be the judge in the last resort, considers he must take into account all the soundings, even when some are abnormal. These soundings are only deleted when they have been proved false without any possible doubt. This scruple often results in festoons of fairly complicated contour lines which are not likely to exist in reality. Once he knows what he must achieve the hydrographer will be better placed than anyone else for smoothing out the irregularities which he thinks should be attributed to opposite errors around the actual sounding. However he will always carefully keep the figures which seem to correspond with the unevenness of the relief. In our opinion, we shall in this way have contour lines which represent the general aspect of the bottom better than a standard chart, although data are left out which could be of value to the mariner who must rely only on the standard data of regulation documents.

As he possesses this main network of smoothed contour lines, the hydraulics Engineer should draw as accurately as possible the contour lines corresponding to C_0, C_1, C_2, \dots , that change not only with the value of γ but also with the period of the wave.

The hydrographer will of course take the plotting sheets into account but he will also draw on his experience and his knowledge of the marine topography. To the too widely spaced contour lines whose depths are given on the charts he will add intermediary and more closely spaced contour lines which will have under 2 mm between them.

We have also seen that the α angle plays an essential role. It would not be sensible to make great efforts to use correct but more intricate methods if the accuracy of angle α was not being sought. It is therefore suggested that on special grids where families of smoothed contour lines are already shown these lines should be graduated in azimuths of their tangents. These azimuths will be between 0° and 179° . The homologous points will subsequently be joined by curves numbered at every 10° only.

Hydrographic offices employing highly qualified draughtsmen are obviously indicated for the successful execution of this work which, moreover, is not entirely graphic and should therefore be supervised by a hydrographic engineer. It could happen in fact that rectification of smoothed contour lines becomes necessary.

Thus a double network of curves is available. *Black* could be used for drawing the contour lines and *red* for the curves of equal azimuth. We shall call these last the "red curves".

Let us imagine that we have a chart which includes all this information. We have then the necessary tools to apply with an improved method our plotting process to the particular case of a given wave.

Generally speaking a line which corresponds to $d = \frac{L_\infty}{2}$ is taken as the initial line and we assume that $C_0 = C_x$. As γ is known, C_1, C_2, C_3, \dots etc. may be deduced. The tables established from (III'') now give d_1, d_2, d_3, \dots etc. If possible we shall show the corresponding contour lines in various colours. Perhaps some have already been plotted but in any case it will be easy to interpolate between lines spaced less than 2 mm apart and to obtain an accuracy compatible with the scale of the chart.

We now start the actual plotting of the orthogonal. In general the azimuth of the first incident ray coming from the open sea has been given. This arrives at A_0 where the tangent's direction is read on the "red" curves. i_0 can therefore be accurately determined.

Let us consider Z_1 . With the help of the red scale and subtracting the azimuth read at A_0 , let us graduate in pencil in terms of α the portion which interests us, that is the portion where it would seem that the wave ray must end (*).

Let us now take *one* of the two graph templates corresponding to the value of $\frac{C_0}{C}$ adopted, for there are two of these graphs which are symmetrical at O in relation to the perpendicular onto the AD side of the rectangle (fig. 10). One is used when the tangents at A_0 and A_1 intersect to the right of the normal and the other for the opposite case. Confusion is avoided by means of the sign of i_0 .

Let us place this template over the chart so that O coincides with A_0 and so that the base (AOD) of the rectangle coincides with the tangent at A_0 , or rather that the incident ray, when prolonged, reaches the values of i_0 on the border scale $A'B'C'D'$.

Then from O let us make a taut wire pivot following the i_0 curve, making a straight line that, through the window $A'B'C'D'$, can be seen to traverse the α scale of Z_1 .

When the values of the i_0 curve scale and of the contour line on meeting the wire are identical, the wire is oriented following A_0A_1 . A_1 is then situated at the intersection of the wire and Z_1 , and here the value of α can be read. With α and i_0 , i_1 is computed by means of a table or a template established for the chosen value of $\frac{C_0}{C}$.

A point on the orthogonal and its corresponding tangent have therefore been determined. From here onwards the operation is restarted and is continued.

In the most general case of a system of complex relationships allowing α and β to be determined in terms of $\frac{C_0}{C}$ the double template is indispensable for accurate resolution and should be inscribed on plexiglass with the negative i_0 curves in red.

We shall thus obtain a kind of protractor, one only for each value of γ , since both faces can be used. A minimum of two is necessary, one with $\gamma > 1$ and the other with $\gamma < 1$ according to whether the depth decreases or increases, in order to be able to apply this process.

However, it is not necessary to increase the number of these protractors. With $\log \gamma = 0.01$, for a wave of $L_\infty = 100$ m ($T = 8$ seconds) we have used a series of contour lines whose depths from $\frac{L_\infty}{2} = 50$ onwards were :

(*) Of course, if along the section of Z_1 involved a sufficiently small curvature allows us to evaluate α practically without error, the plotting of the orthogonal can be achieved by more simple procedures.

33.48 - 27.99 - 24.54 - 21.99 - 19.97 - 18.29 - 16.86 - 15.62 - 14.53 - 13.55 - 12.66 - 11.87 - 11.14 - 10.47 - 9.85 - 9.28 - 8.77 - 8.28 - 7.80 - 7.39 - 6.62 - 6.28.

The contour lines quickly become dense enough, even to the point of being inconvenient. The only template ($\log \gamma = 0.01$) which we worked out without aiming for great accuracy allowed us to plot some very satisfactory orthogonals. However we should add that in the particular case involved the depth was always decreasing, and between 50 and 25 m the slope was steep. In spite of the scale adopted (1/5 000) the contour lines in this area were sufficiently close so that there was no need for intermediary contour lines.

However (if because the slope changes direction, we need to obtain on the orthogonal a point nearer the crest or the thalweg than the last contour line it has been possible to use) it can happen that it is necessary to determine the point where the wave ray intersects a contour line not appearing in the set we have chosen.

Except between Z_0 and Z_1 (*), a solution exists which enables the same protractor to be used.

Let us assume (figure 11) that the contour line Z' under consideration is situated a little beyond the last contour line it has been possible to process. This line's velocity C' is between C_p and C_{p+1} . Let us compute C'_0 so that $\frac{C'_0}{C'} = \gamma$ and let us make the contour line Z'_0 , situated between Z_{p-1} and Z_p , correspond to it. The wave ray, already plotted between A_{p-1} and A_p , intersects Z'_0 at A'_0 where i'_0 is known. A' can therefore be determined from A'_0 , always using the protractor corresponding to γ .

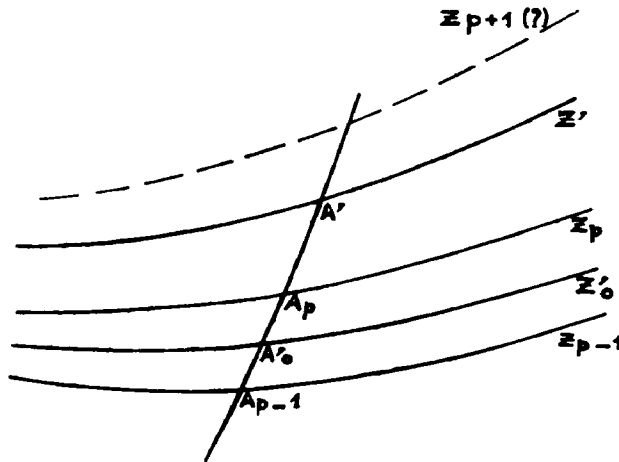


FIG. 11

If Z_{p+1} exists and if the relief between Z_p and Z_{p+1} does not present anomalies it is more logical to determine A' by the intersection of its

(*) Even in this case a less simple but more general solution can be used.

contour line with the arc $A_p A_{p+1}$. Theoretically the point thus determined is not exactly the same as that found by the procedure just specified. We have in effect used Σ surfaces which are not identical. Very fortunately the discrepancy is not perceptible when plotted if the contour lines on which the Σ surfaces are based are sufficiently close.

The case of submarine relief showing crests and thalwegs

Generally speaking, the set of Σ surfaces represents the submarine relief correctly. They link up with one another making an angle which is not zero but which is small since any two surfaces when joining together are both close to the actual topography.

However, we have had to set aside the case when, between the two contour lines on which these surfaces are based, the depth and consequently the velocity pass through a minimum or a maximum. We shall briefly study this case, taking the case of a minimum (a crest). If it were a question of a maximum (a thalweg) our chain of reasoning would be the same.

The depth decreases along the orthogonal A_0A , going through a minimum at A' , and then increases.

Let us consider the vertical cylinder with $A_0A'A$ as directrix which we shall assume to be plotted in the horizontal plane, the surface of still water. This cylinder cuts the submarine surface with a curve G . B_0, B' and B are points at which the verticals of A_0, A' and A cut this surface. Let us develop the cylinder starting from the generatrix A_0B_0 up to the generatrix AB (figure 12).

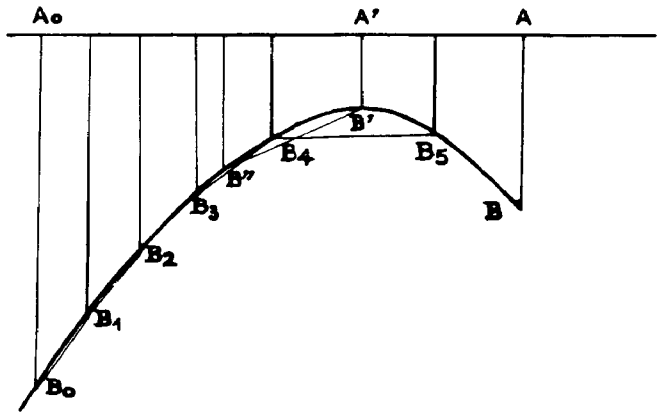


FIG. 12

In figure 12 which represents the developed cylinder, $A_0A'A$ becomes a horizontal straight line and G a new curve $B_0B'B$. The verticals (such as AB) represent the depths.

Velocity could be represented by a similar curve, with a minimum of similar trend at the point homologous to B' , but it would be more flattened

if the metre per second were taken as the unit for velocity and the metre as the unit for depth.

For a wave with an 8-second period, which is what we have been studying, in depths of about 8 metres — more exactly 8.18 metres — (the figures for the depth in metres are the same as those for the speed in metres per second), the velocity, however, increases more slowly than the depth. In depths of 50 m it is only a quarter this figure.

Let us anyhow start by applying our method. B_1, B_2, B_3, \dots , will correspond to contour lines Z_1, Z_2, Z_3, \dots , since γ has a given value. By replacing the actual submarine topography with Σ surfaces we have replaced the B_0B' curve by a broken line $B_0B_1B_2\dots$. The chords such as B_0B_1 are not entirely rectilinear, but if the successive velocities $C_0C_1, C_1C_2, C_2C_3, \dots$ are very close ($\gamma \simeq 1$) they will be very nearly so.

The broken line in question is sufficiently close to the actual curve, but it stops at B_4 for the depth corresponding to B_5 is less than the depth for B' .

We may prolong this line by the horizontal line $B_4B'_5$, the velocity remaining constant during this interval — i.e. there will be no refraction — but this step is only satisfactory when B_4 is very close to B' .

We can equally well proceed as indicated above, i.e. by making a point B'' correspond to B' , so that $\frac{C''}{C'} = \gamma$. This solution is more convenient particularly for the exceptional case of an angular point at B' (as the case of an abrupt ridge in the relief) and in this case A' on the wave ray will also be an angular point.

Let us review the problem anew, assuming that the tangent at B' to the depth curve is horizontal (i.e. that there is no angular point) and eliminating the following two eventualities that are very rarely encountered.

a) B' is not the culminating point of the relief.

b) The crest line passing through B' is not horizontal. In other words the contour line passing through B' is not the crest line.

As a result of these assumptions the orthogonal is tangent at A' to Z' , whence $i' = 90^\circ$ (figure 13).

Let us take the same notations and conventions as used when the formulas (11) and (13) were established. We now no longer know the contour line, but we do know the angle of incidence.

Formula (9) remains valid :

$$di = d\omega + \frac{dC}{C} \tan i \quad (9)$$

As previously let us put :

$$\frac{dC}{C} = -k d\omega$$

However along our orthogonal, k can now no longer be a constant since C is minimum at A' . Again, for reasons of convenience of compu-

tation only, we shall arbitrarily put $k = \frac{m \cos i}{1 + m \sin i}$, m being a constant. For $i = 90^\circ$ this k function is cancelled out. We have :

$$\frac{dC}{C} = \frac{-m \cos i}{1 + m \sin i} d\omega \tag{17}$$

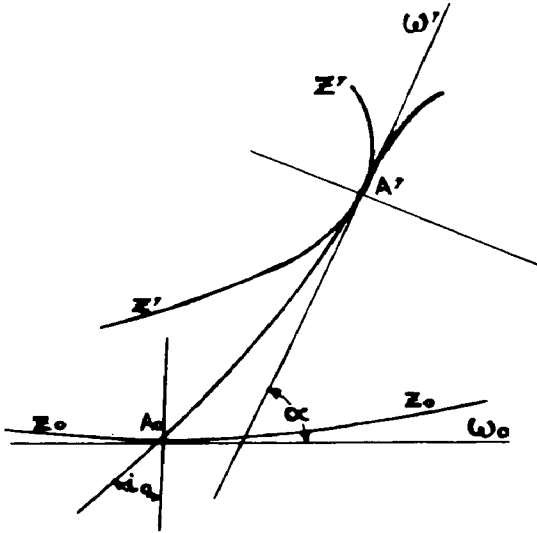


FIG. 13

Replacing $\frac{dC}{C}$ in (9) by this value we get :

$$di (1 + m \sin i) = d\omega$$

Integrating between ω_0 and ω :

$$i - i_0 - m (\cos i - \cos i_0) = \omega - \omega_0$$

At A' , for $i = 90^\circ$ we have :

$$\alpha = \frac{\pi}{2} - i_0 + m \cos i_0. \tag{18}$$

Likewise, by integration of (17) which may be written as :

$$\frac{dC}{C} = -m \cos i di, \tag{17'}$$

we obtain :

$$\log_e \frac{C}{C_0} = -m (\sin i - \sin i_0)$$

And at A' for $i = 90^\circ$:

$$C' = C_0 e^{-m(1 - \sin i_0)} \tag{19}$$

We shall determine m by considering what (17) becomes in the particular case of point A_0 . Re-introducing k we have :

$$\left(\frac{dC}{d\omega} \right)_0 = -k_0 C_0 \tag{17''}$$

The tangent to the orthogonal at A_0 encounters the contour lines of the special grid at points where :

ω — thanks to the red curves,

C — by means of standard tables in terms of depths or more exactly

in terms of $\frac{d}{L_0}$,

are accurately known.

It is therefore possible to plot an arc of the curve representing C in terms of ω . Its slope $\left(\frac{dC}{d\omega}\right)_0$ could then be determined for ω_0 and (17'') will give the value of k_0 .

Now

$$k_0 = \frac{m \cos i_0}{1 + m \sin i_0}$$

whence :

$$m = \frac{k_0}{\cos i_0 - k_0 \sin i_0}$$

Since m is known we also know α ; ω_0 will allow the computation of ω' which is the azimuth of the tangent at A' . A first locus of A' will be the red curve graduated in terms of ω' . A second will be the Z' contour line which is known since (19) gives us C' . The position of A' is therefore determined.

The new fictitious surface used for these computations rests on the Z_0 contour line and on the curve locus of A' points. Along these two curves the surface is tangent to the actual surface, and is therefore extremely close to it and thus is even more satisfactory than the Σ surface.

We shall not continue our research which becomes only of mathematical interest. We must not forget that at the time of the computation of the coefficient of refraction the case of $i = 90^\circ$ was excluded. This case is probably preceded by a dissipation of energy, due to either breakers or slack water.

Improving the standard methods

It is to be hoped that the present somewhat arduous study, although involving only elementary ideas, will encourage some readers to abandon methods open to criticism. However we do not think it idle to state how these methods may be improved by those unwilling to abandon them. At least we should evaluate their reliability in each case.

As it is, a glance at our graphs is most instructive. It is immediately seen that a striking dissymmetry exists between the right and the left sides. This was to be expected. *On a path which is never infinitely small* it is not valid to make the wave ray rotate through the same angle since it is propagating across contour lines whose directions can be very different.

There is another consideration that is also a question of common sense. Except for the angles of incidence already close to 90° the refraction

is small when the topographic configuration makes the orthogonals increasingly perpendicular to the contour lines. This is all the more perceptible when γ is larger than 1, the wave rays having already a tendency to be parallel to the normal. On the contrary, refraction is great when orthogonals make increasingly larger angles of incidence with the contour lines.

In the last case — and although our graphs were plotted with γ greater than 1, i.e. running counter to the effect of rotating contour lines — we see that the i_0 curves deviate appreciably, and even make a pronounced bend in order to come close to an asymptotic curve with which they may be practically merged. The direction of A_0A is therefore dependent decreasingly on the incidence and increasingly on α .

Our evaluation of the construction of the templates makes it possible to give more accurate information of immediate practical interest.

We have thought it preferable to compute in terms of i_0 and α not the angles i and β but the angle Ψ — through which the wave-ray turns, and the angle μ — through which chord A_0M turns when M moves along the orthogonals from A_0 to A (fig. 14). The direction of A_0A and that of the tangent to the orthogonal at A are determined equally easily and within the limits considered smaller figures are found. The scales used for plotting the curves facilitate their reading. In fact we have $\Psi = i - i_0 - \alpha$ and $\mu = \alpha - 90^\circ + i_0$.

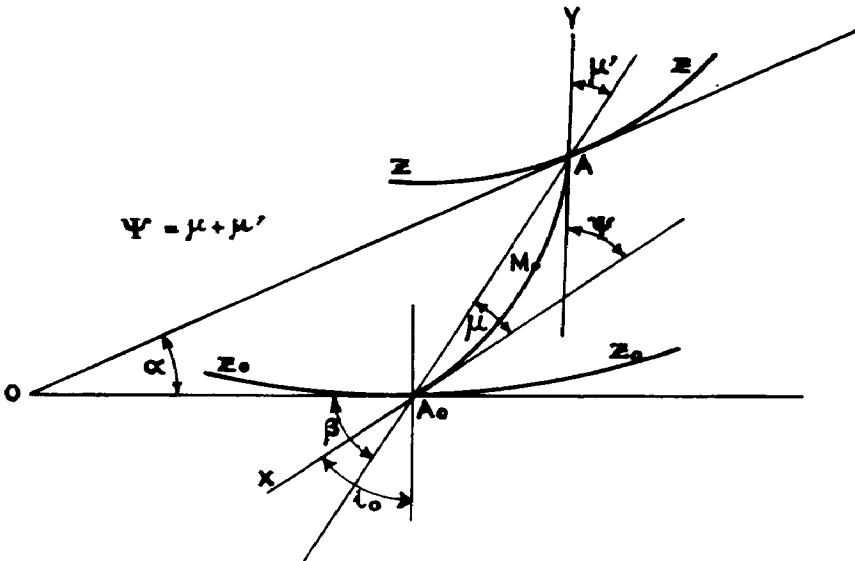


FIG. 14

We see that except in the case where i_0 is very close to α we obtain fairly accurately $\Psi = 2\mu$.

This is yet another fact that could be foreseen : it will suffice to remark that on a short path the orthogonal can be taken as an arc of a circle.

However we have thus shown up a capital fault of the present methods i.e. the need for both the direction of A_0A and that of the tangent at A (i.e. for i) in order to continue the plotting. In fact we use only one direction — the tangent's direction — and as a result, as a rule, we make the orthogonal turn through a 2μ instead of a μ angle if we are dealing with only one arc. For this arc the result will be improved if we divide it into smaller sections and deal with each section separately. However if the arc is divided into n sections the *systematic* error will become smaller but is only divided by \sqrt{n} .

Another error is added (algebraically) when determining this single direction selected as identical to that of the tangent at A . In fact we have seen that this tangent's direction (or else i) is taken as equal to what it would be for $\alpha = 0$ even though α , for a rugged relief, may be relatively large.

This new simplification obviously involves a new error, which also becomes considerable when angle i_0 , α and the interval of the contour lines being considered are large, or finally when the refraction angle Ψ is large. The interval will be distinguished whatever the scale by the absolute value of $1 - \frac{C_0}{C}$.

In the case we have studied, i.e. when the velocities of the contour lines between which a segment of an orthogonal is plotted satisfy the equality $\log \frac{C_0}{C} = 0.01$, we have noted the figures given below.

With the former methods the tangent's direction, accurate for $\alpha = 0$ is only inaccurate by $10' - 12'$ for $\alpha = 10^\circ$ with i_0 between -30° and $+30^\circ$.

The discrepancy increases, as was foreseen, but remains between $37'$ and $39'$ for $\alpha = 50^\circ$ with i_0 varying from -30° to 0° , and between $52'$ and 1° with i_0 varying from -50° to $+20^\circ$.

With the exception, perhaps, of the latter, these discrepancies are acceptable only when they concern the value of the new angle of incidence which is necessary for the computation of the next section of the wave ray.

Unfortunately the direction of the tangent is paramount for determining point A .

The angle errors on the A_0A direction are equal to half the angle Ψ for $\alpha = 0$, nil for $i_0 = 0$. They reach $-6'$ for $i_0 = \pm 10^\circ$, $14'$ for $i_0 = \pm 20^\circ$, $22'$ for $i_0 = \pm 30^\circ$, $32'$ for $i_0 = \pm 40^\circ$.

For $\alpha = 10^\circ$, the absolute value of the total error is $10'$ for i_0 between -10° and $+20^\circ$ and $26'$ for i_0 between -30° and $+40^\circ$. For $\alpha = 50^\circ$ it varies between $33'$ and $36'$ with a minimum of $21'$ for $i_0 = 0$.

Outside these limits the error increases rapidly. It is essential to improve the A_0A direction which is a capital data.

An obvious *improvement* would be obtained by simply taking the value of $\frac{\Psi}{2}$ and not Ψ for μ . A strictly accurate A_0A direction will not be

obtained, for the Ψ value adopted from an increasingly erroneous i value will give a decreasingly accurate μ value.

Nevertheless, for $\alpha = 0$ the error will be absolutely negligible. For $\alpha = 10^\circ$ it remains very small and is $< 5'$ for i_0 between -40° and $+35^\circ$, with a minimum of $1'$ for $i_0 = 0$, and $< 8'$ for i_0 between -50° and $+45^\circ$.

Finally, the error obviously increases for $\alpha = 50^\circ$, but it is $< 22'$, with a minimum of $19'$ for $i_0 = -20^\circ$, when i_0 is between -40° and 0° and $< 30'$ when i_0 is between -50° and $+15^\circ$.

From these figures it will be seen that it is imperative to assign μ its improved value, as shown above.

Now, it seems that the standard method thus modified becomes largely acceptable when the inclination of any contour line on its neighbour is less than 10° .

Otherwise the remedy will consist in plotting a sufficient number of intermediate contour lines, not only because the successive inclination will decrease, but also because, $\frac{C_0}{C}$ tending towards 1, the refraction will decrease and its consequent error.

The drawback is then that the intermediate contour lines are more or less in error. This could only be avoided if special grids were available. These grids are also useful because they allow the accurate determination of the tangent to the contour lines which are necessary whatever method is used.

With our templates and once the network of contour lines has been adopted, the plotting of orthogonals becomes easy and may be carried out by an operator following simple instructions. On the contrary with the standard method which, even when improved, remains tainted with a systematic error, the various elements must be constantly and critically watched with the aid of intricate instructions based on the often difficult evaluation of these discrepancies which can only be eliminated when $\alpha = 0$.

Such directions cannot be deduced directly from the few figures given. Our statements must be interpreted. They have, no doubt, a qualitative general value but our figures are only quantitatively accurate for $\log \frac{C_0}{C} = 0.01$. This means that the value of $\frac{C_0}{C}$ is greater than 1. The case when $\frac{C_0}{C}$ is smaller than 1 (for increasing depths) gives fairly different results.

In figure 14, we started from the incident ray XA_0 . We could change the direction of propagation and start from an incident ray YA . The orthogonal will follow the same path in the opposite direction and turn by exactly the same angle Ψ . Chord AA_0 will also make the same angle μ with YA as with XA_0 , since in practice $\mu = \mu' = \frac{\Psi}{2}$. Finally, α is identical by definition.

On the other hand the initial angle of incidence, in terms of which we have computed the errors, is not the same and this angle will change in proportion as α becomes further apart from zero.

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We hope that this paper which at first glance would seem to be best suited to hydraulics engineers will also be of interest to hydrographers. This is because not only do we suggest that they should accept the fresh responsibilities involved in the setting up of special grids but because they cannot ignore the correct plotting of refraction diagrams any more than mariners can ignore the swell.

We shall give an example.

Amongst the data which must be gathered before a recommended coastal route as well as, above all, an anchorage can be inserted on a chart, the study of the coefficients of refraction corresponding to the directions of predominant waves is, in our opinion, of considerable importance.