ANALYSIS OF SHORT-PERIOD TIDAL OR TIDAL STREAM OBSERVATIONS

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The extension of a tidal station network throughout the world and the development of reliable tide gauges will enable tidal analysis over long periods of observations (of the order of one year) to be made at numerous places. However it will always be worthwhile to get the best from tidal observations obtained during coastal surveys which will be more quickly executed as the methods used become further perfected. The processing of deep-sea tidal and tidal stream observations, which are both of short duration (of the order of several weeks), should moreover be envisaged.

The analysis of water heights, or of current components, obtained during short-period observations presents two main difficulties. The first is an insufficient adjustment of the random discrepancies : the instrumental discrepancies, those due to transcription of readings on the card to be introduced into the computer, and those due to natural and accidental phenomena. On this subject we will limit ourselves to the following remarks. For checking the data, analogue recording might be thought to be more useful than digital recording because it permits the detection of the largest random discrepancies through a quick examination of the curves. In fact, the analogue to digital " conversion ", absolutely necessary for the analysis, introduces the largest errors, and a " sifting " [1] of the input data is in any case necessary. The longer the analysis period is, the weaker is the influence of perturbations either non-periodic or with periods differing from those of the waves analysed. For short periods these perturbations remain important and considerably affect the results.

The second difficulty is the poor separation of waves with large equilibrium coefficients, consequently with ranges that are presumed significant. It is this point which is here specially studied. A convenient criterion for this " separation " is :

$$S = d \frac{\Delta T}{T^2}$$

(1) " Sifting " of a set of readings consists in finding the discrepancies larger than a " standard deviation " without *systematically* reducing these discrepancies. The suspect reading may be either corrected or omitted. " Sifting " thus essentially differs from " smoothing " which systematically affects all the readings of the series.

where d, T, and ΔT are respectively the observational length, the period of the wave to be extracted, and the difference in period with other waves having significent equilibrium coefficient, all expressed in the same unit. In order to define this difficulty clearly, and to show how it can be overcome, we shall consider that the analysis has been made according to the fairly often used " special hours " (2) method, but our general conclusions are valid for any method.

RESIDUAL WAVE — RESIDUAL COEFFICIENT

We shall now briefly describe the classic idea of a " residual wave " and a " residual coefficient ". The water height contributed by a wave $\overrightarrow{A_i}$ in a tidal height $(Y)_t$ at time t is :

$$(Y_i)_t = A_i \cos (q_i t - \beta_i)$$

t being the time reckoned in mean time from the origin of observation, q_i being the speed of the wave in degrees per hour of mean time $(q_i = \frac{360°}{T_i}$, T_i being the period), A_i being the range reckoned in the same unit as $(Y)_t$ and β_i the phase lag in degrees (3); and

$$(Y)_t = A_0 + \sum_{i=1}^{i=m} A_i \cos (q_i t - \beta_i)$$

where A_0 is the height of mean sea level above the height datum, m being the number of waves.

To find constants A_j and β_j of a $\overrightarrow{A_j}$ wave (with speed q_j and period T_j) we shall as a first step consider the set of n observation equations corresponding to times.

$$t;\ t + T_j\ \ldots\ldots;\ t + (n-1)\ T_j$$

These equations will be averaged term by term in order to obtain a " final " equation which will replace this set. This operation is indicated by the operator $\overline{}\, j$.

In this final equation the contribution of wave $\overrightarrow{A_i}$ is :

$$\overline{(Y_i)_t}^{\,j} = A_i \frac{\sin \delta_i^j}{n \sin \frac{\delta_i^j}{n}} \cos \left[q_i t - \left(\beta_i - \frac{n-1}{n} \delta_i^j \right) \right]$$

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(2) The French use the expression "jour special de l'onde" which is the period of the constituent, i.e. $\frac{360°}{q}$, q being the speed. The special hour is $\frac{15°}{q}$.

(3) A_i and β_i are the "raw harmonic constants". The final harmonic constants H_i and g_i are deduced (see IHB Special Publication No. 26) from :

$$H_i = \frac{A_i}{f_i} \qquad g_i = \beta_i + (V_0 + u)_i \text{ Greenwich}$$

f_i being the nodal factor chosen for the central day of the observation period, and $(V_0 + u)_i$ Greenwich for the beginning of this period.

where

$$\delta_i^j = 180^\circ\, n\, \frac{q_i - q_j}{q_j} = 180^\circ\, \frac{T_j - T_i}{T_i} = 180^\circ\, n\, \varepsilon_i^j, \text{ with } \varepsilon_i^j = \frac{T_j}{T_i} - 1$$

The term :

$$C_i^j = \frac{\sin \delta_i^j}{n \sin \frac{\delta_i^j}{n}}$$

is called the "Residual Coefficient". The more the periods T_i and T_j are different and the larger n is (4), the smaller is this residual coefficient. In practice it quickly becomes negligible provided the length of observation is greater than a few days if $\vec{A}_i$ and $\vec{A}_j$ are of different species (5). When both waves are of the same species we can compute C_i^j with fairly good accuracy by the following formula :

$$C_i^j = \frac{\sin \delta_i^j}{\delta_i^j \sin 1^\circ}$$

C_i^j depending only on δ_i^j can be easily tabulated. Finally this coefficient is equal to 1 for $i = j$. The final equation thus becomes :

$$\overline{(Y)}_t^j = A_0 + A_j \cos (q_j t - \beta_j) + \sum_{i \neq j} C_i^j A_i \cos \left[q_i t - \left(\beta_i - \frac{n-1}{n} \delta_i^j \right) \right]$$

and in practice those terms under sign Σ which are not negligible relate to waves of the same species as $\vec{A}_j$ (i.e. to waves for which ε_i^j is small enough for $(\varepsilon_i^j)^2$ to be negligible) and to the harmonic waves of $\vec{A}_j$.

The second stage of the analysis consists in determining the coefficients of the Fourier series expansion, over the interval $[\,0; T_j\,]$, of

$$\overline{(Y)}_t^j$$

which is a function of t. The two coefficients of the fundamental (coefficients of terms with period T_j) are identical to coefficients of the expansion over the same interval of function :

$$A_j \cos (q_j t - \beta_j) + \sum_{i \neq j} A_i^j \cos (q_j t - \beta_i^j)$$

where $A_i^j \cos (q_j t - \beta_i^j)$ can be considered as the partial water height contributed by wave $\vec{A}_i^j$, called the " $\vec{A}_j - \vec{A}_i$ residual wave ". The raw constants of this wave are given by :

$$A_i^j = C_i^j A_i \frac{\sin 180^\circ \varepsilon_i^j}{180^\circ \varepsilon_i^j \sin 1^\circ} \sqrt{1 + \varepsilon_i^j \cos^2 (\beta_i - 180^\circ n \varepsilon_i^j)}$$

$$\tan \beta_i^j = (1 - \varepsilon_i^j) \tan (\beta_i - 180^\circ n \varepsilon_i^j)$$

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(4) We must, however note that when n is close to $k \dfrac{q_j}{q_i - q_j} = k \dfrac{T_i}{T_j - T_i}$, where k is an integer, the residual coefficient is negligible even when n is not very large. This is why, having taken the equilibrium coefficients into account, it is recommended that the length of observations to be analysed be chosen from 14, 15, 29, 58, 87 etc. mean days.

(5) A group of waves of the same species will hereafter be called a " set ".

and almost accurately, since ε_i^j is small :

$$A_i^j = C_i^j A_i \qquad \beta_i^j = \beta_i - 180^\circ n \varepsilon_i^j = \beta_i - \delta_i^j$$

These relations make it possible to take wave $\vec{A}_i^j$ as an intermediate unknown for the computation of $\vec{A}_i$.

The analyses of functions $\overline{(Y)}_t^i$ for $i = 1 \ldots m$, thus provide a system of m vectorial equations for m unknowns. This system is particularly simple when n is very large since the ranges of residual waves are negligible and each equation contains only one unknown (6). When the observation period is short all the unknowns appear in each equation (either directly or through their residual) but there is no *theoretical* difficulty in solving the system. The *practical* difficulty does not arise from bulky computations (easily processed in an electronic computer), but from the fact that where unknowns with close periods appear directly in the equations, these equations have close coefficients. As a result, and due to uncertainties in the data, the error on the unknowns is very large, as will be seen in the following example.

A tide made up of two waves only — $\vec{A}_j$ and $\vec{A}_i$ — is under consideration. The analyses for these waves provide respectively ranges M_j and M_i and phase lags μ_j and μ_i for the fundamental of the Fourier expansion. If these waves have close periods, $\delta_i^j \# \delta_j^i = \delta$ and $C_i^j \# C_j^i = C \# 1$. The raw harmonic constants for $\vec{A}_j$ and $\vec{A}_i$ are the A_j, A_i, β_j, β_i solutions of the system :

$$A_i \cos \beta_i = \frac{M_i}{1 - C^2} \cos \mu_i - \frac{C M_j}{1 - C^2} \cos (\mu_j + \delta)$$

$$A_i \sin \beta_i = \frac{M_i}{1 - C^2} \sin \mu_i - \frac{C M_j}{1 - C^2} \sin (\mu_j + \delta)$$

$$A_j \cos \beta_j = \frac{M_j}{1 - C^2} \cos \mu_j - \frac{C M_i}{1 - C^2} \cos (\mu_i - \delta)$$

$$A_j \sin \beta_j = \frac{M_j}{1 - C^2} \sin \mu_j - \frac{C M_i}{1 - C^2} \sin (\mu_i - \delta)$$

We see that the coefficients of the known terms (M_i, $M_j \ldots$), which are deduced from observations (and so subject to errors) are very large and consequently the unknowns are ill determined.

We may also add that the equations of the final system, having close coefficients, may be replaced by only one of them (or by an equation obtained by term by term averaging). It follows that the system then includes fewer equations than unknowns. Further relations, other than the observation equations, must be introduced — that is an *additional assumption* has to be made.

(6) One of the good points about the "special hours method" is that it establishes equations having only one unknown. The figure m being theoretically infinite, in practice an assumption must be made on the set of waves involved in the tide in question. This assumption is not necessary in the "special hours method" (if n is large), but it is necessary when the "least square method" is used.

APPROXIMATE CONSTANTS — DIFFERENCE TIDE

It should first be remarked that the use of approximate constants (7) introduces an additional assumption. These constants are used in two ways. In the first case, the terms under sign Σ in the final equations are computed. There remains only one unknown per equation, as in the case when n is very large. The assumption made is that the residual wave of the "approximate" wave (the wave whose constants are the approximate constants) is identical to the residual of the corresponding true wave. It is seen that the larger n is the less important is the assumption, since the ranges of these two residuals, and consequently the range of their difference, decreases as n increases. In the second case, an approximate tide is computed with approximate constants, and thereafter the "difference tide" which is the difference as a function of time between actual and calculated heights; and this "difference tide" is analysed by the "special hours method". The assumption made is that two waves $\overrightarrow{A_i}$ and $\overrightarrow{A_j}$ with very close periods have one and the same " complementary wave " $\vec{a_i}$ (8), and this decreases by one the number of unknowns in the final system, and thus permits the omission of the equation corresponding to $\vec{a_i}$ or to $\vec{a_j}$.

CONCORDANCE ON SETS OF WAVES

The most convenient additional assumption to introduce is that of "similar tides" which is thus defined (7).

Two tides, in ports A and B, are respectively of the form :

$$(Y)_t = A_0 + \sum_{i=1}^{i=m} A_i \cos (q_i t - \beta_i)$$

$$(Z)_t = B_0 + \sum_{i=1}^{i=m} B_i \cos (q_i t - \gamma_i)$$

They are similar if ratios $\frac{B_i}{A_i}$ and $\frac{\gamma_i - \beta_i}{q_i}$ are independent of i. It is obvious that when these conditions are realised the tidal curve as a function of time at port B is derived from the one at port A by a translation equal to

(7) See " La Méthode des Concordances et l'analyse harmonique par les constantes approchées " (*Annales Hydrographiques* 1956 and *International Hydrographic Review*, Vol. XXXIV, No. 1). The use of these constants has many other advantages in particular for the sifting of observations and for computation.

(8) If $(\overrightarrow{A_j})_a$ is the approximate wave of $\overrightarrow{A_j}$, the complementary wave (the unknown of the "difference tide") is $\vec{a_j} = \overrightarrow{A_j} - (\overrightarrow{A_j})_a$.

$\frac{\gamma_i - \beta_i}{q_i}$ following the time axis, and by an affinity following the height axis with a $\frac{B_i}{A_i}$ ratio. Conversely, if the two curves are thus derived from one another the tides are similar and the harmonic constants at B are easily derived from constants at A. In effect, since the above conditions are satisfied for the constituent waves they are also satisfied for their residuals. It suffices to analyse in both ports the same length of observation for one wave only in order to obtain the value of ratios.

In reality, strictly similar tides only exist in ports both very close to each other and in the same ocean area. For a long time this similarity has been used to link the various tidal stations of the same bathymetric survey. However it is reasonable to suppose that the response of an ocean area to forces with periods close to one another are themselves also close, that is to say that at two points, A and B, in this region, the conditions of similarity are satisfied for a partial tide made up of a *set of waves* (5), of close periods, represented by the wave whose assumed range is the largest in the set. The closer the ports are, and the narrower the period band covered by the set, the better is the similarity. The method of "concordance on a set of waves" is thus applied, as follows.

The unknown tide at port B and the tide observed at the standard port A at the same times are separately analysed. The analysis deals with only a certain number of waves distributed in the tide spectrum, each wave representing a set whose bandwidth increases as the observation length decreases. For each wave ratios $\frac{B_i}{A_i}$ and $\frac{\gamma_i - \beta_i}{q_i}$ are then computed and are used for the determination of constants of the waves of the set in port B. It is clear that in this method, as in the approximate constants method (first case), the influence of the additional assumption decreases as the length of observation increases, i.e. this assumption becomes less necessary (9).

It may happen that the observed tide at standard port A is not available, but only this port's constants. It will then suffice to predict the tide at A at the same times for which it is being observed at B, and to proceed as before (10). Finally we may have to use as a reference tide an artificial tide predicted from the expansion of the tide generating potential.

The introduction of an additional assumption through the "concordance on sets of waves" is especially convenient for it does not necessitate any special programme. Programmes optimised for analysis and prediction are used *whatever the method used,* and these are already familiar to the user. Moreover, the range ratio $\frac{B_i}{A_i}$ and the phase difference $\gamma_i - \beta_i$ can be

(9) As the length of observations increases the analysis gives a spectrum which passes from a band spectrum to a line spectrum.

(10) The observed reference tide or the predicted reference tide has each its own advantage in use. The observed tide takes into account waves which in reality exist in both the A and B tides but which have been omitted in the analysis for A. The predicted tide represents very accurately the sum of the constituents used and the uncertainty of these constituents has practically no effect on the constants deduced for B.

directly applied to constants H_i and g_i of the standard port, thus avoiding passing from raw constants to final constants (3).

EXAMPLE OF CONCORDANCE ON SETS OF WAVES

Harmonic constants computed for a 370-day period are available for both Brest (at the extreme tip of Brittany) and Pointe de Grave (at the mouth of the Gironde). In both these ports the tide has a fairly large mean range (5.4 m at Brest and 4.2 m at Port Bloc). The two stations are about 210 miles apart : they are situated in the same ocean area, but the first lies at the extremity of a peninsula where the continental shelf is broad, the other lies at the inner point of a gulf at the mouth of an important river and where the continental shelf is narrow. The method of "concordance on sets of waves" was applied for determining the constants for Pointe de Grave, starting from the constants for Brest and using a number of observations all starting from 1 July 1963, and of the following length in days : 7; 15; 22; 30; 37; 44; 52; 59; 67; 74. The waves analysed were the principal semi-diurnals and the principal diurnal (here relatively small). These constants permitted the plotting of vectors $\overrightarrow{A_i}$ which were compared to vectors $\overrightarrow{A_i'}$ deduced from constants computed over 370 days and considered accurate. A "relative modulus" was examined, and this in hundredths is the ratio :

$$\frac{|\overrightarrow{A_i} - \overrightarrow{A_i'}|}{|\overrightarrow{A_i'}|}$$

These comparisons gave the following results. For the M_2 constituent the relative modulus is 2.5 for a 7-day observation and remains close to 0.5 for the observations longer than 15 days. For the N_2 wave this modulus is 2 for any length of observation. It is particularly interesting to consider the separation of constituents K_2 and S_2. Obviously for a 7-day period the relative modulus gives the high figure 7, but it is almost impossible to separate waves of such close period over so short a length of observation. Over 15 days this relative modulus is 3, and it remains at 1 for the longer periods. In fact it is seen that the modulus $|\overrightarrow{A_i} - \overrightarrow{A_i'}|$ of the discrepancy vector remains unchanged when the periods analysed become longer whilst the vector itself takes inconsistent positions. This seems to reflect an inaccuracy in the observations which had not first been subjected to a sifting, and had not been corrected for atmospheric pressure.