THE SHAPE OF THE EARTH GIVEN BY ARTIFICIAL SATELLITES

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Before going into the heart of the matter it will be necessary to make a rapid review of what the standard methods yielded before the use of artificial satellites.

When the shape of the earth's surface is spoken about it is obviously not a question of the physical surface which is subject to local geographical irregularities (such as mountains and valleys, etc.), but of the surface of the geoid, which is the equipotential surface of the gravitional field passing through a point taken as origin, a surface which is assigned zero altitude. It happens that the general shape of the geoid is very little different to that of an ellipsoid of revolution flattened at the poles. Consequently, the geoid can be determined in two stages. First of all, by determining this mean ellipsoid which is taken as the mathematical reference surface, and on whose surface triangulations are computed, and then by reckoning at each point the altitude of the geoid above or below the reference ellipsoid.

During the last century the first of these problems, the determination of an ellipsoid of revolution which on the whole deviates the least from the geoid, was solved in many different ways. At that time this determination was carried out by measuring length AB of an arc of meridian (figure 1) whose range in latitude $\varphi_A - \varphi_B$ was determined by another way through astronomical observations (the arc-measuring method). The length a of the equatorial radius was then computed and also α, the value of the flattening being equal to $\frac{a - b}{a}$, a and b being the meridional ellipse's semi-axes.

At the beginning of the 20th century in the U.S.A. HAYFORD instigated a new method — the method of surfaces — and in 1909 published the following result :

$$a = 6\,378\,388 \text{ metres} \qquad \alpha = \frac{1}{297.0}$$

which was adopted in 1924 by the International Association of Geodesy to define an international reference ellipsoid. It is well known that the IHB has computed a table for this ellipsoid for use in particular in their work on preparing the General Bathymetric Chart of the Oceans (GEBCO).

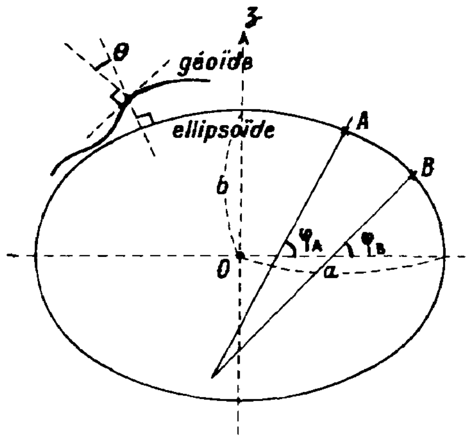

FIG. 1

HAYFORD's method consisted of deriving a and α so as to reduce the *plumb line deflections* in their aggregate to the minimum, i.e. the small θ angles made at each point of the geoid by the normal to this surface and the normal to the ellipsoid dropped at this point, and this for an extensive area — on this occasion United States territory — entirely covered by a triangulation net computed from the geographical position of a base station. These deflections may be directly obtained without reference to the ellipsoid by evaluating the attraction exerted on the plumb line mass by the surrounding topographical features, however applying large corrections, made necessary by the action of compensating underlying masses, to the deflections thus calculated, according to a specific theory of the earth-crust equilibrium, i.e. Isostasy.

However Hayford's computations were only worked out from the triangulation net and the plumb line deflections concerning U.S. territory only. This is why, quite recently, the Hayford method has been re-studied both in the U.S.A. and in the U.S.S.R. with more numerous and more accurate modern data, including the results obtained by ISOTOV and KRASSOWSKY furnished by measurements of gravity intensity. All this recent work has led to the conclusion that Hayford's a value was too high by about 180 metres and that the correct α value was slightly lower than $\frac{1}{298}$.

Mention has just been made of gravimetric methods which consist in determining the g value of the gravity field at numerous stations. Formerly very difficult, these measurements are nowadays greatly facilitated by the use of gravimeters on shore, at sea, or even airborne. Conveniently reduced to zero height, and freed from the attraction of surrounding topographical masses, corrected for the effect of underlying compensating masses involved by the isostatic theory, the g values are introduced into a relation established in the 18th century by CLAIRAUT. This relation, for an ellipsoid of revolution with an equatorial radius a and a flattening α, uniformly rotating about its minor axis with an angular speed ω, gives the g value in terms of its value

g_e at the equator and of latitude φ for the observational point :

$$g = g_e \left[1 + \left(\frac{5}{2} \frac{\omega^2 a}{g_e} - \alpha \right) \sin^2 \varphi \right] \tag{1}$$

Thus an accurate value for both the flattening α and the equatorial gravity can be computed :

$$g_e = 978.049 \text{ gals.} ^{(*)}$$

As to the value of the equatorial radius a, the gravimetric method does not allow this to be obtained with sufficient accuracy.

Let us now see how until recently the second problem was resolved — the problem of determining the altitude of the geoid in relation to the reference ellipsoid. The solution is virtually immediate for a region where numerous plumb line deflections have been determined. This solution is found by comparing the geographic coordinates — the latitude and the longitude — obtained by astronomical observations to their values computed at the surface of the ellipsoid, starting from the geographic coordinates of a base station and an observed azimuth. The plumb line deflection, i.e. the angle between the normal to the ellipsoid and the normal to the geoid, is the slope of the latter surface in relation to the former. Starting from a point of origin it is possible to compute step by step the altitudes of points on the geoid in relation to corresponding points on the ellipsoid.

This result, however, may also be obtained gravimetrically. For numerous points carefully distributed over the earth's surface we can compute the *gravity anomaly* — i.e. the difference between on the one hand the *observed* g value corrected for both altitude and the effects of topographical and compensating masses and on the other hand the theoretical g value *reckoned* on the International ellipsoid by the Clairault formula. A formula, established by STOKES (1849), then gives the altitude of the geoid above and below the ellipsoid in terms of anomalies.

Before proceeding further it should be noted that until the use of satellites there had been no determination of the geoid that was both total and accurate. Stokes' formula, on which the gravimetric method is based, requires, indeed, a knowledge of numerous values of g measured in *all* parts of the globe.

Thus, from this point of view, the major part of the oceans and even of certain continents is at present unexplored so that only small " pieces " of the geoid can be obtained by gravimetry with any accuracy, and this only for regions well provided with g values and yet far from those where g is unknown or little known. However these more or less accurate pieces of geoid are well " positioned " in relation to the International reference ellipsoid, but this is not the case when the method used is the plumb line deflection method. This last method is in fact based on the availability of a triangulation net computed on an ellipsoid determined from the latitude, the longitude and an azimuth obtained by astronomical observation at the net's base station. But for the different countries the ellipsoids are not

(*) Gal. is the abbreviation for GALILEO and is the C.G.S. unit for acceleration. It is the acceleration of a mobile of uniformly accelerated motion whose speed increases by 1 cm per second.

necessarily the same, and moreover to adopt a geographical position astronomically determined at a base station amounts to assuming that at this station there is no plumb line deflection, i.e. that the ellipsoid is tangent to the geoid. This assumption being far from justified, triangulation nets are computed on surfaces which do not agree at the junction points of two national networks so that important discrepancies in geodetic positions are found there (*). Of course, this is also the case for the " pieces " of the geoid determined in each country by the plumb line deflection method.

Relative order has been brought to this confusion by grouping the nets of several neighbouring countries and by adopting for the whole one and the same ellipsoid as well as a sole base station (datum). This has been done for the U.S.A., Canada and Mexico, and also for Western Europe with the exception of Great Britain.

There exists, it is true, a procedure which in theory at least would make it possible to express all the geodetic points of the universe in one and the same system. This is to choose a single ellipsoid, which would naturally be the International one, and to position the relevant portion of this ellipsoid suitably at each base station in relation to the geoid, and then to transfer the national triangulations onto the International ellipsoid by means of differential formulae. The correct positioning at the base station of the ellipsoid in relation to the geoid is possible, thanks to Stokes' formula, and is done by measuring numerous g values in a fairly wide area around this point. In fact Stokes' formula gives the deviation in altitude of the two surfaces. Moreover by means of derivations VENING MEINESZ and de GRAAFF HUNTER have deduced from Stokes' formula two other formulae that supply the components of the plumb line deflection along the meridian and the prime vertical.

It must nevertheless be acknowledged that this programme which aimed at setting up a world-wide geodetic system, and which had earlier been advocated by Professor W. HEISKANEN, has lost a great deal of its interest since it has been known how to put artificial satellites into orbit. These, we know, make it possible to connect entire blocks of triangulation over either sea or desert regions, more particularly by star background photography.

Leaving aside this use of satellites for long distance geodetic connections we come to the determination of the Earth's shape — i.e. of the geoid — by the analysis of the motions of artificial satellites.

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Let O be the Earth's centre of gravity, Oz its axis of rotation, Ox and Oy in the equator and forming with Oz a trihedral triangle (figure 2). Let us imagine that the earth is limited by a *surface rotating about Oz, a surface close to a sphere with O as centre and a as radius*. It can be shown that the potential of the attraction that this sphere exerts on a point M at a distance

(*) Admiral NARES, a past Director of the IHB, studied these discrepancies at the junction of Western European national nets (" Bulletin géodésique ", 1949).

r from O and of latitude φ is expressed by a development in terms of the increasing powers of $\left(\frac{a}{r}\right)$ in the form of :

$$V = f\frac{M}{r}\left[1 - \sum_{n=2}^{n=\infty} J_n \left(\frac{a}{r}\right)^n P_n(\sin\varphi)\right] \tag{2}$$

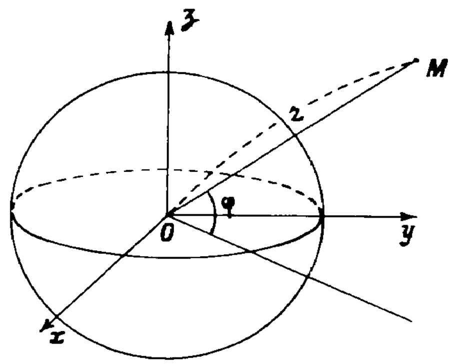

FIG. 2

In this expression f is the gravitation constant, and M the Earth mass. Each term of $\left(\frac{a}{r}\right)^n$ defines what is called a *spherical harmonic of the n^{th} order* in which $P_n(x)$ is the LEGENDRE polynomial of the n^{th} order given by the relation :

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Thus

$$P_0(x) = 1\ ;\quad P_1(x) = x\ ;\quad P_2(x) = \frac{1}{2}(3x^2-1),\quad \text{etc.}$$

The J_ns are numerical constants which remain small when the geoid does not deviate too much from the sphere of radius a.

The equation for the geoid is obtained by making V equal to the constant $\frac{fM}{a}$. If then all these J_n are zero the geoid will merge with the sphere of radius a whose meridian is the circle (M_0). If all these J_n, excepting only J_p, are presumed zero we are defining an *elementary geoid of revolution* of the p^{th} order whose meridian becomes more complicated as p increases. For instance, figure 3 shows the meridians (M2), (M3), (M4) and (M5) of the elementary geoids of the 2nd, 3rd, 4th and 5th order.

By superimposing an infinity of elementary geoids it is obvious that almost any form of the *general geoid* could be obtained, always supposing that it revolves around Oz, that O is its centre of gravity, and that it remains close to the sphere having centre O and radius a. Of course, it is the spherical harmonics having the largest J coefficient which give the surface

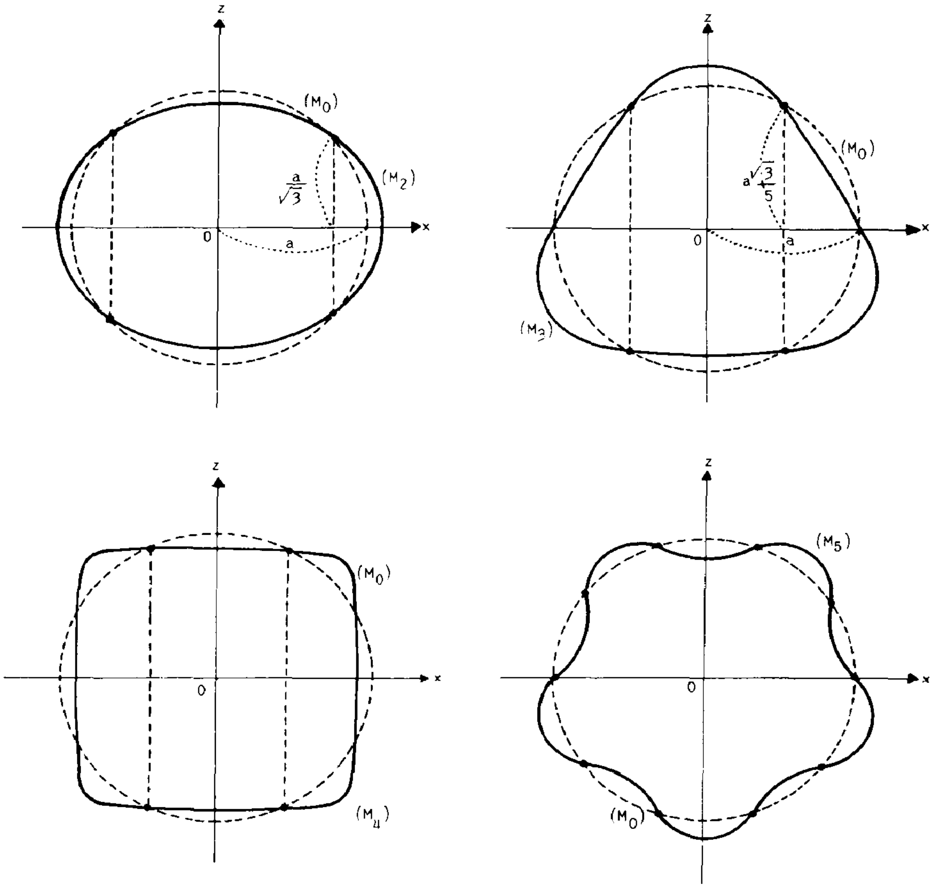

FIG. 3

its general aspect since we know that the real geoid is little different from an ellipsoid of revolution flattened at the poles. J_2 should have a value well above the other Js. In fact, the J_2 constant is of the same magnitude as the α flattening of the mean ellipsoid and we can demonstrate the relation

$$J_2 = \frac{2}{3}\left[\alpha - \frac{m}{2} + \alpha\left(\frac{m}{7} - \frac{\alpha}{2}\right) + \ldots\right] \tag{3}$$

in which

$$m = \frac{\omega^2 a^3 (1 - \alpha)}{f\,M}$$

ω being the speed of the Earth's rotation.

To define the shape of the true geoid is thus entirely a question of determining the value of the J constants, and it is here that artificial satellites can be brought in.

If the geoid were perfectly spherical (i.e. with all Js zero) the satellite's path would be an ellipse with one of its foci the centre of the Earth O, and whose shape and orientation in space would remain unchanged while the Earth would turn interior to it. However the geoid differs considerably from a sphere and, this being so, the satellite has a motion called a *disturbed* motion. The elliptical path, still having a focus at O, becomes distorted and moves in its plane, while the orientation of this plane in space also varies more slowly or less slowly. In particular, the ON_1 nodal line, i.e. the intersection of the ellipse's plane with the plane of the terrestrial equator (figure 4), rotates on an average in the equator and, for a satellite launched in an easterly direction, as is always the case, this rotation is *anticlockwise* (the opposite to the earth's rotation).

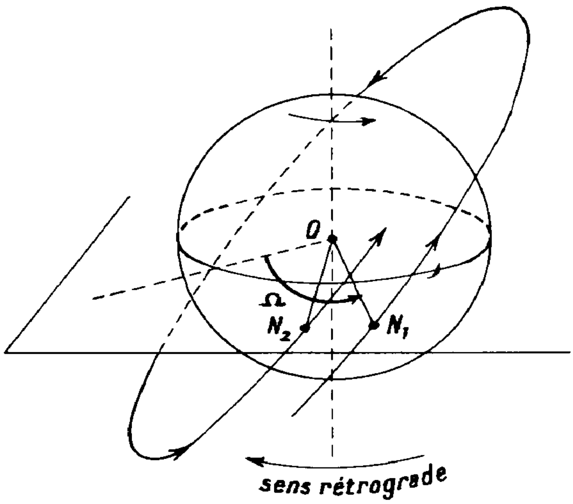

FIG. 4

The mean value of this disturbance is of the order of 3° to 4° per day and can be determined with accuracy since the observations can be extended over a long period. This mean value may be expressed in terms of J coefficients by Celestial Mechanics methods, the disturbing function here being constituted by the sum of spherical harmonics of the 2nd, 3rd, etc., order. As a first approximation, by neglecting the harmonics of a higher order than the 2nd and also the J_2^2 terms, the angle Ω_m defining in the equatorial plane the mean position of the nodal line is expressed in terms of time t by the relation :

$$(\Omega_m)_t = (\Omega_m)_0 - \frac{3}{2}\, n\, J_2 \frac{a^2}{a'^2} \frac{\cos i}{(1 - e^2)^2}\, t$$

in which n is the mean motion of the satellite, a' the semi-major axis of the elliptical path (*), e its eccentricity, i the inclination of its plane relative to the equatorial plane. This expression shows that the retrogressive motion is faster for inclinations tending towards zero; that it is non-existent for Polar orbits ($i = 90°$); and that it has a tendency to slow down for small eccentricities and also for large semi-axes a'.

(*) Between n and a' there is the relation $n^2 a'^3 = f\,M$ (f being the gravitation constant, M the mass of the Earth).

If all the Js and the powers of J_2 are taken into account the expression of $(\Omega_m)_t$ obviously becomes more complicated. However the variation $\dfrac{d\Omega m}{dt}$ may be expressed as a function of J, and as this is observable — with accuracy when over a sufficiently long period — by using the data from a sufficient number of satellites it is possible to determine the J coefficients' values and thus obtain the shape of the geoid's meridian.

A list of the published, although not definitive, J_n values is given below :

$$J_2 = +\,1082.86 \times 10^{-6} \qquad J_6 = +\,0.72 \times 10^{-6}$$
$$J_3 = -\,2.45 \times 10^{-6} \qquad J_7 = +\,0.41 \times 10^{-6}$$
$$J_4 = -\,1.03 \times 10^{-6} \qquad J_8 = +\,0.34 \times 10^{-6}$$
$$J_5 = -\,0.05 \times 10^{-6}$$

The constant J_2 has been named the *factor of geopotential ellipticity*. This constant is bound to the mean ellipsoid's moments of inertia (C around the axis of rotation, A around an axis located in the equator) by the relation :

$$J_2 = \frac{C - A}{M\,e^2}$$

The International Astronomical Union (IAU) considers J_2 as one of the *fundamental astronomical constants* and after discussion of all the results obtained has adopted the value :

$$J_2 = 1082.7 \times 10^{-6}$$

The IAU also decided that for computing the flattening the following formula that is very close to (3) should be used :

$$\alpha = \frac{3}{2} J_2 + \frac{1}{2} m' + \frac{9}{8} J_2^2 + \frac{15}{28} J_2 m' - \frac{39}{56} m'^2 \tag{4}$$

with :

$$m' = \frac{a\,\omega^2}{g_e}$$

and for computing equatorial gravity (*)

$$g_e = \frac{f M}{a^2}\left[1 - \mu_a + \frac{3}{2} J_2 - m' + \frac{27}{8} J_2^2 - \frac{6}{7} J_2 m' + \frac{47}{56} m'^2\right] \tag{5}$$

in which μ_a is the relative mass of the terrestrial atmosphere (**). It should be noted that up to now g_e had only been obtained by measurement made on the Earth's surface.

Assuming that $a = 6\,378\,165$ m, we thus compute :

$$\alpha = \frac{1}{298.25} \quad \text{and} \quad g_e = 978.031 \text{ gals.}$$

(*) Computation to be made by successive approximations.

(**) The relations (4) and (5) both take into account the fact that not only is the motion of the Moon on the celestial sphere well known nowadays but also that, thanks to Radar, the distance from Earth to Moon can be accurately measured.

Although the J_2 value is by far the largest of the J values it is seen that J_3 is not negligible and is higher than the succeeding values.

It is this J_3 constant which will give the geoid a pear-shaped look with its peak at the North Pole. Finally, taking the first eight harmonics into account we arrive at the meridian shown in figure 5, in which the dotted line represents the meridian of the ellipsoid of revolution with a $\frac{1}{298.25}$ flattening. The scale for the deviations of the two surfaces is much exaggerated in order to make the figure clearer. The true shape at the South Pole is obviously not concave but convex.

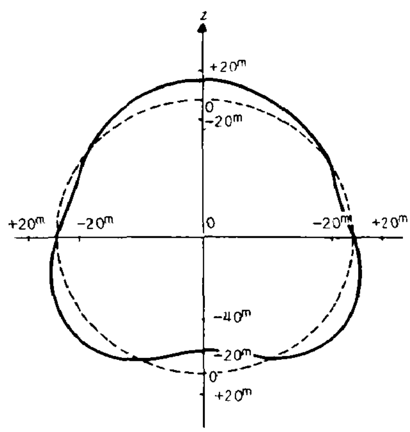

FIG. 5

Expression (2) of the potential of the gravitational force depends only on latitude φ because we have expressly assumed the surface of the geoid to be a surface of revolution around Oz. However we may free ourselves from this assumption and seek a more general shape for the geoid by putting terms depending on longitude λ in the expression of V.

It is well known that a function $f(\varphi, \lambda)$ of the coordinates of a point on a sphere defined by latitude φ and longitude λ may be expanded into a converging series of spherical functions :

$$f(\varphi, \lambda) = \sum_{n=0}^{n=\infty} Y_n$$

with

$$Y_n = \sum_{p=0}^{p=n} P_n^p (\sin \varphi) \left[C_{np} \cos p\lambda + S_{np} \sin p\lambda \right]$$

the $P_n^p(x)$ s indicating the *Legendre auxiliary functions* of order n and rank p(*) and the C_{np}s and S_{np}s designating convenient numerical coefficients.

(*) The auxiliary function $P_n(x)$ is bound to the Legendre polynomial $P_n(x)$ by the relation :

$$P_n^p(x) = \frac{2^n . n\,!}{(n+1) \dots (n+p)} (x^2 - 1)^{p/2} \frac{d^p}{d\,x^p} P_n(x)$$

Thus we may write :

$$V = \frac{fM}{r}\left[1 + \sum_{n=2}^{n=+\infty}\sum_{p=0}^{p=n}\left(\frac{a}{r}\right)^n P_n^p\ (\sin\varphi)\ \{C_{np}\cos p\lambda + S_{np}\sin p\lambda\}\right] \quad (6)$$

If the geoid is not too far apart from the sphere, coefficients C_{np} and S_{np} remain small, and obviously we have $J_n \equiv C_{n0}$, and the numerical values of C_{n0} are those given above for the J_ns.

The question of finding the C_{np}s and the S_{np}s starting from the observed retrogression of the nodal line is rather more complicated than finding the J_ns. Studies of this kind are continuing and the results so far obtained should not be considered definitive.

According to KAULA (1963), however, here is what the geoid would be when represented by contours, starting from an ellipsoid with a $\frac{1}{298.25}$ flattening and an equatorial radius $a = 6\ 378\ 165$ m (Figure 6).

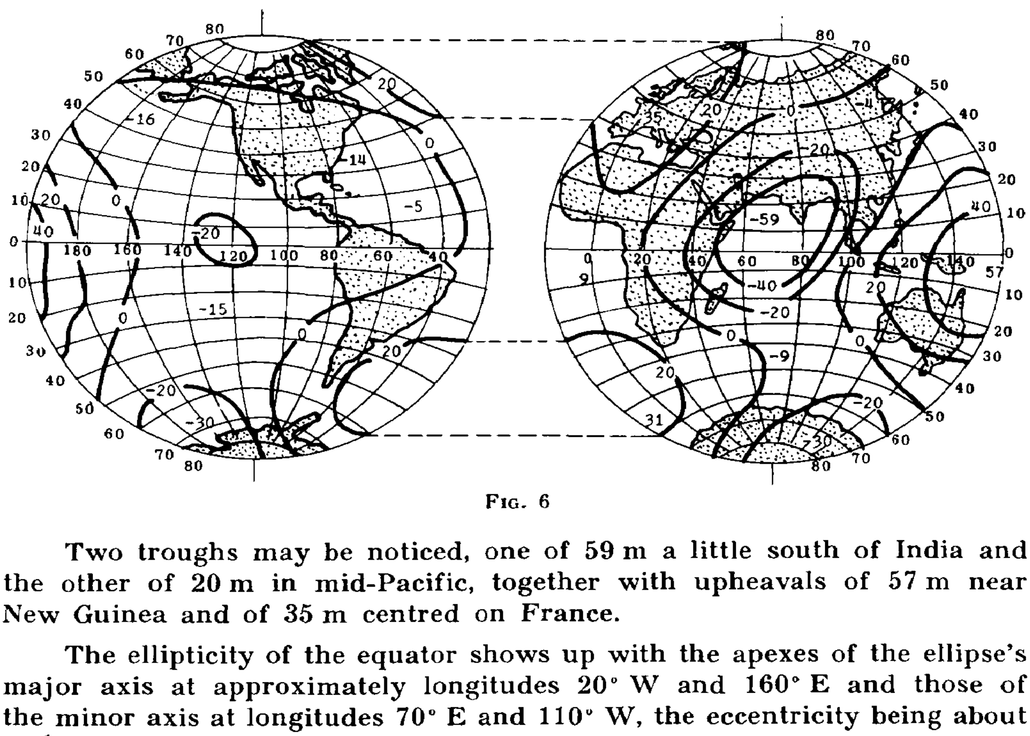

FIG. 6

Two troughs may be noticed, one of 59 m a little south of India and the other of 20 m in mid-Pacific, together with upheavals of 57 m near New Guinea and of 35 m centred on France.

The ellipticity of the equator shows up with the apexes of the ellipse's major axis at approximately longitudes 20° W and 160° E and those of the minor axis at longitudes 70° E and 110° W, the eccentricity being about $\frac{1}{50\ 000}$. Its pear-shaped nature is also noticeable, for the South Pole is depressed by about 25 m and the North Pole raised by approximately 15 m. However the stalk of the pear has undergone a twisting so that the maximum and the minimum of upheaval or of depression are located far from the poles.

It is obvious that as new determinations are carried out the "chart" of the geoid will be defined more completely, but already we may expect that it will retain the same general aspect.