STEP ADJUSTMENT
OF CONDITIONED OBSERVATIONS (*)

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1. — INTRODUCTION

In an article published in the review *Engenharia* (São Paulo, January 1948) M. A. MACHADO, engineer at the Geographical and Geological Institute of the State of São Paulo, presented a method of adjusting the geometric figures of triangulation making it possible to obtain by a simple and elegant procedure results which are in perfect agreement with those given by the least squares method. The general principle on which this method is based will not however be found in this article.

By studying the mathematics of the adjustment of normally distributed observations, so well set out by TIENSTRA and based on entirely new ideas, we finally found a way to generalize the Machado method.

In this article we shall not comment upon the basic ideas introduced by TIENSTRA. We prefer to follow a mathematical development based on the classical theory established by Gauss and Legendre. It will be noticed, however, that some points are common to this article and the Tienstra theory, with the exception that TIENSTRA used the RICCI calculus notation whereas we have adopted the standard matrix notation.

2. — TWO-STEP ADJUSTMENT

Let us first show the standard adjustment procedure expressed in matrix form. The condition equations can be written as follows :

(*) Paper presented at the 1st Brazilian Cartographic Congress (Savador, Bahia, 1962), with some alterations which take into account the article " The Electronic Computer in Survey Adjustments " by SYBREN H. de JONG, and SMITH S. TEZEAN (*The Canadian Surveyor*, Vol.XIX, No. 1, March 1965).

$$\begin{array}{l} a_{11} v_1 + a_{12} v_2 + \ldots\ldots + a_{1n} v_n = w_1 \\ a_{21} v_1 + a_{22} v_2 + \ldots\ldots + a_{2n} v_n = w_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} v_1 + a_{m2} v_2 + \ldots\ldots + a_{mn} v_n = w_m \end{array}$$

By making (*)

$$\left| \begin{array}{l} a_{11}\ a_{12} \ldots\ldots a_{1n} \\ a_{21}\ a_{22} \ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{m1}\ a_{m2} \ldots\ldots a_{mn} \end{array} \right| = \mathbf{A} \qquad \left| \begin{array}{c} v_1 \\ v_2 \\ . \\ . \\ v_n \end{array} \right| = \mathbf{V} \quad \text{and} \quad \left| \begin{array}{c} w_1 \\ w_2 \\ . \\ . \\ w_m \end{array} \right| = \mathbf{W}$$

the system of condition equations will be written :

$$\mathbf{AV} = \mathbf{W} \tag{2a}$$

The weights can be put in the following matrix form :

$$\left| \begin{array}{ccccc} p_1 & 0 & 0 & 0 \ldots\ldots & 0 \\ 0 & p_2 & 0 & 0 \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots\ldots & p_n \end{array} \right| = \underset{(nn)}{\mathbf{P}} \tag{2b}$$

and the correlative coefficients will be expressed by the vector :

$$\left| \begin{array}{c} k_1 \\ k_2 \\ . \\ . \\ k_m \end{array} \right| = \underset{(m\ 1)}{\mathbf{K}}$$

The least squares postulate is defined by

$$\sum_{i=1}^{n} p_i v_i^2 = \mathbf{V^*PV} = \text{minimum} \tag{2c}$$

Differentiating both this expression and (2*a*) we have respectively :

$$\begin{array}{l} \mathbf{V^*P}d\mathbf{V} = 0 \\ \mathbf{A}d\mathbf{V} \quad = 0 \end{array} \tag{2d}$$

According to Legendre's method, and using the K correlative coefficients, we have :

$$\mathbf{V^*P}d\mathbf{V} - \mathbf{K^*A}d\mathbf{V} = 0$$

or

$$(\mathbf{V^*P} - \mathbf{K^*A})d\mathbf{V} = 0$$

whence :

$$\mathbf{V^*P} - \mathbf{K^*A} = 0$$

The corrections will then be given by :

$$\mathbf{V} = \mathbf{P}^{-1}\mathbf{A^*K}$$

and by putting

$$\mathbf{AP}^{-1}\mathbf{A^*} = \mathbf{N}$$

(*) We have retained the author's matrix notation; i.e. strokes in bold face type for matrices and asterisks for transposed matrices.

We obtain according to (2a) the system of normal equations :

$$NK = W$$

Let us now make the adjustment with the help of this same system of condition equations by making up the following two separate groups :

$$\left.\begin{array}{l} a_{11}v_1 + a_{12}v_2 + \ldots\ldots + a_{1n}v_n = w_1 \\ a_{21}v_1 + a_{22}v_2 + \ldots\ldots + a_{2n}v_n = w_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{r1}v_1 + a_{r2}v_2 + \ldots\ldots + a_{rn}v_n = w_r \end{array}\right\} \text{GROUP I}$$

and

$$\left.\begin{array}{l} a_{(r+1)1}v_1 + a_{(r+1)2}v_2 + \ldots\ldots + a_{(r+1)n}v_n = w_{r+1} \\ a_{(r+2)1}v_1 + a_{(r+2)2}v_2 + \ldots\ldots + a_{(r+2)n}v_n = w_{r+2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots\ldots + a_{mn}v_n = w_m \end{array}\right\} \text{GROUP II}$$

Expressing the matrices corresponding to these groups by :

$$\underset{(rn)}{A_1} \qquad \underset{(r1)}{W_1} \qquad \text{and} \qquad \underset{(m-r)n}{A_2} \qquad \underset{(m-r)1}{W_2}$$

the system of condition equations could be put in the form :

$$\begin{vmatrix} A_1 \\ A_2 \end{vmatrix} V = \begin{vmatrix} W_1 \\ W_2 \end{vmatrix} \tag{2e}$$

Introducing the matrix of the weights (2b) and in addition the column vectors :

$$\begin{vmatrix} k_1 \\ k_2 \\ . \\ . \\ k_r \end{vmatrix} = K_1 \quad \text{and} \quad \begin{vmatrix} k_{r+1} \\ k_{r+2} \\ . \\ . \\ k_m \end{vmatrix} = K_2$$

the least squares condition will also be expressed by (2d) whereas, according to (2e), we shall write

$$\begin{vmatrix} A_1 \\ A_2 \end{vmatrix} dV = 0$$

Applying the Lagrange method we shall have :

$$V^*PdV - \begin{vmatrix} K_1^* \\ K_2^* \end{vmatrix}^* . \begin{vmatrix} A_1 \\ A_2 \end{vmatrix} dV = 0$$

or

$$V^*P - \begin{vmatrix} K_1^* \\ K_2^* \end{vmatrix}^* . \begin{vmatrix} A_1 \\ A_2 \end{vmatrix} = 0$$

whence

$$V = P^{-1} \begin{vmatrix} A_1^* \\ A_2^* \end{vmatrix}^* . \begin{vmatrix} K_1 \\ K_2 \end{vmatrix} = P^{-1} \begin{vmatrix} A_1^* & A_2^* \end{vmatrix} \begin{vmatrix} K_1 \\ K_2 \end{vmatrix}$$

or again

$$V = P^{-1}(A_1^* K_1 + A_2^* K_2) \tag{2f}$$

By entering in (2*e*) the expression of V given by (2*f*) we have the system :

$$\begin{vmatrix} A_1 \\ A_2 \end{vmatrix} (P^{-1}A_1^* K_1 + P^{-1}A_2^* K_2) = \begin{vmatrix} W_1 \\ W_2 \end{vmatrix}$$

and by developing we obtain :

$$\begin{aligned} A_1P^{-1}A_1^* K_1 + A_1P^{-1}A_2^* K_2 &= W_1 \\ A_2P^{-1}A_1^* K_1 + A_2P^{-1}A_2^* K_2 &= W_2 \end{aligned} \tag{2g}$$

which is the system of normal equations.

By making, for reasons of simplification,

$$A_1P^{-1}A_1^* = N_{11} \qquad A_1P^{-1}A_2^* = N_{12}$$

$$(2h) \qquad\qquad (2i)$$

$$A_2P^{-1}A_1^* = N_{21} \qquad A_2P^{-1}A_2^* = N_{22}$$

the system (2*g*) is transformed into :

$$\begin{aligned} N_{11}K_1 + N_{12}K_2 &= W_1 \\ N_{21}K_1 + N_{22}K_2 &= W_2 \end{aligned} \tag{2j}$$

From this system's first equation we take :

$$K_1 = N_{11}^{-1} W_1 - N_{11}^{-1} N_{12}K_2$$

where by making

$$N_{11}^{-1} W_1 = \overline{K}_1 \tag{2k}$$

and

$$-N_{11}^{-1} N_{12}K_2 = \Delta K_1 \tag{2l}$$

we obtain

$$K_1 = \overline{K}_1 + \Delta K_1$$

where $\overline{K}_1$ represents the correlative coefficients when only the condition equations of Group I are taken into account. Entering this expression in the (2*f*) system we have :

$$V = P^{-1}A_1^* \overline{K}_1 + P^{-1}A_1^* \Delta K_1 + P^{-1}A_2^* K_2$$

or, taking expression (2*l*) into account,

$$V = P^{-1}A_1^* \overline{K}_1 + P^{-1}(A_2^* - A_1^* N_{11}^{-1} N_{12})K_2 \tag{2m}$$

Comparing this expression with (2*f*) we may draw the following conclusions :

a) $P^{-1}A_1^* K_1$ corresponds to $P^{-1}A_1^* \overline{K}_1$, $\overline{K}_1$ being the value of the correlative coefficient obtained when only the condition equations of group I are taken into account, whereas K_1 is the value obtained when all the condition equations of the system are taken into account.

b) $P^{-1}A_2^*$ corresponds to $P^{-1}(A_2^* - A_1^* N_{11}^{-1} N_{12})K_2$ where K_2 is the same in both expressions and where A_2^* is changed to $A_2^* - A_1^* N_{11}^{-1} N_{12}$.

Let us now analyse the correction $P^{-1}A_1^* N_{11}^{-1} N_{12}$ which we shall call C_2. This correction is merely the solution by the least squares method of a multiple system similar to that arising from the condition of the first step and whose form is :

$$A_1C_2 = N_{12}$$

In reality this is a multiple system since C_2 and N_{12} are matrices and not single vectors. In fact each column of N_{12} represents the residuals obtained by correcting, according to the first-step equations, the coefficients of each second-step equation which are the elements of the rows of A_2. The columns of C_2 are the solutions obtained taking into account their corresponding columns in N_{12} and they represent the distribution, according to the least squares principle, of the above-mentioned residuals among the second-step coefficients which are the elements of A_2.

Thus we arrive at obtaining that the corrected coefficients satisfy the equations of the first group, that is that $A_1 A_2^* = 0$ is obtained.

This equality is very easily demonstratable since we may write :

$$A_1 \bar{A}_2^* = A_1(A_2^* - A_1^* N_{11}^{-1} N_{12}) = N_{12} - N_{11} N_{11}^{-1} N_{12} = 0$$

The matrices A_1 and $\bar{A}_2$ consequently represent orthogonal systems.

It is interesting to note that if we take into account the coefficients equal to zero appearing in both steps in the groups of equations the matrices A_1 and A_2 have the same number of columns. If coefficients of the first group that are not zero correspond to coefficients equal to zero in the second group these last will undergo corrections which will make them differ from zero. On the other hand the second group coefficients, to which coefficients equal to zero of the first group correspond, will not undergo any change. We are then able to draw the following conclusion. *According to the least squares postulate, any adjustment can be carried out in two steps provided that the coefficients corresponding to the second step, and which enter into the correction computation, are corrected according to the least squares principle to include the conditions of the first step.* This important conclusion can be generalised for any number of steps as soon as the coefficients corresponding to a step are corrected according to the conditions of the preceding steps.

Thus we may analyse several of the preceding expressions in the light of the principles of the statistical theory for random variables. In fact v_1, v_2, ... v_r can be considered as independent random variables which are observed in order to determine a physical value. If we make p_i correspond to each of such v_i variables these can be considered as normally distributed variables represented by a Gauss curve whose parameters have been homogenized according to the p_1, p_2, ... p_r weights. This being so, w_1, w_2, ... w_r which are linear functions of v_1, v_2, ... v_r will also be random variables, also with a normal distribution, whose parameters will be determined by the Gauss law concerning propagation of errors, in view of the fact that all the observations have the same weight equal to unity. Taking all the independent observations into account, and attributing to them unity weights and variances, the theory of statistics establishes that :

a) N_{11} is the matrix whose diagonal is made up by the variances, and the other elements are the covariances between the w_i residuals of the group I equations;

b) N_{12} is the covariance matrix between the w_i residuals of group I and the w_j residuals of the group II equations;

c) $N_{11}^{-1} N_{12}$ is consequently a covariance ratio which may be considered as a measure of correlation between the group I and the group II variables.

To complete the solution of this problem it remains to be shown how K_2 is determined. If we designate the vector of the corrections determined at the first step by V_1, taking only the group I condition equations into account, and the corrections obtained at the second step are designated by V_2, according to (2*m*) we have :

$$V_1 = P^{-1}A_1^* \overline{K}_1$$

and

$$V_2 = P^{-1}(A_2^* - A_1^* N_{11}^{-1} N_{12})K_2 \tag{2n}$$

thus

$$V = V_1 + V_2$$

From (2*e*) we then have

$$\begin{vmatrix} A_1 \\ A_2 \end{vmatrix} \begin{vmatrix} V_1 + V_2 \end{vmatrix} = \begin{vmatrix} W_1 \\ W_2 \end{vmatrix}$$

or, by developing :

$$A_1V_1 + A_1V_2 = W_1$$
$$A_2V_1 + A_2V_2 = W_2$$

But since in the first step only the group I condition equations are considered we necessarily have $A_1V_1 = W_1$, which leads to $A_2V_1 = 0$. This means that the first step corrections remain invariant. This being so, we may deduce from the last equation :

$$A_2V_2 = W_2 - A_2V_1 = \overline{W}_2 \tag{2o}$$

By multiplying both sides of (2*n*) by A_2, and by taking (2*o*) into account we obtain the equation :

$$A_2P^{-1}(A_2^* - A_1^* N_{11}^{-1} N_{12})K_2 = \overline{W}_2 \tag{2p}$$

whose solution yields K_2. Expression (2*o*) shows that $\overline{W}_2$ can easily be computed by using observations corrected in the first step to form the condition equations.

If the coefficients obtained by correcting A_2^* are designated by $\overline{A}_2^*$ we can put (2*p*) in the form :

$$A_2P^{-1}\overline{A}_2^* K_2 = \overline{W}_2 \tag{2q}$$

3. — GENERALIZING THE PROCEDURE

Let us take a system of condition equations separated into several groups as given below, the numbers of rows and columns being indicated between brackets at the right and following the same order as the elements of the equations. For example, (*tn*) corresponds to matrix A_3, (*n*1) to vector V, (*t*1) to vector W_3.

$$\begin{array}{ll}
A_1V = W_1 & (rn)\ (n1)\ (r1) \\
A_2V = W_2 & (sn)\ (n1)\ (s1) \\
A_3V = W_3 & (tn)\ (n1)\ (t1) \\
\cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots \\
A_kV = W_k & (un)\ (n1)\ (u1)
\end{array}$$

Designating by :

V_i = partial vector corresponding to the solution obtained at step i ($i = 1, 2, 3, \ldots\ldots k$);

$\overline{A}_i$ = matrix of coefficients corrected at step $i = A_i - C_i$, C_i being the correction matrix;

$\overline{K}_i$ = vector of the partial correlative coefficients at step i;

$\overline{W}_i$ = vector of the known term at step i, taking the corrections up to step $i - 1$ into account;

generalizing for case of n steps, we can write the following expressions which, according to the theorem we have demonstrated, are valid for the case of two steps :

$$V = V_1 + V_2 + V_3 + \ldots + V_k$$

$$\begin{array}{lll}
V_1 = A_1^* \overline{K}_1 & A_1 A_1^* \overline{K}_1 = W_1 & \overline{K}_1 = N_{11}^{-1} W_1 \\
V_2 = \overline{A}_2^* \overline{K}_2 & A_2 \overline{A}_2^* \overline{K}_2 = \overline{W}_2 & \overline{K}_2 = \overline{N}_{22}^{-1} \overline{W}_2 \\
V_3 = \overline{A}_3^* \overline{K}_3 & A_3 \overline{A}_3^* \overline{K}_3 = \overline{W}_3 & \overline{K}_3 = \overline{N}_{33}^{-1} \overline{W}_3 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
V_k = \overline{A}_k^* \overline{K}_k & A_k \overline{A}_k^* \overline{K}_k = \overline{W}_k & \overline{K}_k = \overline{N}_{kk}^{-1} \overline{W}_k
\end{array}$$

If we now make

$$A_i \overline{A}_i^* = \overline{N}_{ii}^{-1}$$

the A_i^* and $A_i\overline{A}_i^*$ expressions can be written in the following manner :

$$\left.\begin{array}{r}
\overline{A}_2^* = A_2^* - A_1^* N_{11}^{-1} N_{12} \\
A_2 \overline{A}_2^* = \overline{N}_{22} = N_{22} - N_{21} N_{11}^{-1} N_{12}
\end{array}\right\} \text{for } i = 2$$

$$\left.\begin{array}{r}
\overline{A}_3^* = A_3^* - A_1^* N_{11}^{-1} N_{13} - A_2^* N_{22}^{-1} N_{23} \\
A_3 \overline{A}_3^* = \overline{N}_{33} = N_{33} - N_{31} N_{11}^{-1} N_{13} - N_{32} \overline{N}_{22}^{-1} N_{23}
\end{array}\right\} \text{for } i = 3$$

or, in a general way

$$\left.\begin{array}{r}
\overline{A}_i^* = A_i^* - \sum_j A_j^* \overline{N}_{jj}^{-1} N_{ji} \\
A_i \overline{A}_i^* = \overline{N}_{ii} = N_{ii} - \sum_j N_{ij} \overline{N}_{jj}^{-1} N_{ji}
\end{array}\right\} \left\{\begin{array}{l}
i = 1, 2, 3, \ldots k \\
j = 1, 2, 3, \ldots k-1
\end{array}\right.$$

It is easy to deduce $\overline{W}_i$ since we know that it is equal to the value of W_i modified by the corrections made up to the $i - 1$ step, or again, to the residual obtained by introducing the corrected values into the condition equations. We can then write :

$$\overline{W}_i = W_i - A_i (V_1 + V_2 + \ldots V_{i-1})$$

If the condition equation systems lead to normal equations made up of triple sets, or if $N_{ij} = 0$ for $\pm i, \mp j > 1$, the C_{ii} matrix representing the correction of matrix N_{ii} is made up of a single matrix $N_{ij} \overline{N}_{jj}^{-1} N_{ij}$.

4. — PRACTICAL EXAMPLE FOR THE CASE OF TWO STEPS

To make the application of the method clearer we shall apply it to the case of adjustment of a doubly braced quadrilateral whose condition equations are :

$$\left\{\begin{array}{rrrrrrrrcl} v_1 + & v_2 + & v_3 + & v_4 + & v_5 + & v_6 + & v_7 + & v_8 & = & w_1 \\ v_1 + & v_2 & & - & v_5 - & v_6 & & & = & w_2 \\ & & v_3 + & v_4 & & - & v_7 - & v_8 & = & w_3 \\ d_1v_1 - & d_2v_2 + & d_3v_3 - & d_4v_4 + & d_5v_5 - & d_6v_6 + & d_7v_7 - & d_8v_8 & = & w_4 \end{array}\right.$$

In the first step let us consider the first three condition equations thus making up group I. Group II will be made up of the last equation alone. Again let us, for simplicity's sake, make P = 1 (unit matrix). This being so, from (2*k*) and the first formula in (2*h*) we shall have :

$$A_1A_1^* \bar{K}_1 = W_1 \tag{4a}$$

But the three first condition equations allow us to write :

$$A_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \end{vmatrix} \tag{4b}$$

By introducing this matrix and its transposed matrix into (4*a*), making the products and writing the results explicitly we have the equations :

$$\begin{vmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix} . \begin{vmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \bar{k}_3 \end{vmatrix} = \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix}$$

whose solution is obviously :

$$\bar{k}_1 = w_1/8; \; \bar{k}_2 = w_2/4 \text{ and } \bar{k}_3 = w_3/4 \tag{4c}$$

The V_1 corrections of the first step will then be given by expression (2*m*) where we shall make $P^{-1} = I$ and we shall neglect the K_2 term. Thus

$$V_1 = A_1\bar{K}_1$$

The elements of V_1 vector can be found by replacing in this expression A_1 by (4*b*) transposed, and K_1 by its (4*c*) elements. We thus obtain :

$$\begin{aligned} v_1 &= v_2 = w_1/8 + w_2/4 \\ v_3 &= v_4 = w_1/8 + w_3/4 \\ v_5 &= v_6 = w_1/8 - w_2/4 \\ v_7 &= v_8 = w_1/8 - w_3/4 \end{aligned}$$

In the practical example whose solution is given in form 4-1, suggested by Engineer MACHADO we have written the computed values for these corrections in the " corrections " column.

In this same form we then see the computation of the logarithmic closure which gives $\bar{W}_2$ immediately, since logarithms are taken for angles after their first correction.

To continue the second-step computation it is first necessary to form the matrix which is the coefficient of K_2 in expression $(2p)$ where we consider $P = 1$, as elsewhere. As to the elements we see that A_2 is the row vector made up of the d_i coefficients of v_i in the last condition equation, A_2^* being the column vector resulting from the transposition of A_2. A_1^* is the matrix obtained by transposing $(4b)$. This being so, there remains to compute only N_{11} with the help of the first formula of $(2h)$ and N_{12} with the first of $(2i)$. We then have :

$$N_{11} = A_1 A_1^* = \begin{vmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

whence

$$N_{11}^{-1} = \begin{vmatrix} 1/8 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{vmatrix}$$

and

$$N_{12} = A_1 A_2^* = \begin{vmatrix} d_1 - d_2 + d_3 - d_4 + d_5 - d_6 + d_7 - d_8 \\ d_1 - d_2 - d_5 + d_6 \\ d_3 - d_4 - d_7 + d_8 \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

thus

$$N_{11}^{-1} N_{12} = \begin{vmatrix} a/8 \\ b/4 \\ c/4 \end{vmatrix}$$

Consequently

$$A_1^* N_{11}^{-1} N_{12} = \begin{vmatrix} a/8 + b/4 \\ a/8 + b/4 \\ a/8 + c/4 \\ a/8 + c/4 \\ a/8 - b/4 \\ a/8 - b/4 \\ a/8 - c/4 \\ a/8 - c/4 \end{vmatrix}$$

Again, we have :

$$A_2^* - A_1^* N_{11}^{-1} N_{12} = \begin{vmatrix} d_1 - (a/8 + b/4) \\ -d_2 - (a/8 + b/4) \\ d_3 - (a/8 + c/4) \\ -d_4 - (a/8 + c/4) \\ d_5 - (a/8 - b/4) \\ -d_6 - (a/8 - b/4) \\ d_7 - (a/8 - c/4) \\ -d_8 - (a/8 - c/4) \end{vmatrix}$$

FORM 4-1

ANGLES

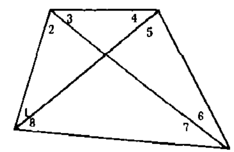
$$-w_1 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 - 360°$$
$$-w_2 = 1 + 2 - 5 - 6$$
$$-w_3 = 3 + 4 - 7 - 8$$

$$(A_1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \end{bmatrix} ; V_1 = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{bmatrix} ; W_1 = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

COTÉS

$$\overline{W}_2 = \log \frac{\sin \bar{2} \sin \bar{4} \sin \bar{6} \sin \bar{8}}{\sin \bar{1} \sin \bar{3} \sin \bar{5} \sin \bar{7}}$$

$$\frac{A_2}{W} = [d_1 - d_2\ d_3 - d_4\ d_5 - d_6\ d_7 - d_8]$$



	Angles observés		Corrections $V_1 = A_1^{\bullet}\bar{K}_1$			Angles corrigés	Log sin +	Log sin -	d_1 $(A_2^{\bullet})$		C_2 $A_1^{\bullet}N_{11}^{-1}N_{12}$			d_1' $(\bar{A}_2^{\bullet})$	$d_1 d_1'$ $A_2\bar{A}_2^{\bullet}$	Corr. $V_2 = \bar{A}_2^{\bullet}K_2$	Angles compensés
1	44° 49' 16''		0,13	-0,50		15'',68	9,8481240		21,2		4,43	31'',75		- 6,12	- 129,7	-0'',03	15'',60
2	42 46 15		0,12	-0,50		14,62		9,9489910	- 10,8		4,42	-31,75		-38,13	411,8	-0,19	14,43
3		22° 42' 02''	0,13		0,25	02,38	9,5864936			50,4	4,43		1,45	-56,28	2836,5	0,28	02,66
4		49 42 27	0,12		0,25	27,37		9,8823845		-17,9	4,42		1,45	-12,03	215,3	-0,06	27,31
5	47 06 41		0,13	0,50		41,63	9,9966461		- 2,7		4,43	31,75		33,48	9 ,	0,17	41,80
6	10 28 48		0,12	0,50		48,62		9,2598214	-113,9		4,42	31,75		-77,73	8853,4	-0,39	48,23
7		20 56 43	0,13		-0,25	42,88	9,5532465			55,1	4,43		-1,45	58,08	3200,2	0,29	43,17
8		51 27 47	0,12		-0,25	46,87		9,8933212		-16,8	4,42		-1,45	13,83	237,3	-0,07	46,80

107 35 31 + 72 24 29 = 179 59' 60''
107 35 29 + 72 24 30 = 179 59 59
$w_2 =$ -2 $w_3 =$ +1 359 59 59
$1 =$ +1

$$\bar{K}_1 = \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \bar{k}_3 \end{bmatrix} = A_1 A_1^{\bullet} W_1 = \underbrace{\begin{bmatrix} 1/8 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}}_{N_{11}^{-1}} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0,125 \\ -0,50 \\ -0,25 \end{bmatrix}$$

00,00 8,9845102 | 8,9845181
102
$\overline{W}_2 =$ 79

10,4 32,5 = 42,9
-116,6 38,3 = 78,3
$-b_2$ -127,0
$-c$ = -5,8
$-a$ = -35,4

$$N_{12} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad N_{11}^{-1}N_{12} = \begin{bmatrix} a/8 = 4,43 \\ b/4 = 31,75 \\ c/4 = 1,45 \end{bmatrix}$$

$\Sigma dd' = 15710,2$

$$K_2 = \frac{\overline{W}_2}{\Sigma dd'} = 0,00503$$

The vector whose elements are defined in the right-hand side of this expression is designated d' in the form. The sum of the products dd' shown there is equal to

$$A_2(A_2^* - A_1^* N_{11}^{-1} N_{12}) = A_2 \bar{A}_2^*$$

which is the coefficient of K_2 for $P = 1$ in $(2p)$. This being so, to obtain K_2 it is only necessary to divide $\overline{W}_2$ by dd'. This value multiplied by each value of d' will give the second corrections.

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