PROPAGATION OF RANDOM ERRORS ALONG AN ANALOGIC AERIAL TRIANGULATION STRIP

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1. — INTRODUCTION

From the reports on the International Congress of Photogrammetry held in Lisbon in September 1964 one can see that discussions on the merits of analytical *versus* analogic aerial triangulation are far from being exhausted. Thus I believe further study of error propagation along a strip of analogic aerial triangulation is still useful.

The majority of the explanations on this subject that I know of are usually difficult to follow because the excessive amount of algebra hides the main physical aspect of the problem. Hence I propose in this article to derive the formulae expressing the propagation of accidental errors directly from the double accumulation of these errors.

2. — PROPAGATION OF ERRORS ALONG THE STRIP

In order to simplify our development, errors in azimuth, scale and general tip (Φ) will be treated separately. In addition we will only consider points along the axis of the strip and take the second nadir pass point N_2 as origin of the coordinates.

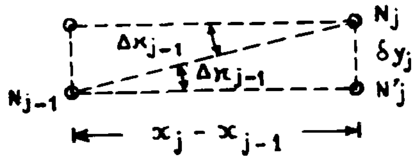

FIG. 2.1

Let us assume (figure 2.1) that stereo model $(j-1, j)$ is oriented without any error. Then if we rotate camera $(j-1)$ by $\Delta\varkappa_{j-1}$ about its vertical axis, there will be a y-parallax $N_jN'_j$. This parallax must be eliminated by moving camera j by an amount dby such that point N'_j coincides with N_j. Figure 2.1 shows that the displacement on the stereo model will be :

$$\delta y_j = (x_j - x_{j-1}) \tan \Delta\varkappa_{j-1} + dy_j$$

where dy_j is the accidental error arising from the y-parallax elimination. But as $\Delta\varkappa_{j-1}$ is small, we can write :

$$\delta y_j = (x_j - x_{j-1})\, \Delta\varkappa_{j-1} + dy_j \tag{2a}$$

Now we see that a new y-parallax will appear in N_{j-1}, exactly equal to δy_j, and its elimination is accomplished by a rotation of the projecting camera j about its vertical axis. Obviously this rotation is also equal to $\Delta\varkappa_{j-1}$. Hence, if we designate by $d\varkappa_j$ the accidental error which arises when we turn camera j, we can write :

$$\Delta\varkappa_j = \Delta\varkappa_{j-1} + d\varkappa_j$$

Thus we can see that the swing error of any camera can be expressed as a function of the swing error of the preceding one. Therefore the following expressions can be written :

$$\begin{array}{lcccc}
\Delta\varkappa_2 & = & & & d\varkappa_2 \\
\Delta\varkappa_3 & = & \Delta\varkappa_2 & + & d\varkappa_3 \\
\Delta\varkappa_4 & = & \Delta\varkappa_3 & + & d\varkappa_4 \\
\dots & & \dots & & \dots \\
\dots & & \dots & & \dots \\
\Delta\varkappa_{i-1} & = & \Delta\varkappa_{i-2} & + & d\varkappa_{i-1}
\end{array}$$

By adding these expressions we obtain :

$$\Delta\varkappa_{i-1} = d\varkappa_2 + d\varkappa_3 + d\varkappa_4 + \dots + d\varkappa_{i-1} \tag{2b}$$

which is the swing error of camera $(i - 1)$.

From figure 2.1 one can see that error δy_j will be integrally transmitted from pair $(j - 1, j)$ to $(j, j + 1)$, since the bridging technique consists in equalizing coordinates of the pass points. Consequently the total error in pass point N_i will be given by the sum of all expressions similar to (2a) from $j = 3$ to $j = i$, that is to say

$$\Delta y_i = \sum_{j=3}^{i} (x_j - x_{j-1})\, \Delta\varkappa_{j-1} + \sum_{j=3}^{i} dy_j$$

But dy_j is usually very small and its random characteristic permits us to neglect its summation in the above expression. In addition if we assume that the strip has n photographs, the accuracy will not be jeopardized if we replace $(x_j - x_{j-1})$ by its mean value between pass points N_2 and N_{n-1}, that is

$$(x_j - x_{j-1}) \approx (x_{n-1} - x_2) \,/\, (n - 3) = B \tag{2c}$$

Then

$$\Delta y_i = B \sum_{j=3}^{i} \Delta\varkappa_{j-1}$$

Now, by taking into account (2b) for $i = j$, we have

$$\Delta y_i = B \sum_{j=3}^{i} \sum_{m=2}^{j-1} d\varkappa_m$$

where the double accumulation of accidental errors is apparent. The development of this expression is

$$\Delta y_i/B = (i-2)d\varkappa_2 + (i-3)d\varkappa_3 + \dots\ d\varkappa_{i-1} \tag{2d}$$

Δy_i is obviously the measure of lateral bending of the strip and the swing error $\Delta\varkappa_{i-1}$ is the difference between the azimuth of the stereo model $(i-1, i)$ and that of the first one. Hence, if we have control points on the first and last stereo models we can find directly the value Δy_n and the azimuth differences. These will be the closing errors which will be designated, respectively, by w_y and w_A. Therefore equations (2b) and (2d) for $i = n$ will be the condition equations written below in matrix form :

$$\begin{Vmatrix} n-2 & n-3 & \dots\dots & 1 \\ 1 & 1 & \dots\dots & 1 \end{Vmatrix} \{d\varkappa\} = \begin{Bmatrix} w_y/B \\ w_A \end{Bmatrix}$$

where $\{ d\varkappa \}$ is the column vector of the swing corrections. The corresponding correlative equations are

$$\{d\varkappa\} = \begin{Vmatrix} n-2 & 1 \\ n-3 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \end{Vmatrix} \begin{Bmatrix} C_1 \\ \\ \\ \\ C_2 \end{Bmatrix} \tag{2e}$$

The system of normal equations will be obtained by multiplying both members of (2e) by the transposed matrix of that multiplying vector $\{ C \}$. Then we have

$$\begin{Vmatrix} \sum_{i=2}^{n-1} (n-i)^2 & \sum_{i=2}^{n-1} (n-i) \\ \sum_{i=2}^{n-1} (n-i) & n-2 \end{Vmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} w_y/B \\ w_A \end{Bmatrix}$$

By applying formulae which give the sum of the whole numbers from 1 to $(n-2)$ and the sum of the squares of these numbers the above equations can be written as follows :

$$\begin{Vmatrix} (n-1)\,(n-2)\,(2n-3)\,/\,6 & (n-1)\,(n-2)\,/\,2 \\ (n-1)\,(n-2)/2 & n-2 \end{Vmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} w_y/B \\ w_A \end{Bmatrix}$$

The solution of this system is

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Vmatrix} 12/(n-1)\,(n-2)\,(n-3) & -6/(n-2)\,(n-3) \\ -6/(n-2)\,(n-3) & 2(2n-3)/(n-2)\,(n-3) \end{Vmatrix} \begin{Bmatrix} w_y/B \\ w_A \end{Bmatrix} \tag{2f}$$

These correlative factors being known, we can derive the formula which gives Δy_i. Let us write (2*e*) explicitly, from $d\varkappa_2$ to $d\varkappa_{i-1}$:

$$\begin{aligned} d\varkappa_2 &= (n-2) \quad C_1 + C_2 \\ d\varkappa_3 &= (n-3) \quad C_1 + C_2 \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ d\varkappa_{i-1} &= (n-i+1)C_1 + C_2 \end{aligned}$$

By multiplying the first of these equations by $(i - 2)$, the second by $(i - 3)$ and so on, we have :

$$\begin{aligned} (i-2)d\varkappa_2 &= [in - 2(i+n) + 4] C_1 + (i-2)C_2 \\ (i-3)d\varkappa_3 &= [in - 3(i+n) + 9] C_1 + (i-3)C_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ d\varkappa_{i-1} &= [in-(i-1)(i+n)+(i-1)^2] C_1 + \qquad C_2 \end{aligned}$$

By adding these equations we have, according to (2*d*),

$$\Delta y_i/B = [ni(i-2) - (i+n) \sum_{m=1}^{i-2} (1+m) + \sum_{m=1}^{i-2} (1+m)^2] C_1 + \sum_{m=1}^{i-2} mC_2$$

Applying the general formulae giving the sums indicated above and performing some tedious algebraic transformations, we come to

$$\Delta y_i = \frac{B}{6} \{-C_1 i^3 + 3(nC_1+C_2)i^2 - [(9n-7)C_1+9C_2]\, i + 6(n-1)C_1+C_2\} \qquad (2g)$$

and we have $\Delta y_i = 0$ for $i = 2$.

Since the first correction is to be applied to the ordinate of N_3, the accuracy will not be jeopardized if we write any abcissa x in terms of the mean distance between two successive pass points. Thus,

$$x = (i-2)B$$

and we have

$$i = x/B+2$$

And by introducing this value of i into (2*g*), we come to the polynomial

$$\Delta y = b_3x^3 + b_2x^2 + b_1x \qquad (2h)$$

$x = 0$ corresponding to the first pass point N_2. Parameters b_3, b_2 and b_1 could be expressed in terms of the coefficients of i^q $(q = 0,1,2,3)$ in formula (2*g*), but since these parameters can be determined directly, there is no interest in deriving these expressions.

Let us now study the scale error. The scale denominator of the first stereo model, which we designated by K_{12}, can be directly found by means of the ground control points. However, when we transfer the scale from pair (1,2) to pair (2,3), some accidental error arises which we designate by dK_2. Thus the scale denominator of pair (2,3) will be $K_{12} + dK_2$. In transferring scale from (2,3) to (3,4) a new error dK_3 will arise, and so on. Hence a general expression can be written for the scale denominator of stereo model $(i - 1, i)$:

$$K_{i-1,\,i} = K_{12} + dK_2 + dK_3 + \dots + dK_{i-1} \qquad (2i)$$

Now, if X_2, X_3, etc. are the natural scale values of the instrumental abcissae x_2, x_3, etc., we have for each pair, according to its own scale :

$$\begin{aligned} X_3 - X_2 &= (x_3 - x_2\ )K_{23} \\ X_4 - X_3 &= (x_4 - x_3\ )K_{34} \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ X_i - X_{i-1} &= (x_i - x_{i-1})K_{i-1,i} \end{aligned}$$

Addition of these expressions gives us

$$X_i - X_2 = \sum_{j=3}^{i} (x_j - x_{j-1})K_{j-1,j}$$

or, according to ($2i$) for $i = j$,

$$X_i - X_2 = \sum_{j=3}^{i} (x_j - x_{j-1})(K_{12} + \sum_{m=2}^{j-1} dK_m)$$

But we can replace, as we did before, the coefficients $(x_j - x_{j-1})$ of ΣdK_m by their mean value B, and we have

$$X_i - X_2 = (x_i - x_2)K_{12} + B \sum_{j=3}^{i} \sum_{m=2}^{j-1} dK_m$$

Now it must be noted that $K_{12}x_2 = X_2$ since it was assumed that x_2 is correct. Therefore we can write.

$$X_i - K_{12}x_i = B \sum_{j=3}^{i} \sum_{m=2}^{j-1} dK_m$$

where the double accumulation of random errors is once more apparent. In addition, as x_i is the instrumental abcissa, it is easy to understand that $(X_i/K_{12} - x_i)$ is the error Δx_i we are looking for. Consequently, if we divide both members of the above expression by K_{12} and put $dK/K = d\lambda$, we obtain

$$\Delta x_i/B = \sum_{j=3}^{i} \sum_{m=2}^{j-1} d\lambda_m$$

or, by developing,

$$\Delta x_i/B = (i-2)d\lambda_2 + (i-3)d\lambda_3 + \dots + d\lambda_{i-1} \tag{2j}$$

If we now put

$$K_{i-1,i}/K_{12} - 1 = \Delta\lambda_{i-1} \tag{2k}$$

expression ($2i$) can be changed into

$$\Delta\lambda_{i-1} = d\lambda_2 + d\lambda_3 + \dots + d\lambda_{i-1} \tag{2l}$$

It is seen, merely by inspection, that ($2j$) and ($2l$) are of the same form as ($2d$) and ($2b$), respectively. Therefore the same reasoning will drive us to another polynomial similar to ($2h$), that is to say to :

$$\Delta x = c_3x^3 + c_2x^2 + c_1x \tag{2m}$$

Height errors still remain, and these will be studied for the aerial levelling and $bz = 0$ methods of bridging. Both methods are based on bz values established beforehand; hence to eliminate y-parallaxes in each stereo model it will always be necessary to modify the tip φ of the aft projecting camera. Consequently, camera i will have tip φ in the $(i - 1, i)$ stereo model and tip φ' in stereo model $(i, i + 1)$. The difference $\varphi' - \varphi = \delta\varphi$ can be taken as the angle between the projection planes of stereo models $(i - 1, i)$ and $(i, i + 1)$, as is shown in figure 2.2. If we designate by Φ_{12} the residual general tip of the first stereo model we will have for stereo model (2,3) :

$$\Phi_{23} = \Phi_{12} + \delta\varphi_2$$

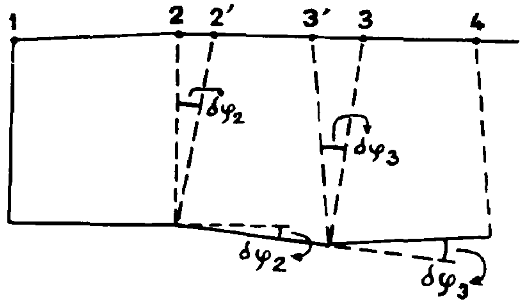

FIG. 2.2

But the errors of setting tips φ_2 and φ_2' will appear in $\delta\varphi_2$. So, if we designate by $d\varphi_2$ the total error of setting, the above expression must be changed into

$$\Phi_{23} = \Phi_{12} + \delta\varphi_2 + d\varphi_2$$

The same reasoning may be generalized in order to find the following equalities :

$$\begin{aligned}
\Phi_{23} &= \Phi_{12} + \delta\varphi_2 + d\varphi_2 \\
\Phi_{34} &= \Phi_{23} + \delta\varphi_3 + d\varphi_3 \\
&\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots \\
\Phi_{i-1,\,i} &= \Phi_{i-2,\,i-1} + \delta\varphi_{i-1} + d\varphi_{i-1}
\end{aligned}$$

Adding these expressions gives us

$$\Phi_{i-1,\,i} = \Phi_{12} + \sum_{m=2}^{i-1} \delta\varphi_m + d\varphi_2 + d\varphi_3 + \ldots + d\varphi_{i-1} \qquad (2n)$$

Now figure 2.3 shows us that the *local* altimetric error at pass point N_j is given by

$$\delta z_j = (x_j - x_{j-1})\, \Phi_{j-1,\,j}$$

or, according to $(2n)$ where we make $i = j$,

$$\delta z_j = (x_j - x_{j-1}) \left(\Phi_{12} + \sum_{m=2}^{j-1} \delta\varphi_m + \sum_{m=2}^{j-1} d\varphi_m \right)$$

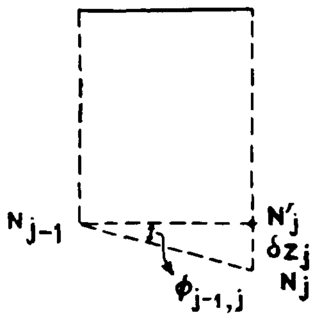

FIG. 2.3

It is now very clear that the altimetric error at pass point N_i will be equal to the sum of all values of δz_j, from $j = 3$ to $j = i$, i.e.

$$\Delta z_i = \sum_{j=3}^{i} (x_j - x_{j-1}) \left(\Phi_{12} + \sum_{m=2}^{j-1} \delta\varphi_m + \sum_{m=2}^{j-1} d\varphi_m \right)$$

Here also the values of $(x_j - x_{j-1})$ multiplying $\Sigma d\varphi_m$ can be replaced by their mean value B, and the above expression will be changed into

$$\Delta z_i = \Phi_{12}(x_i - x_2) + \sum_{j=3}^{i} (x_j - x_{j-1}) \sum_{m=2}^{j-1} \delta\varphi_m + \mathrm{B} \sum_{j=3}^{i} \sum_{m=2}^{j-1} d\varphi_m$$

Therefore, if we put

$$\left[\Delta z_i - (x_i - x_2)\, \Phi_{12} - \sum_{j=3}^{i} (x_j - x_{j-1}) \sum_{m=2}^{j-1} \delta\varphi_m \right] / \, \mathrm{B} = \theta_i \tag{2o}$$

we can write

$$\theta_i = \sum_{j=3}^{i} \sum_{m=2}^{j-1} d\varphi_m = (i-2)d\varphi_2 + (i-3)d\varphi_3 + \dots + d\varphi_{i-1} \tag{2p}$$

Comparing (2*p*) with (2*d*) we see that they are analogous, and the same will be observed if we transpose the first two terms of the second member of (2*n*) and compare the result with (2*b*). Thus a similar development will give us

$$\theta_i = a'_3\, x^3 + a'_2\, x^2 + a'_1\, x$$

and if we put

$$\| \, a'_3 \quad a'_2 \quad a'_1 \, \| = \frac{1}{\mathrm{B}} \, \| \, a_3 \quad a_2 \quad a_1 \, \|$$

the above polynomial and formula (2*o*) give us

$$\Delta z = (x_i - x_2)\, \Phi_{12} - \sum_{j=3}^{i} (x_j - x_{j-1}) \sum_{m=2}^{j-1} \delta\varphi_m + a_3 x^3 + a_2 x^2 + a_1 x \tag{2q}$$

This expression is a general one, since if the bridging is done without establishing the bz values beforehand we need only make $\delta\varphi = 0$ in order to use the formula.

It can now be pointed out that however formulae (2h), (2m) and (2q) have been derived for pass points on the strip's axis, these formulae can be considered as continuous functions of x without jeopardizing accuracy. There it is possible to admit that the centre of gravity of the control points in the first stereo model is the origin of all computations. Hence the following expressions can be derived from the above-mentioned formulae :

$$\partial\, \Delta y \,/\, \partial\, x \;=\; 3b_3x^2 + 2b_2x + b_1 \;=\; \Delta A \qquad \text{(azimuth error)}$$
$$\partial\, \Delta x \,/\, \partial\, x \;=\; 3c_3x^2 + 2c_2x + c_1 \;=\; \Delta\lambda \qquad \text{(scale error)}$$
$$\partial\, \Delta z \,/\, \partial\, x \;=\; 3a_3x^2 + 2a_2x + a_1 \;=\; \Delta\Phi \qquad \text{(tip error)}$$

3. — SOME PRACTICAL CONSIDERATIONS

We believe it would be useful at this point to show with figures an interesting result of our reasoning. In table XII of BACHMANN's book *Théorie des Erreurs de l'Orientation Relative* (1943), we find the five parameters obtained from 25 relative orientations of the same stereo model. That table also shows us all the parameters' deviations from their mean values. The complete independence of each one of these 25 operations permits us to assume that they are the relative orientations of 25 stereo models of an aerial triangulation having 27 photograms. Thus it is possible to consider the deviations shown in that table as bridging errors in $\varkappa$, φ, ω, by and bz, this last parameter not having been established beforehand. It is well known that in this kind of bridging a strong bending appears in the strip due to the earth curvature effect and other small systematic errors. We shall here investigate the effect of the doubly accumulated random errors which has a considerable role in that bending.

The second column of table 3-I shows the deviations of BACHMANN's table concerning the parameter φ. The third and fourth columns contain the figures resulting from single and double accumulations, respectively, of the deviations. Consequently the third column is, according to (2n), equal to $\Phi_{i-1,\,i}$ for $\Phi_{12} = \delta\varphi = 0$, and the fourth column records the values of θ_i computed by formula (2p). Obviously the last figures recorded in these columns are the closing errors w_Φ and w_Θ. It is evident that in this special case w_Φ should be equal to zero because it is the sum of deviations from an arithmetical mean, thus the figure we see there is a rounding off error which we will take as w_Φ.

Replacing w_y/B and w_A in expression (2f) by θ_i and w_Φ, respectively, we can find C_1 and C_2 for $n = 27$ and compute $d\varphi_c$ by expression (2e) where we replace $d\varkappa$ by $d\varphi$. Columns 5, 6 and 7 of table 3-I show the computed values of $d\varphi_c$, Φ_c and θ_c.

We are now in a position to correct the strip's bending, that is to say, the altimetric errors resulting from that bending. These corrections are

TABLE 3-I

i	$d\varphi$	$\Delta\Phi$	θ	$d\varphi_c$	$\Delta\Phi_c$	θ_c	Δz	Δz_c	diff.
2	– 0,2	– 0,2	– 0,2	– 5,3249	– 5,3249	– 5,3249	0,0	– 1,1	1,1
3	– 1,1	– 1,3	– 1,5	– 4,8788	– 10,2037	– 15,5286	– 0,3	– 3,1	2,8
4	– 6,5	– 7,8	– 9,3	– 4,4328	– 14,6365	– 30,1651	– 1,9	– 6,0	4,1
5	– 5,4	– 13,2	– 22,5	– 3,9867	– 18,6232	– 48,7883	– 4,5	– 9,8	5,3
6	2,3	– 10,9	– 33,4	– 3,5406	– 22,1638	– 70,9521	– 6,8	– 14,2	7,4
7	– 0,8	– 11,7	– 45,1	– 3,0945	– 25,2583	– 96,2104	– 9,0	– 19,2	10,2
8	– 2,0	– 13,7	– 58,8	– 2,6485	– 27,9068	– 124,1172	– 11,8	– 24,8	12,0
9	– 17,3	– 31,0	– 89,8	– 2,2024	– 30,1092	– 154,2264	– 18,0	– 30,8	12,8
10	– 7,0	– 38,0	– 127,8	– 1,7563	– 31,8655	– 186,0919	– 25,6	– 37,2	11,6
11	– 9,5	– 47,5	– 175,3	– 1,3102	– 33,1757	– 219,2676	– 35,1	– 43,9	8,8
12	4,6	– 42,9	– 218,2	– 0,8642	– 34,0399	– 253,3075	– 43,6	– 50,7	7,1
13	9,2	– 33,7	– 251,9	– 0,4181	– 34,4580	– 287,7655	– 50,4	– 57,6	7,2
14	16,2	– 17,5	– 269,4	0,0280	– 34,4300	– 322,1955	– 53,9	– 64,4	10,5
15	– 0,4	– 17,9	– 287,3	0,4741	– 33,9559	– 356,1514	– 57,5	– 71,2	13,7
16	– 10,2	– 28,1	– 315,4	0,9202	– 33,0357	– 389,1871	– 63,1	– 77,8	14,7
17	– 2,6	– 30,7	– 346,1	1,3662	– 31,6695	– 420,8566	– 68,2	– 84,2	16,0
18	– 5,8	– 36,5	– 382,6	1,8123	– 29,8572	– 450,7138	– 76,5	– 90,1	13,6
19	– 5,9	– 42,4	– 425,0	2,2584	– 27,5988	– 478,3126	– 85,0	– 95,7	10,7
20	6,0	– 36,4	– 461,4	2,7045	– 24,8943	– 503,2069	– 92,3	– 100,6	8,3
21	3,6	– 32,8	– 494,2	3,1505	– 21,7438	– 524,9507	– 98,8	– 105,0	6,2
22	– 6,5	– 39,3	– 533,5	3,5966	– 18,1472	– 543,0979	– 106,7	– 108,6	1,9
23	17,0	– 22,3	– 555,8	4,0427	– 14,1045	– 557,2024	– 111,2	– 111,4	0,2
24	10,5	– 11,8	– 567,6	4,4888	– 9,6157	– 566,8181	– 113,5	– 113,4	0,1
25	7,9	– 3,9	– 571,5	4,9348	– 4,6809	– 571,4990	– 114,3	– 114,3	0,0
26	4,6	0,7	– 570,8	5,3809	0,7000	– 570,7990	– 114,2	– 114,2	0,0

$C_1 = -0,4460769$; $C_2 = 5,8270000$ $d\varphi_c = (n - i)C_1 + C_2$

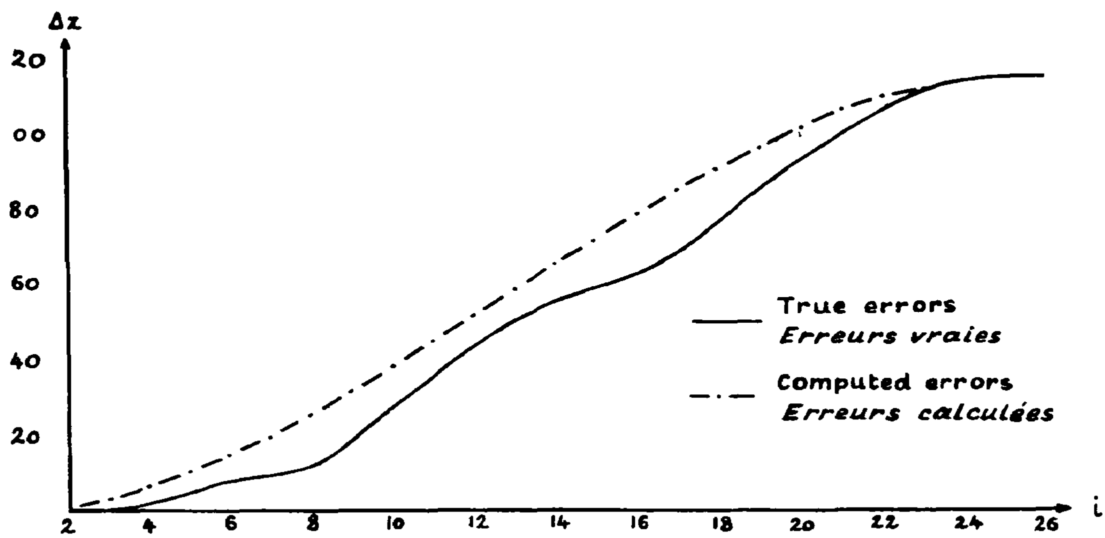

FIG. 3.1

equal to the quantity between brackets in formula (2*o*) where $\Phi_{12} = \delta\varphi = 0$ and where B is the mean distance between two consecutive pass points. Consequently we have :

$$\Delta z_i = B\theta_i$$

But θ_i is expressed in centesimal minutes and the above formula must be changed into

$$\Delta z_i = (\theta_i^c / 6366)B$$

Finally, by assuming B = 1273.2 m (0.2×6366) we can write

$$\Delta z_i = 0.2\theta_i$$

We have computed with this formula the figures recorded in columns 8 and 9 of table 3-I by using the true and computed values of θ_i, respectively. The last column shows the final height deviations. For the hypothetical strip chosen as example high values of the height deviations are the consequence of the deviations of the relative orientation which reach about 20^c.

Figure 3.1 gives a visual idea of the results.