THE ANALYSIS OF CURRENT OBSERVATIONS

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Abstract

A method is described for the approximate separation of the periodic and aperiodic processes contained in current observations. The short period irregularities, assigned mainly to turbulence, are first removed by applying a smoothing operator to the data. The low frequency band of the aperiodic process is then isolated with the help of a low pass filter. This low frequency variation is removed from the observations and the residues are analyzed for the presence of tidal constituents.

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The observation of currents in the open sea is a difficult task and it can be done only for relatively short intervals of time; on the other hand the information which it yields on the mass transport, the tidal oscillations and the turbulence of the flow is of great value. It is therefore quite evident that the data obtained should be subjected to the most thorough analysis possible.

We assume that a current measurement embodies the linear superposition of three independent processes : turbulence fluctuations, periodic (mainly tidal) oscillations and a "steady" flow.

The turbulence fluctuations are usually of a short period compared to those of the tidal oscillations. The tidal oscillations are found over a range of frequencies extending from 1 cycle/day to 8 cycles/day; they are of constant amplitude and phase and they constitute the regular portion of the record. The steady flow is created by gradients in pressure, salt or temperature, by the wind etc., and it cannot be expected to be constant in time : over a short interval, we may then consider the steady flow to be an aperiodic function of time. Its irregularities unfortunately cover the whole range of frequencies and certainly overlap the tidal band.

The three processes cannot be separated exactly from a short record of observations but such a separation should at least be attempted.

Some necessary mathematical notions

In spectral analysis a function may be defined in time (t) or in frequency (σ); any of the two definitions specify the function completely and the choice of one variable in preference to the other is a matter of convenience. We write the time representation of an abstract function $\mathcal{Z}$ as $z(t)$ and its frequency representation (or *spectrum*) as $Z(\sigma)$. The transition from one representation to the other is effected by a Fourier transformation:

$$Z(\sigma) = \int_{-\infty}^{\infty} z(t) e^{-2\pi i \sigma t}\, dt$$

$$z(t) = \int_{-\infty}^{\infty} Z(\sigma) e^{2\pi i \sigma t}\, d\sigma$$

An abstract linear operator $\mathcal{F}$ may also have a t and a σ representation as well, these we write as $f(t)$ and $F(\sigma)$. We define the operation:

$$\mathcal{F} \cdot \mathcal{Z}$$

to mean

$$\int_{-\infty}^{\infty} f(t - t_0) z(t) dt$$

in time, and

$$F(\sigma) \cdot Z(\sigma)$$

in frequency.

As a mathematical quantity, a current observation is a two dimensional vector and it can be written as:

$$\underline{z}(t) = [x(t), y(t)]$$

The pair of real numbers $x(t)$ and $y(t)$ are its components in the x and y directions. We assume that the vertical component of the velocity vector can be measured independently of its horizontal components.

A two dimensional vector is mathematically equivalent to a complex number and we therefore identify $\underline{z}(t)$ with the complex number $z(t)$ which can have any of the following representations:

$$z(t) = \begin{cases} [x(t), y(t)] \\ x(t) + iy(t) \\ |z(t)|\, e^{i\zeta(t)} \end{cases} \qquad i = \sqrt{-1}$$

$|z(t)|$ is the *magnitude* of $z(t)$ while $\zeta(t)$ is its *phase*. There are related to x and y through the relations:

$$|z(t)| = [x(t)^2 + y(t)^2]^{1/2}$$

$$\zeta(t) = \text{arc tan}\, y(t)/x(t)$$

The pair representation of complex numbers is useful for practical calculations while the other two representations serve for algebraic manipulations. A complex number may be stored in a computer as the couple of real numbers:

$$(x, y) \quad \text{or} \quad (|z|, \zeta)$$

The first encoding is to be preferred since linear operations on complex numbers imply pairs of identical operations on their x and y components. In general the complex number $ae^{i\delta}$ where a itself is complex may be interpreted as a vector $\underline{a}$ rotated counter-clockwise by an angle δ.

Our basic assumptions and these newly introduced notions permit us to write a given current observation at a time t as:

$$z(t) = d(t) + a_0(t) + \sum_{l=-n}^{n} a_l \, e^{2\pi i \sigma_l t} \qquad l \neq 0$$

where:

$d(t) \equiv$ the instantaneous value of the turbulence process;
$a_0(t) \equiv$ the "steady" flow at time t;
$a_l \equiv$ the complex amplitude of the lth tidal constituent;
$\sigma_l \equiv$ its frequency;
$n \equiv$ the number of tidal constituents.

We note

$$\sigma_{-l} = -\sigma_l$$

by definition
and

$$a^*_{-l} \neq a_l$$

The latter relation holds because $z(t)$ is a complex quantity.

The smoothing of the data

If $d(t)$ contains only high frequencies it may be removed from $z(t)$ by applying a smoothing operator $\mathfrak{S}$ to the sequence of observations.

The presence of short period oscillations implies that $z(t)$ is sampled at short intervals of time, let us say Δt.

We choose as a smoothing operator :

$$\mathfrak{S} = (\mathfrak{A}_k/k)^3 \qquad (1)$$

$\mathfrak{A}_k$ indicates in time the summation of k consecutive observations; $\mathfrak{A}_k/k$ therefore simply means an averaging of k consecutive observations. The exponent indicates the repeated application of such an operator to the same portion of the sequence of the observations. The spectrum of (1) is

$$S(\sigma) = \left(\frac{\sin \pi k \, \Delta t \, \sigma}{k \sin \pi \, \Delta t \, \sigma}\right)^3$$

which is shown in fig. 1. We note that it effectively cuts off all frequencies beyond 1 cycle/hour.

The application in time of the operator (1) is effected in the following way. We select a portion of the sequence of observations

$$\{ z(j \Delta t) \}$$

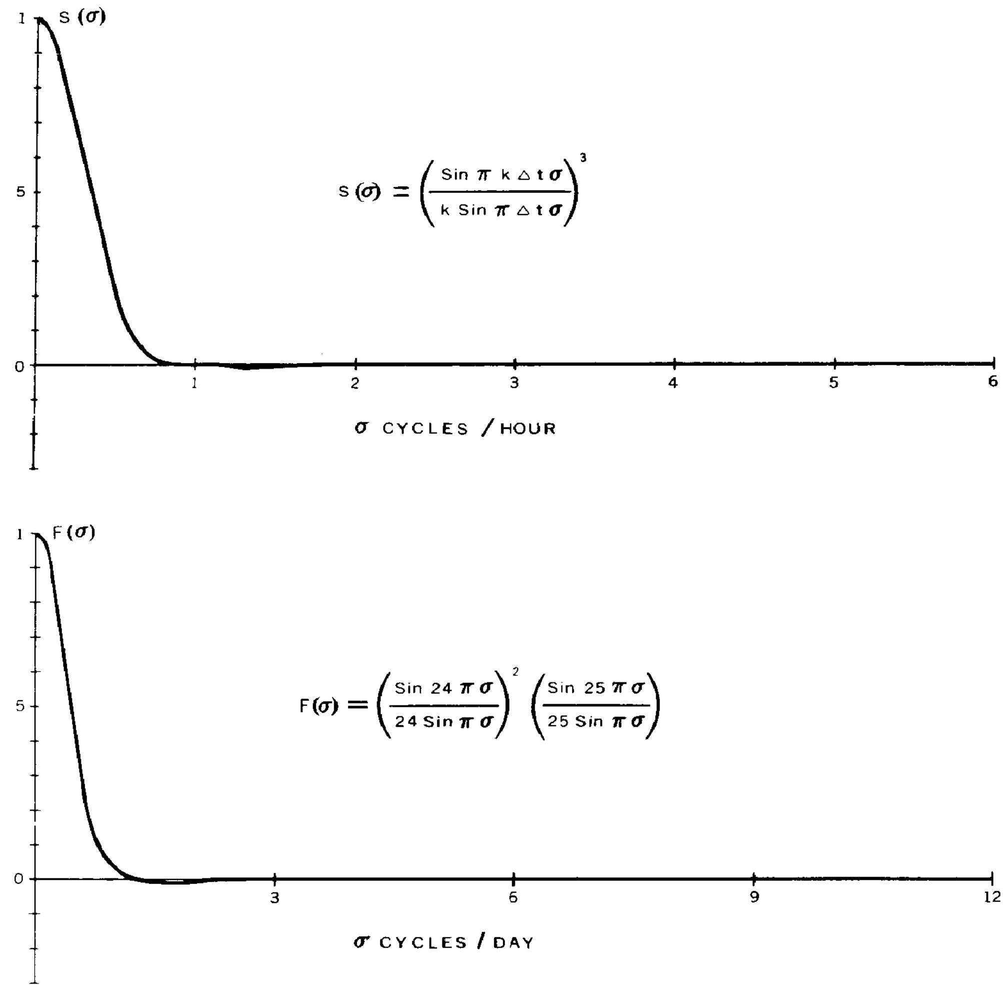

Fig. 1. — The spectra of the smoothing operator $(\mathcal{A}_k/k)^3$ and of the low pass filter $(\mathcal{A}_{24}/24)^2\,(\mathcal{A}_{25}/25)$.
On the scale used, the spectra of the smoothing operators for $k = 12$, 6 and 4 are nearly identical.

which contains $3(k-1)+1 = 3k-2$ elements. We write explicitly this subsequence as:

$$z_1, z_2, \ldots, z_k, z_{k+1}, \ldots, z_{2k}, z_{2k+1}, \ldots, z_{3k-2} \tag{2}$$

Applying $\mathcal{A}_k/k$ to $\mathcal{Z}$ means in time computing the sequence of averages:

$$\bar{z}_1 = (z_1+z_2+\ldots+z_k)/k,\ \bar{z}_2 = (z_2+\ldots+z_{k+1})/k,\ \ldots,\ \bar{z}_{2k-1} = (z_{2k-1}+\ldots+z_{3k-2})/k$$

$(\mathcal{A}_k/k)^2\,\mathcal{Z}$ means

$$\bar{\bar{z}}_1 = (\bar{z}_1+\bar{z}_2+\ldots+\bar{z}_k)/k,\ \bar{\bar{z}}_2 = (\bar{z}_2+\ldots+\bar{z}_{k+1})/k,\ \ldots,\ \bar{\bar{z}}_k = (\bar{z}_k+\ldots+\bar{z}_{2k-1})/k$$

and finally $(\mathcal{A}_k/k)^3\,\mathcal{Z}$ stands for :

$$\bar{\bar{\bar{z}}} = (\bar{\bar{z}}_1 + \bar{\bar{z}}_2 + \ldots + \bar{\bar{z}}_k)/k \tag{3}$$

The application of $(\mathcal{A}_k/k)^3$ to a sequence of $3k-2$ consecutive observations results then in a single number $\bar{\bar{\bar{z}}}$ which does not contain any contribution from $d(t)$. Table 2 gives some additional details on this smoothing operator. We have avoided giving its time representation because it is complicated as well as superfluous; it is preferable to restrict ourselves to its operational aspect displayed in (3).

The smoothed sequence can be sampled at larger time intervals than Δt, which is of order of a few minutes; we choose this new time step to be one hour and we write the smoothed sequence as:

$$\{ z(j) \} \qquad j = -N_1, \ldots, 0, \ldots, N_1$$

In what follows whenever the time step Δt is not mentioned, this implies that it is equal to one hour and we write:

$$z(j) = a_0(j) + \sum_{l=-n}^{n} a_l \, e^{2\pi i \sigma_l j}$$

The low frequency band

We may now apply a low pass filter to the smoothed observations. Our choice is

$$\mathbf{F} = (\mathcal{A}_{24}/24)^2 \, (\mathcal{A}_{25}/25)$$

whose spectrum is

$$F(\sigma) = \left(\frac{\sin 24 \, \pi \sigma}{24 \sin \pi \sigma} \right)^2 \left(\frac{\sin 25 \, \pi \sigma}{25 \sin \pi \sigma} \right)$$

$F(\sigma)$ is displayed in fig. 1 : it cuts off the frequencies which equal or exceed 1 cycle/day. This includes the tidal frequencies.

The operation $\mathbf{F} \cdot \mathbf{Z}$ results in time into the sequence:

$$\{ a_0'(j) \} \qquad j = N, \ldots, 0, \ldots, N \qquad N < N_1$$

$2N+1$ is the number of hourly observations left after smoothing and low passing.

With the help of the $a_0'(j)$'s we may form the sequence of the residues

$$\{ z_r(j) \} = \{ z(j) - a_0'(j) \}$$

an element of which we may write as

$$z_r(j) = \sum_{l=-n}^{n} a_l \, e^{2\pi i \sigma_l j} + \varepsilon(j)$$

where :

$$\varepsilon(j) = a_0(j) - a_0(j)$$

The notation reflects our belief that the sequence of the residues contains primarily the contribution of the tidal constituent.

The constituent ellipse

A given constituent l contributes the terms

$$z_l(j) = a_{-l}\, e^{-2\pi i \sigma_l j} + a_l\, e^{2\pi i \sigma_l j} \tag{4}$$

to the current at the instant j. (4) is a periodic function of j and its period T_l is equal to:

$$T_l = 1/\sigma_l$$

Over this time interval the tip of the vector represented by (4) draws an ellipse whose semi-major and semi-minor axes are of length:

$$M_l = |a_l| + |a_{-l}| \tag{5}$$

$$m_l = |a_l| - |a_{-l}| \tag{6}$$

The major axis is inclined to the x axis by an angle θ_l which is given by

$$\theta_l = \frac{1}{2}(\alpha_l + \alpha_{-l}) \tag{7}$$

and the value j_0 (not necessarily an integer) of j for which the vector (4) points along the semi-major axis is:

$$j_0 = \frac{1}{4\pi\sigma_l}(\alpha_{-l} - \alpha_l) \tag{8}$$

α_l and α_{-l} are the *phases* of a_l and a_{-l} since we may write these quantities as:

$$a_l = |a_l|\, e^{i\alpha_l}\,; \qquad a_{-l} = |a_{-l}|\, e^{i\alpha_{-l}}$$

Formulas (5) to (8) indicate that all the elements of the constituent ellipse can be evaluated from the quantities a_l and a_{-l}. These formulas were quoted without proof by TAYLOR (1920). We set out to give this proof.

(4) can be rewritten as

$$z_l(j) = |a_{-l}|\, e^{i(\alpha_{-l} - 2\pi\sigma_l j)} + |a_l|\, e^{i(\alpha_l + 2\pi\sigma_l j)} =$$

$$= e^{\frac{1}{2} i(\alpha_l + \alpha_{-l})}\left[|a_{-l}|\, e^{-i(2\pi\sigma_l j + \frac{1}{2}(\alpha_l - \alpha_{-l}))} + |a_l|\, e^{i(2\pi\sigma_l j + \frac{1}{2}(\alpha_l - \alpha_{-l}))}\right]$$

which we interpret as the vector

$$|a_{-l}|\, e^{-i(2\pi\sigma_l j + \frac{1}{2}(\alpha_l - \alpha_{-l}))} + |a_l|\, e^{i(2\pi\sigma_l j + \frac{1}{2}(\alpha_l - \alpha_{-l}))} \tag{9}$$

rotated by an angle $\frac{1}{2}(\alpha_l + \alpha_{-l})$.

The vector (9) can be written more concisely as

$$M_l \cos \eta_l + i\, m_l \sin \eta_l$$

where:

$$M_l \equiv |a_l| + |a_{-l}|$$

$$m_l \equiv |a_l| - |a_{-l}|$$

$$\eta_l \equiv 2\pi\sigma_l j + \tfrac{1}{2}(\alpha_l - \alpha_{-l})$$

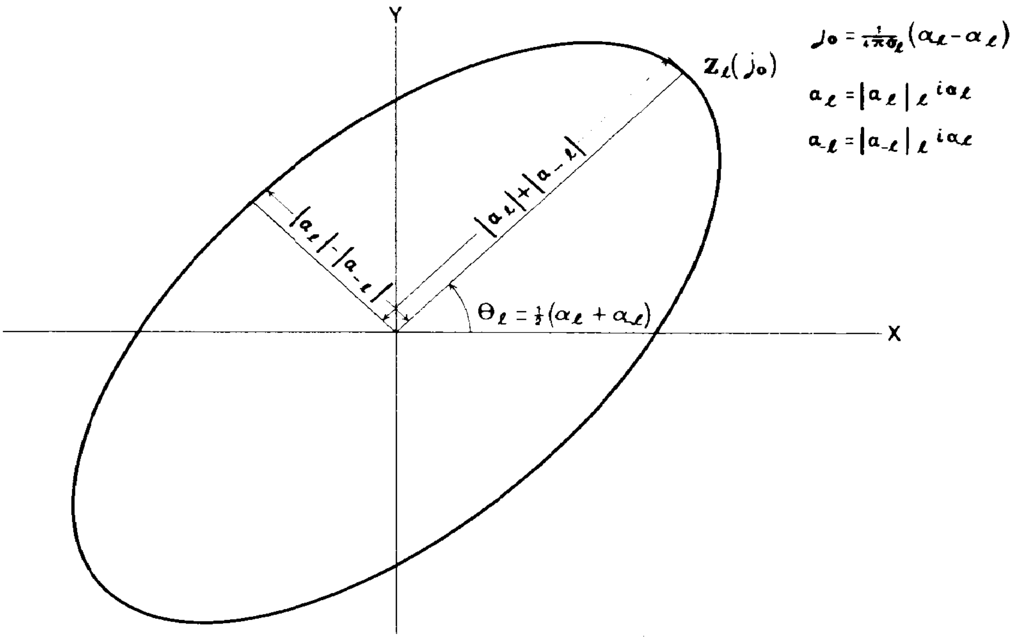

FIG. 2. — The elements of the ellipse drawn by a given tidal constituent of frequency σ_l, over a period T_l, in terms of the magnitude and phase of the complex numbers a_l and a_{-l}.

Its x and y components are

$$x = M_l \cos \eta_l$$
$$y = m_l \sin \eta_l$$

and these satisfy the equation of an ellipse:

$$\frac{x^2}{M_l^2} + \frac{y^2}{m_l^2} = 1$$

The tip of the vector $z_l(j)$ will therefore draw an ellipse with semi-major and semi-minor axes of length M_l and m_l, inclined at an angle $\frac{1}{2}(\alpha_l + \alpha_{-l})$ to the x axis, over the time interval T_l. This proves formulas (5) to (8). (8) is verified by noting that $\eta_l = 0$ when $j_0 = \dfrac{1}{4\pi\sigma_l}(\alpha_{-l} - \alpha_l)$ and at this instant the vector $z_l(j)$ has components $(M_l \cos \theta_l, M_l \sin \theta_l)$.

Fig. 2 shows in a schematic fashion the elements of the tidal stream ellipse in terms of the phase and the magnitude of the complex numbers a_l and a_{-l}. If $m_l > 0$, the rotation of the constituent, vector is counter-clockwise, if $m_l < 0$, it is clockwise. If $m_l = 0$, the constituent ellipse is a straight line.

Our next task is to obtain estimates of a_l and a_{-l}.

The tidal constituents

The easiest way to obtain estimates of these quantities is to apply the least square condition to the sequence of the residues, i.e.

$$\sum_{j=-N}^{N} |\varepsilon(j)|^2 = \text{Min.}$$

which is equivalent to

$$\frac{\partial}{\partial a_m^*} \sum_{j=-N}^{N} |\varepsilon(j)|^2 = 0$$

or

$$Z(\sigma_m) = \sum_{l=-n}^{n} a_l A_{lm} \tag{10}$$

where

$$Z(\sigma_m) = \sum_{j=-N}^{N} z_r(j) e^{-2\pi i \sigma_m j}$$

$$A_{lm} = \sum_{j=-N}^{N} e^{2\pi i j(\sigma_l - \sigma_m)} = \frac{\sin[(2N+1)\pi(\sigma_l - \sigma_m)]}{\sin \pi(\sigma_l - \sigma_m)}$$

(10) stands for the matrix equation:

$$Z = aA \tag{11}$$

Z and a are row vectors with $2n$ components and A is a $2n \times 2n$. This matrix can be partitioned in the following way:

$$A = \begin{array}{c} l \\ = \\ 1 \\ 2 \\ \cdot \\ \cdot \\ n \\ -1 \\ -2 \\ \cdot \\ \cdot \\ \cdot \\ -n \end{array} \overset{m = 1\ 2\ \dots\ n \qquad -1\ -2\ \dots\ -n}{\left(\begin{array}{c|c} A_+ & A_- \\ \hline A_- & A_+ \end{array} \right)}$$

where we have used the following symmetry properties of the elements of the A matrix:

$$A_{l,-m} = A_{-l,m} \qquad A_{l,m} = A_{-l,-m}$$

a and Z are partitioned in a similar fashion

$$a = (a_+, a_-)$$
$$Z = (Z_+, Z_-)$$

The + sign refers to the positive values of the index and the — sign to its negative values.

(11) can be extended into

$$Z_+ = a_+ A_+ + a_- A_-$$
$$Z_- = a_+ A_- + a_- A_+$$

Addition and subtraction of these two equations yield

$$c = a_1(A_+ + A_-)$$
$$s = a_2(A_+ - A_-)$$

which has for solution

$$a_1 = c(A_+ + A_-)^{-1}$$
$$a_2 = s(A_+ - A_-)^{-1} \qquad (12)$$

where we have defined

$$a_1 \equiv \frac{1}{2}(a_+ + a_-)$$

$$a_2 \equiv \frac{1}{2}(a_+ - a_-)/i$$

$$c_m \equiv \frac{1}{2}[Z(\sigma_m) + Z(-\sigma_m)] = \sum_{=-N}^{N} z_r(j) \cos 2\pi \sigma_m j$$

$$s_m \equiv \frac{1}{2}[Z(\sigma_m) - Z(-\sigma_m)]/i = \sum_{j=-N}^{N} z_r(j) \sin 2\pi \sigma_m j$$

(12) is in a format suitable for practical computations.

The matrix elements

$$(A_+ \pm A_-)_{lm} = \frac{\sin[(2N+1)\pi(\sigma_l - \sigma_m)]}{\sin \pi(\sigma_l - \sigma_m)} \pm \frac{\sin[(2N+1)\pi(\sigma_l + \sigma_m)]}{\sin \pi(\sigma_l + \sigma_m)}$$

and the inverse matrices $(A_+ \pm 2A_-)^{-1}$ must be calculated for each analysis since 2N+1 varies from one set of observations to another.

a_1, a_2, c and s are stored in a computer as rows of n pairs of real numbers. Similarly $z_r(j)$ and $a_0'(j)$ are strings of 2N+1 pairs of real numbers; the c and s vectors are obtained by multiplying the x and y components of $z_r(j)$ by $\cos 2\pi\sigma_m j$ or $\sin 2\pi\sigma_m j$ and summing. The c and s vectors components stand for:

$$c_m = \left[\sum_{j=-N}^{N} x_r(j) \cos 2\pi\sigma_m j, \quad \sum_{j=-N}^{N} y_r(j) \cos 2\pi\sigma_m j\right], \text{ etc.}$$

In this fashion we obtain values for a_1 and a_2; from these we deduce:

$$a_+ = a_1 + ia_2 \qquad a_- = a_1 - ia_2$$

a_1 and a_2 are stored as the pairs (a_{1x}, a_{1y}) and (a_{2x}, a_{2y}) and

$$ia_2 = i(a_{2x}, a_{2y}) = (-a_{2y}, a_{2x})$$

so that

$$a_+ = (a_{1x} - a_{2y}, a_{1y} + a_{2x})$$
$$a_- = (a_{1x} + a_{2y}, a_{1y} - a_{2x})$$

We may estimate also $\varepsilon(j)$ through

$$\varepsilon(j) = z_r(j) - \sum_{l=-n}^{n} a_l e^{2\pi i \sigma_l j}$$

The expected error on the estimate of the complex amplitude of the tidal constituent is of the order of

$$\Delta a \sim v/(2N+1)^{\frac{1}{2}}$$

where v is the standard deviation which is estimated from:

$$v^2 = \frac{1}{2N} \sum_{j=-N}^{N} |\varepsilon(j)|^2$$

Once $a_0'(j)$, $\varepsilon(j)$, a_l and a_{-l} are obtained, these quantities may be interpreted physically bearing in mind the limitations of the analysis.

Example

We have subjected a set of current observations in the Strait of Belle Isle to this type of analysis. The observations were initiated at 0900 hours on the 18th of August 1963 and they extended over 32 days and 20 hours. The depth at which they were taken was 13 metres and the interval of sampling was $\Delta t = 5$ minutes.

Fig. 3 shows a portion of these observations before and after the smoothing by $(\mathcal{A}_{12}/12)^3$. The periodicities and the trends are apparent in the unsmoothed data in spite of the marked blur brought about by the short period fluctuations. X and Y in the diagram indicate the components $x(t)$ and $y(t)$ of the current vector, XBAR and YBAR stand for $x(j)$ and $y(j)$ in the notation utilized in the paper.

Fig. 4 shows first the result of the application of $(\mathcal{A}_{24}/24)^2(\mathcal{A}_{25}/25)$ to $(x(j), y(j))$. The curves labelled XAO and YAO stand for the x and y components of $a_0'(j)$. We note the tendency of the steady flow to reverse its direction and vary its intensity in time. We show as well the x and y components of $z_r(j)$ which are labelled XR and YR; this function does exhibit marked periodicities and it is searched for tidal constituents.

The curves XPR and YPR in fig. 5 show the x and y components of the function $a_0'(j) + \sum_{l=-n}^{n} a_l e^{2\pi i \sigma_l j}$ obtained from the least square condition and the low passing; they are put immediately between the x and y components of $z(j)$. This function is defined only over the inner portion of data due to the unavoidable loss of data introduced by the use of the smoothing and low pass filters.

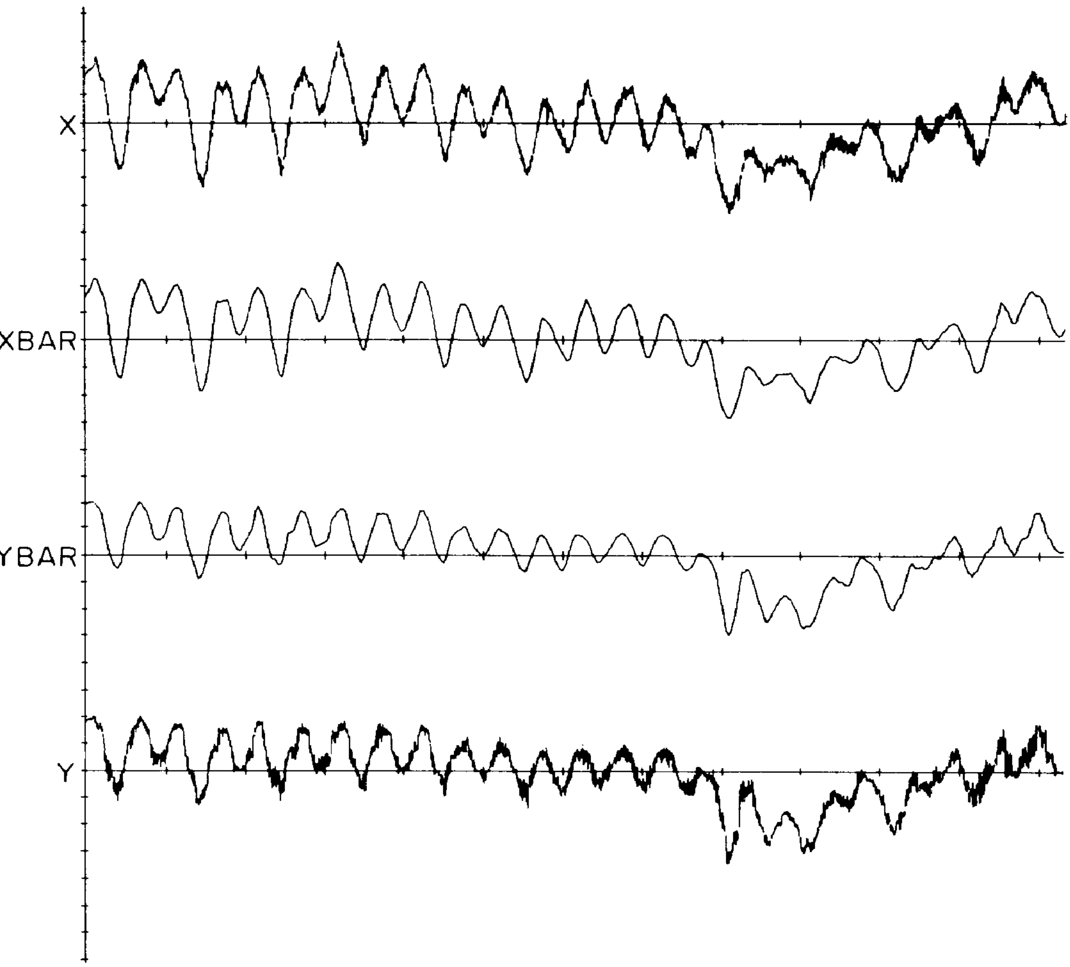

FIG. 3. — The application of the smoothing operator $(\mathfrak{A}_{12}/12)^3$ to a portion of the observations on the current in the Strait of Belle Isle taken at five minute intervals.

$\varepsilon(j)$ contains some contributions from parts of the low frequency band because of the imperfection of the low pass filter. For instance 72.1 % of an oscillation of period $\frac{1}{2}$ cycle/day contained in $a_0(j)$ will be left in $z_r(j)$ while 27.9% of it has been put into $a_0'(j)$ by applying $(\mathfrak{A}_{24}/24)^2(\mathfrak{A}_{25}/25)$ to $\mathfrak{Z}$. This phenomenon could have been avoided by using a better low pass filter but as a rule such a filter would be too costly on data. $\varepsilon(j)$ also contains the fluctuations of $a_0(j)$ which have frequencies overlapping those of the tidal constituents: such a type of oscillation of frequency 2 cycles/day is quite obvious in the diagram. We note that these oscillations do vary in time and therefore they reveal themselves as noise. They are also short frequencies superimposed on the lower frequency oscillations which are interpreted as reflecting possibly the lower part of the spectrum of the turbulence which could have escaped the smoothing operator. Finally $\varepsilon(j)$ contains some portions of the tidal constituents themselves, which cannot be resolvea by the analysis or which were only imperfectly resolved, on account of the distortion introduced by the vagrancies of the steady flow.

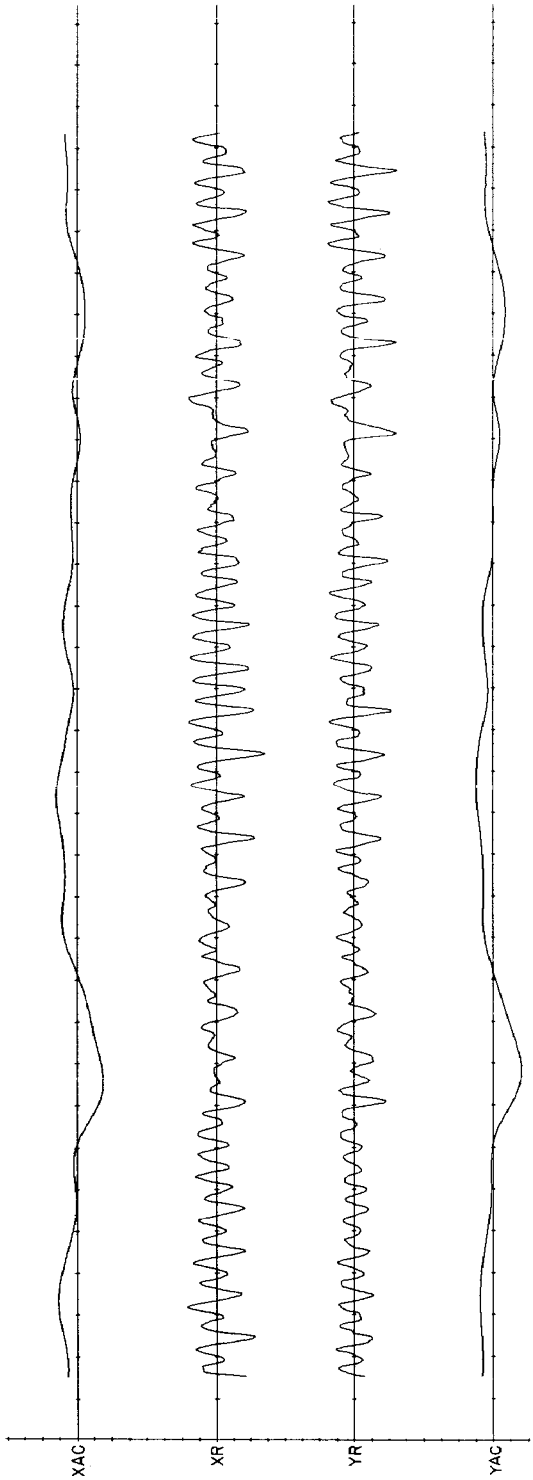

Fig. 4. — The x and y components of $a_0(j)$ and $z_r(j)$. The markings on the horizontal scale denote the calendar days measured from 0900 hours August 18, 1963. The first mark corresponds then to 0900 hours August 19. The interval Δj between two such marks is 24. The interval between two vertical markings is 1 knot.

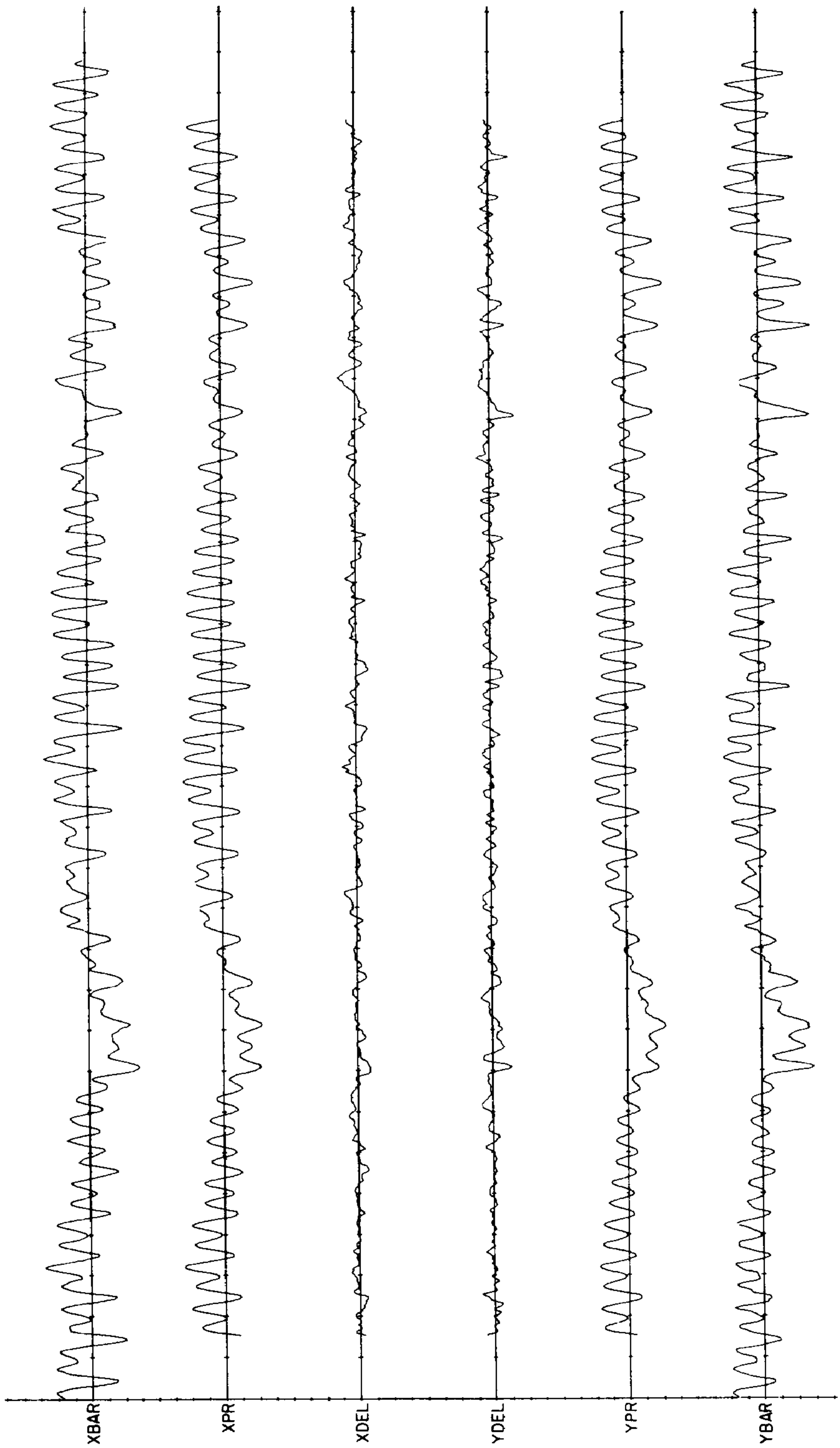

FIG. 5. — The x and y components of $z(j)$, $a_0(j) + \sum_{l=-n}^{n} a_l \, e^{2\pi i \sigma_l j}$ and $\varepsilon(j)$.

TABLE 1

Values of the elements of the constituent ellipses obtained from the analysis of a set of observations on the currents in the Strait of Belle Isle (51°26′ N, 56°48 W) at a depth of 13 metres, taken every five minutes over an interval of 32 days and 20 hours, starting at 0900 hours August 18, 1963.

$V_l \equiv$ the longitude on the celestial equator, measured from the meridian of Greenwich, of the fictitious star corresponding to the given constituent.

l	Name	Frequency σ_l Cycles/hour	Semi-major axis M_l knots	Semi-minor axis m_l knots	Probable Error on M_l and m_l knots	Inclination of Ellipse θ_l degrees	Greenwich phase lag $g_l = V_l + \sigma_l j_o$ degrees	Probable error on θ_l and g_l degrees
1	M_2	.08051	1.22	—.09	± .02	220.4	208.3	± 1.0
2	S_2 (K_2)	.08333	.52	—.07	± .02	218.5	240.9	± 2.2
3	N_2	.07900	.27	—.04	± .02	227.3	221.4	± 4.2
4	K_1 (P_1)	.04178	.47	—.07	± .02	220.0	140.0	± 2.4
5	O_1	.03873	.52	.00	± .02	218.1	095.4	± 2.2
6	NO_1	.04027	.11	—.07	± .02	205.9	153.9	±10.5
7	Q_1	.03722	.06	—.04	± .02	258	081	±30
8	OO_1	.04483	.03	—.03	± .02	321	011	—
9	J_1	.04329	.05	.01	± .02	291	181	—
10	M_4	.16102	.07	—.01	± .02	206	105	±16
11 MS	MS_4	.16384	.04	—.01	± .02	211	129	—
12	MN_4	.15951	.08	—.02	± .02	230	103	±14
13	M_8	.32205	.01	.00	± .02	174	324	—

All in all such an analysis conveys the impression that the noisiness of the record is quite high and this most likely holds for the majority of current observations near the surface. This implies that the three processes present in currents can only be separated with the help of very long series of observations which, for the moment, are not available.

The standard deviation of the observations is equal to .38 knot and the expected error on the magnitude of the complex amplitude is .01 knot.

The values of the elements of the ellipse of the tidal constituents which are separable, are given in table 1. The hourly values of $a_0'(j)$ are not listed; these can be read off the plots of XAO and YAO noting that each division of the vertical scale stands for 1 knot.

ADDITIONAL NOTES

The Filters

The filters which we propose are relatively simple but they introduce an appreciable distortion in the spectra of the functions to which they are applied.

The distortion in the amplitude of the tidal constituents is of no consequence since the distortion can be evaluated accurately and the analyzed amplitudes accordingly readjusted. If a power spectrum analysis of $a_0'(j)$ was performed, which is not recommended, $a_0'(j)$ could be readjusted to contain the whole low frequency band of $a_0(j)$.

The number of observations needed for the application of a filter of the form:

$$\mathfrak{A}_k^p \mathfrak{A}_q$$

is equal to $(k-1)p+(q-1)+1$. Table 2 lists some suggested smoothing

TABLE 2

Characteristics of the filters suggested in the paper

Δt	Operator	Interval of observation lost by its use	% Reduction in the amplitude of the constituents of frequency							
			Smoothing operator							
minutes		minutes	σ =			1	2	4	8	cycles-day
5	$(\mathfrak{A}_{12}/12)^3$	165				.82	3.31	12.84	44.96	
10	$(\mathfrak{A}_6/6)^3$	150				.76	3.31	12.60	44.32	
15	$(\mathfrak{A}_4/4)^3$	135				.62	3.16	12.18	43.24	
			Low pass filter							
hour		hours	σ =	1/15	1/2	1	2	4	8	cycles-day
1	$(\mathfrak{A}_{24}/24)^2(\mathfrak{A}_{25}/25)$	70		2.3	72.1	100	100	100	100	

operators as well as their distortion characteristics; it gives also the same information for the low pass filter.

In our method the filtering entails a total loss of 70 hours + 1 hour 45 minutes = 71 hours 45 minutes, or about three days of observations. For this method to be practicable one needs at least five days of observations. For a shorter interval one has to have recourse to filters even worse than those proposed.

The choice of the constituents

Very few tidal constituents can be separated from observations covering an interval of only a few days. Table 3 gives a list of the constituents which we propose to look for whenever it is possible.

In the course of a given analysis we must decide which ones of this set are separable: this is achieved by submitting the frequencies of pairs of constituents to the Rayleigh's test

$$| \sigma_l - \sigma_{l+1} | \, 2N \gtrless 1$$

2N+1 being the number of hourly observations available for the application of the least square requirement.

Two constituents can be separated if the difference of their frequencies multiplied by 2N is larger than 1 in absolute value. To apply the test and pick the frequencies of interest we use the pair arrangement displayed in table 3. The names in the pair stand for the frequencies.

TABLE 3

List of the tidal constituents to be tested for separability

Name	Frequency cycles/hour	Pair Arrangement
M_2	.08051	Zero, M_2
S_2	.08333	M_2, S_2
L_2	.08202	S_2, L_2
N_2	.07899	M_2, N_2
μ_2	.07769	N_2, μ_2
K_1	.04178	M_2, K_1
O_1	.03873	K_1, O_1
NO_1	.04027	O_1, NO_1
Q_1	.03722	O_1, Q_1
OO_1	.04483	K_1, OO_1
J_1	.04329	K_1, J_1
M_4	.16102	M_2, M_4
MS_4	.16384	M_4, MS_4
MN_4	.15951	M_4, MN_4
M_8	.32205	M_4, M_8

We calculate the frequency difference of the first pair and apply Rayleigh's test. If the test succeeds we retain the second constituent of the pair; if it fails we move on to the next pair and repeat the test. In this way we obtain, arranged by species and in their relative order of importance, a list of the constituents which may be separated from a given set of data.

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Reference

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