# ON THE ANALYSIS OF NON-EQUIDISTANT OBSERVATIONS OF THE TIDE

by Vidar Hisdal
Norsk Polarinstitutt, Oslo

## INTRODUCTION

The "classical" methods of computing harmonic tidal constituents are based on equidistant observation series (generally hourly readings). During the last few years, however, the application of digital electronic computers has made possible a more direct and flexible attack on the problem of tidal analysis, and, in principle, equidistant series are no longer necessary to make the calculations feasible. This fact, however, has been of little practical consequence, since the heights of the tide to be analysed are generally read at constant time intervals from continuous recordings.

In the present paper we shall consider a special group of series of tidal observations, viz. series containing severals gaps, too large to allow reliable interpolations, and with greatly varying time intervals between the individual readings. Prior to the introduction of digital computers such series were practically impossible to analyse.

Due to observational difficulties, tidal data brought home by expeditions to polar regions are frequently of the last-mentioned type. Even if an expedition has had an automatic tide gauge at its disposal (and the topographic conditions have permitted a satisfactory mounting !), various complications, especially the presence of drift ice, may make it difficult to obtain a continuous record, and to keep a constant reference level for a sufficient number of days. By means of modern pressure type gauges these difficulties may to some extent be eliminated.

Relatively few expeditions, however, are equipped with gauges of self-registering type. Tidal observations carried out by topographic and hydrographic expeditions working in the Arctic during summer time, to establish datums of the surveys, very often consist of frequent readings of a tide staff during a few days. Due to lack of time, such series are in most cases too short to allow reliable estimates to be made even of the principal harmonic constituents. By reading the staff whenever time permits, i.e. at varying intervals, ranging from one hour to a few days, during the whole stay of the expedition at the main base, and analysing the data by means of the method described below, valuable information about the major tidal constituents could be obtained for regions where at present little is known about the character of the tide. Since the observation hours are not fixed, the extra work that such investigations would entail, should be quite modest, and also could easily be included in the programme of biological, geological, and other expeditions that are staying at a permanent base for a few weeks or more.

## METHOD OF ANALYSIS

Supposing the harmonic representation of the tidal height ($h$) to be valid, we have the well-known expression :

$$h = z_0 + \sum_{j=1}^{N} f_j H_j \cos (\sigma_j t - g_j) \qquad (1)$$

where $z_0$ is the mean sea level, in terms of a chosen zero level, N is the necessary number of harmonic constituents, $f_j$, $H_j$, $\sigma_j$ and $g_j$ are respectively the node factor, amplitude, angular speed, and a further specified phase lag of the constituent $j$, and $t$ is the time. Provided the observation period is long enough — for series with large gaps this may mean half a year or more — all essential constituents may be introduced in the above equation, and the harmonic constants ($H_j$, $g_j$) may be found by means of the method of least squares, i.e. by minimizing the sum of the squared differences between the observed tidal heights and the corresponding heights given by equation (1) (cf. MURRAY, 1964). If a modern digital computer is available, the non-uniform time interval between the readings creates no special difficulty for the computation of the coefficients of the normal equations.

We confine ourselves to these brief comments concerning the comparatively long sequences of observations. As previously mentioned, it is the frequently occurring short series of tidal readings that especially interest us here. In this case we cannot introduce more than a few main constituents in our equation, and have to try in one way or another to correct for the influence of the most important of the minor constituents.

A possibility which in this connection immediately suggests itself, is to apply the ideas behind the so-called "Admiralty method of prediction" (see e.g. DOODSON and WARBURG, 1941). This method has previously been applied by PRATT (1960) as an alternative basis for his tidal analysis of gravity measurements from an Antarctic ice shelf station, and by HISDAL (1965) when analysing two short tide records, also from an ice shelf station. These investigations were concerned with equidistant series. However, this is not a necessary pre-condition.

The Admiralty method is based on the two principal semidiurnal and diurnal constituents, $M_2$, $S_2$ and $K_1$, $O_1$ respectively. Assuming certain equilibrium relations to be valid, which in most cases may be considered a fair approximation, these constituents are modified in such a way that, to a large extent, the effect of the minor constituents is taken into consideration. If $h_t$ is the height of the observed tide, and $s_0$ is a mean sera level, we may write :

$$h_t = r_t + s_0 + \sum_{j=1}^{4} B_jC_jH_j \cos (\sigma_j t - g_j - b_j - c_j) \tag{2}$$

Here $B_j$, $C_j$, $b_j$ and $c_j$ are parameters varying with time. Besides correcting for nodal variations, they serve the above-mentioned purpose of making allowance for the influence of constituents other than the four principal ones. Data for computing these parameters may be found in the Admiralty Tide Tables, Part III (1941). Variations that are not covered by the last term of the equation, the sum of the four modified constituents, are symbolised by the term $r_t$. This quantity may include such influences as those due to an insufficient allowance for the minor harmonic constituents, possible shallow water tides, meteorological effects, and inaccurate readings. Evidently, unless the time interval between the readings is not large, successive values of $r_t$ are not independent of each other. On the other hand, the factors responsible for $r_t$ should not be dominating. Thus, there should be no pronounced distortion of the tidal profile due to shallow water effects.

It goes without saying that the expression (2) may to a certain extent be further adapted to the observation series to be analysed. In some cases it may be desirable to change the values of $s_0$ in the course of the observation period, e.g. if the tide staff is toppled over and there is no fixed mark on land to which one may refer the old as well as the new readings. It goes without saying that $s_0$ in equation (2) is to be considered only as an estimated mean sea level for the observation period, or parts of it, and may deviate systematically from the annual mean sea level.

We may frequently utilize the observation material better if we take into consideration that some observations are more important than others, or in other words, that they contain more information concerning the task of separating the harmonic constituents.

It is quite clear that if a day contains one observation only, this observation, other things being equal, should be given greater weight than observations from days where the height of the tide is read every few hours. In all cases where the length of the time interval between successive readings differs widely, the validity of the result of the analysis should improve by applying some system of weights reflecting the relative importance of the individual observations.

The problem of determining an appropriate weight system is evidently very complicated, and depends on several factors. A thorough treatment of the subject would lead far beyond the scope of the present work. It seems reasonable to assume, however, that the irregularity of the time intervals may to a great extent be allowed for by letting the weights be dependent only on the length of the time intervals. An example is given below (see "Application").

For the process of calculation equation (2) may conveniently be written:

$$h_i - r_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ik}x_k + \dots + a_{i9}x_9 \quad \text{(weight } w_i\text{)}, \tag{3}$$

where $i = 1, 2, 3, \dots, n$ ($n$ is the number of observations), $a_{i1} = 1$, $x_1 = s_0$,

$$a_{i2} = [B_1C_1 \cos (\sigma_1 t - b_1 - c_1)_i]\,; \; x_2 = H_1 \cos g_1;$$

$$a_{i3} = [B_1C_1 \sin (\sigma_1 t - b_1 - c_1)]_i; \; x_3 = H_1 \sin g_1,$$ and so on.

We assume that $r_i$ is a comparatively small residual term, and that the "best" estimates of $s_0$ and the four sets of harmonic constants ($H_j$, $g_j$) are those found by solving the equations (3) according to the method of least squares, i.e. by minimizing

$$\sum_{i=1}^{n} w_i r_i^2$$

Following the general theory we put

$$d_{mk} = \sum_{i=1}^{n} w_i a_{im} a_{ik}, \quad \text{and} \quad e_m = \sum_{i=1}^{n} w_i a_{im} h_i$$

where the subscripts $m$ and $k$ take the values 1, 2, 3, ... 9. Furthermore, if $\mathbf{D}$ denotes the square matrix corresponding to the determinant $D = |d_{mk}|$, and we introduce the two column vectors :

$$\mathbf{e} = (e_1, e_2, \dots e_9) \quad \text{and} \quad \mathbf{x}' = (x_1', x_2', \dots x_9')$$

where $x_k'$ is a least squares estimate of $x_k$, we may write the normal equations in matrix notation :

$$\mathbf{D}\,\mathbf{x}' = \mathbf{e} \tag{4}$$

It is probably in most cases convenient to apply the set $(h, t, w, B_j, C_j, b_j, c_j)_i$ as input data on the computer. Since the number of observations is not very great, the last four times four parameters are comparatively quickly determined. (More than one third of them may be considered as constants). The calculation of the coefficients of (4) (i.e. $d_{mk}$, $e_m$) involves no special problems for modern computers.

Concerning the task of solving sets of linear equations on digital computers, several techniques exist (cf. e.g. RALSTON and WILF, 1960). For the application presented below the so-called GAUSS-JORDAN elimination method was used.

If the station considered is situated "not too far" from one or more stations where the tidal constituents are relatively well established, e.g. on the basis of about a year's observations (unfortunately this is not often the case in polar regions), it may be reasonable, instead of the equlibrium relationships, to introduce regional relationships between the principal constituents and the most important minor ones. Except that other parameters would have to be introduced in the equation system, the way of solving the problem would be analogous to that outlined above.

Generally speaking, the observation period should not be shorter than two weeks, and preferably not shorter than four weeks. This is on the assumption that the meteorological conditions influencing the height of the sea level are not too variable, and that there are few large gaps in the observation series. However, even relatively short series containing large gaps may give valuable information concerning the principal constituents, provided their amplitudes are not too small, and the readings are taken during periods for which the phase differences between constituents with equal species number differ considerably.

## APPLICATION

During a combined glaciological-meteorological expedition to Van Keulenfjorden, Svalbard, in the summer of 1964, a tide staff was set up in a fairly well-sheltered place at the expedition base Slettebu (77°32′ N, 15°21′ E). The staff was read at irregular intervals over a period of 25 days. The distribution of the observations with time appears from Fig. 1. The readings were taken merely when time permitted, and involved practically no additional work.

The chief aim of the tidal observations was to establish a mean sea level for the levelling of a neighbouring glacier. However, in order to get some further information about the character of the tide in the area, it was found worth while to try to estimate the principal harmonic constituents as well. This analysis at the same time may be expected to yield the "best" estimate of the mean sea level.

The calculations were carried out, in accordance with the method outlined above, on an IBM 1620 II computer which was fast enough for our purpose. In the first analysis all observations were given equal weights. In the second one each observation ($h_i$) was weighted according to the following simple scheme :

| Smallest time difference between $h_i$ and $h_{i\pm1}$ .................... | 1-3 | 4-8 | 9-15 | $\geqslant$ 16 hours |
|---|---|---|---|---|
| Number of cases ................<br>Weight ......................... | 160<br>1 | 6<br>2 | 3<br>3 | 1<br>4 |

(i.e. weight equal to square root of lower limit of time difference in hours).

Evidently in our case the number of "isolated" observations is so small that the weighting procedure will be of no great importance.

The analysis of the weighted series gave the following amplitudes H (in cm) and phase lags $\varkappa$ (referred to the local meridian) :

| $M_2$ | | $S_2$ | | $K_1$ | | $O_1$ | |
|---|---|---|---|---|---|---|---|
| H | $\varkappa$ | H | $\varkappa$ | H | $\varkappa$ | H | $\varkappa$ |
| 48.6 | 23.3° | 18.0 | 58.0° | 7.5 | 220.7° | 3.8 | 159.0° |

(These values differ inconsiderably from those found by analysing the non-weighted series).

In order to judge the reliability of our result we have made a comparison with data from Longyearbyen (78°14′ N, 15°39′ E), a mining settlement situated in a fjord farther north. Here an automatic tide gauge has been in operation since the summer of 1956. The gauge is mounted near the outlet pipes of the cooling water from the power station, causing the sea in the immediate vicinity to be ice-free even during the middle of winter (HORNBAEK, 1963). By means of the marigrams we have read the height of the tide at this station for the same dates and hours at which the tide was observed at Slettebu, and made the sequence of readings (weighted and non-weighted) the subject of an analysis corresponding to that applied for the latter station. The principal harmonic constituents obtained on the basis of the weighted series, and those resulting from an analysis of hourly heights (DOODSON's method) for the period 1 July 1956 to 21 June 1957 (HORNBAEK, 1963) are as follows :

| | $M_2$ | | $S_2$ | | $K_1$ | | $O_1$ | |
|---|---|---|---|---|---|---|---|---|
| | H | $\varkappa$ | H | $\varkappa$ | H | $\varkappa$ | H | $\varkappa$ |
| Irreg. intervals 25 days | 50.2 | 24.1° | 20.4 | 66.3° | 7.0 | 230.0° | 3.7 | 171.0° |
| Hourly heights One year | 52.2 | 27.1° | 19.9 | 70.8° | 6.9 | 236.2° | 3.1 | 92.2° |

We see that, except for the phase lag of $O_1$, the difference between the two sets of data for Longyearbyen is not very great. (The agreement is slightly poorer if we use the values found when analysing the non-weighted series). The amplitude of $O_1$ is apparently too small in relation to the magnitude and character of our observation material to allow a reliable estimate of this constituent. There is a fair correspondance between the constituents found for Slettebu and Longyearbyen. It does not seem likely, therefore, that much would have been achieved by using regional relationships based on the data from Longyearbyen when analysing the observation series for Slettebu.

Finally we shall study a bit more closely the difference ($r_i$) between the observed and the calculated heights of the tide. (Strictly speaking it is a "least squares estimate" of this difference we consider).

As would be expected in the present case, the decrease of the standard deviation of $r_i$ when the observation series are weighted is very slight — from 7.00 to 6.96 cm for Slettebu, and from 8.27 to 8.06 cm in the case of Longyearbyen.

The variation with time of $r_i$ for Slettebu is shown, together with the observed tidal heights, in fig. 1, while fig. 2 gives a picture of the frequency distribution. It should be mentioned in this connection that the variations with time of $r_i$ for Slettebu and Longyearbyen (data not given) are surprisingly parallel, and hence is sufficient to study $r_i$ for the former station. Furthermore, it should be noted that the "isolated" observations are in most cases so close to the calculated ones (see fig. 1) that it would scarcely have had any notable effect on our results if we had attached still greater weights to these observations.

The variation of $r_i$ around the zero line is more or less smooth, and no doubt a strong persistence tendency is present (contributing, among other things, to the irregular form of the frequency distribution). This may be

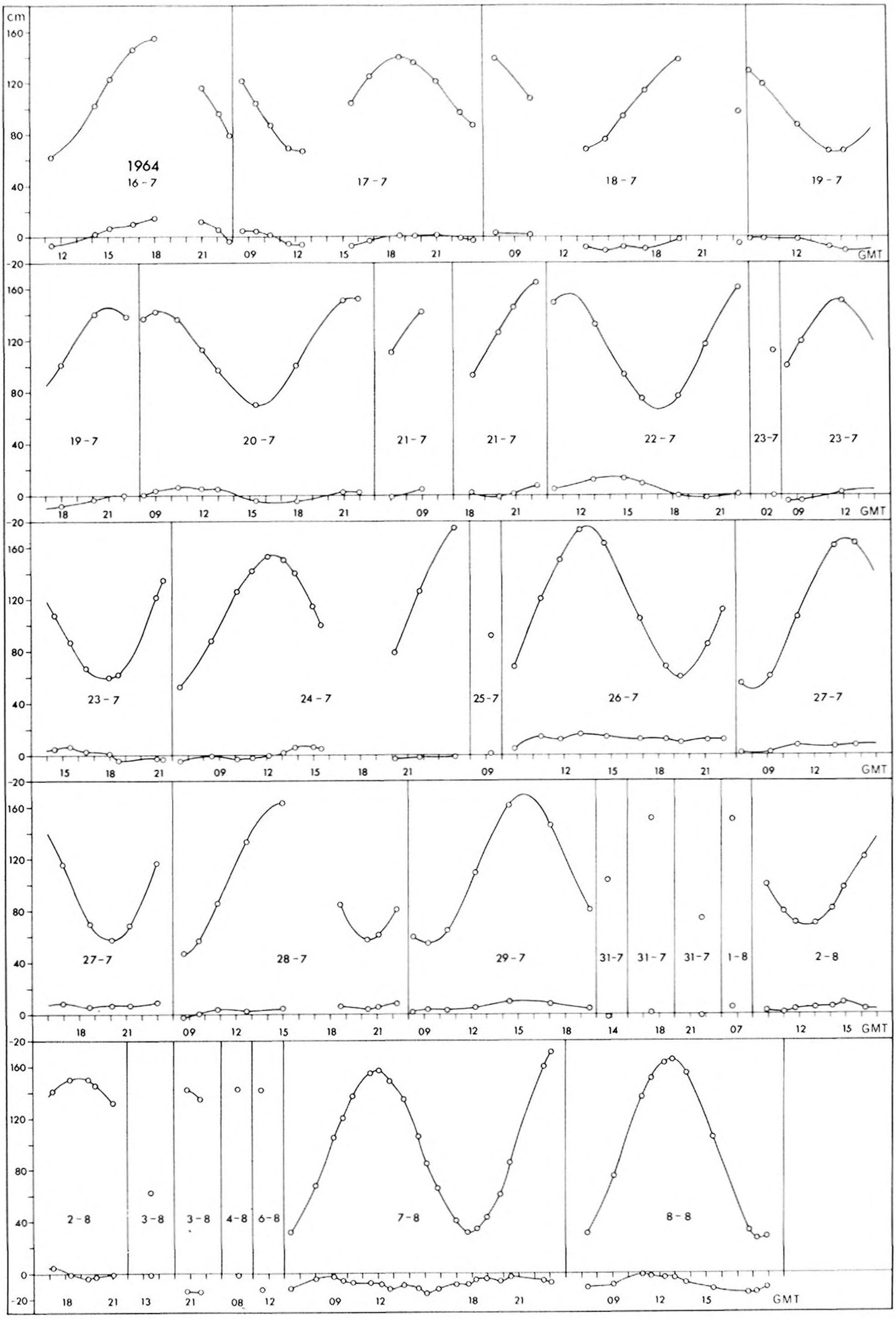

FIG. 1. — Upper points : Observed heights of the tide.
Lower points (close to the zero line) : Difference between observed and computed tides ($r_t$). Points separated by time intervals less than three hours are connected by smooth curves.

partly due to an insufficient allowance for the minor harmonic constituents, but in all probability the meteorological effects — wind and air pressure — are most important in this connection.

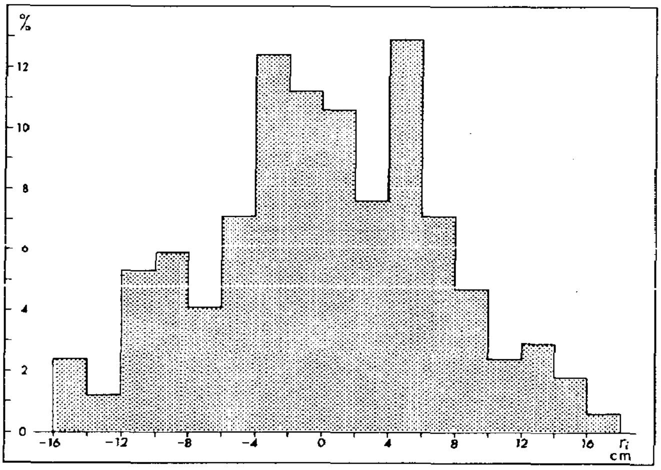

Fig. 2. — Frequency distribution of the difference between observed and computed heights of the tide ($r_i$).

There are several reasons why it seems hopeless in our case to reach conclusive results as to the separate influences of wind and pressure. Firstly, we have the fact that these influences are of very complex nature, and that the two elements are not independent of each other, the wind being normally stronger and of a more permanent direction for low than for high pressure situations. Secondly, we have in our case the additional difficulty that the observation series is short and contains several gaps, and, also, the meteorological observation material available for the area considered is rather limited. In the following we have tentatively used the air pressure as an indicator of the meterological effects.

As the difference between the air pressures at Slettebu and the meteorological station at Isfjord Radio (78°04′ N, 13°38′ E) is very small, we may use the more complete pressure series from the latter station (three-hourly values). We choose the five periods with the most pronounced positive values of $r_i$ (i.e. observed elevations are higher than the calculated ones), and correspondingly the five periods with the most pronounced negative values of $r_i$.

For the first group we find that in four cases the pressure is considerably lower than the mean pressure for the whole observation period, and in one case, 22 July 11h-17h GMT, the pressure is slightly higher (2-4 mb) than the mean. A probable explanation of this exception is found if we look at the wind conditions, as they appear from the weather maps and from the observations made at Isfjord Radio. (Slettebu is not representative as far as the wind is concerned, because of the mountain ranges bordering the fjord). From the morning of 21 July to the afternoon of the following day the wind in the ocean area adjacent to the coast is south-southwest and the force is 6 to 7 Beaufort (strong breeze to near gale). This is the longest spell of so strong winds encountered during the whole observation period, and may well have resulted in a considerable pile-up of water along the coast. The mean pressure for the four former periods lies 9.6 mb below the total mean, or 8.7 mb below if all five periods are considered.

Concerning the second group, the pressure is in all five cases considerably higher than the total mean. The difference is on an average 12.1 mb. It may be noted that for both groups the pressure difference in mb is of the same order of magnitude as the difference between observed and calculated tidal height in cm, as would be required by a quasi-static relationship.

These results suggest that the magnitude and variation of $r_i$ to a great extent reflect disturbances caused by the meteorological conditions.

Finally, concerning the determination of an annual mean sea level for Slettebu, it is assumed that a fairly reliable estimate is obtained by correcting the calculated $s_0$ for this station by means of the difference between the mean sea level found for Longyearbyen on the basis of the series corresponding to that applied for Slettebu, and that found on the basis of the annual series.

## ACKNOWLEDGEMENT

I am indebted to Mr. O. Liestøl (Norsk Polarinstitutt) and his assistants for reading the tide staff during the last part of the observation period. My thanks are also due to Mr. P. Ofstad (Oslo University) for valuable suggestions and considerable help in connection with the analysis of the data.

## REFERENCES

Admiralty Tide Tables, Part III, 1941. Hydrogr. Dept., London.

Doodson, A. T. and Warburg, H. D. (1941) : *Admiralty manual of tides* Hydrogr. Dept., London.

Hisdal, V. (1965) : *On the tides at Norway Station and adjacent coastal areas of Antarctica.* Norsk Polarinstitutt, Skrifter Nr. 133, Oslo.

Hornbaek, H. (1963) : *Tidal observations in Svalbard.* Norks Polarinstitutt, Årbok 1962, Oslo.

Murray, M.T. (1964) : A general method for the analysis of hourly heights of tide. *Int. Hydrogr. Rev.,* Vol. XLI, No. 2, Monaco.

Pratt, J. G. D. (1960) : Tides at Shackleton, Weddell Sea. *Trans-Ant. Exp.,* 1955-58, *Scient. Rep.* No. 4, London.

Ralston, A. and H.S. Wilf (1960) : *Mathematical methods for digital computers.* John Wiley & Sons, New York.