THE MUNK-CARTWRIGHT METHOD FOR TIDAL PREDICTION AND ANALYSIS

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1. — INTRODUCTION

In the course of a Tidal Symposium held in Paris in May 1965 MUNK and CARTWRIGHT presented a new method for tidal prediction and analysis to which they gave the name " response method ". Others proposed to call this new solution the " convolution method " since the tidal heights are obtained by a weighted sum of terms of the equilibrium tide. This last name seems the more appropriate one, but this interesting method could also be called the " generalised Laplace method ".

The paper [1] presented by these two eminent scientists is of an extremely high level and they end it with a quotation from Hilaire BELLOC (1925) : " When they pontificate on the tides it does no great harm, for the sailorman cares nothing for their theories, but goes by real knowledge ".

However I think that it is possible to explain the method in a somewhat less sophisticated way, and also to extract from their article a little of this " real knowledge " so dear to sailors. This is the aim of the present article.

2. — BASIC CONCEPT

Let us take the height of the lunar equilibrium tide, which is dependent on the cube of the parallax, as point of departure. This height can be expressed as a function of the latitude φ of the place, the declination $\delta_☾$ and the hour angle $t_☾$ by the well-known formula :

$$y' = \frac{3La^3}{4Tr^3}\, a \left[\frac{1}{3} (3\sin^2\varphi - 1)(3\sin^2\delta_☾ - 1) + \sin 2\varphi \sin 2\delta_☾ \cos t_☾ + \cos^2\varphi \cos^2\delta \cos 2t_☾ \right] \qquad (2a)$$

where

L	=	the mass of the Moon;
T	=	the mass of the Earth;
a	=	the radius of the Earth (presumed spherical);
r	=	the distance between the centre of the Earth and that of the Moon.

The geodetic coefficients of (2a) are the same, to within a few constants, as those which were found by MUNK and CARTWRIGHT who followed a somewhat transcendental procedure.

In order to simplify matters let us put :

$$a/r_{☾} = P_{☾} \qquad \text{(horizontal parallax)}$$

and write

$$3La^4/4Tr^3 = 3LaP_{☾}/4T = (3La\bar{P}_{☾}^3/4T)\,(P_{☾}/\bar{P}_{☾})^3 = K_{☾}(P_{☾}/\bar{P}_{☾})^3$$

$\bar{P}$ being the mean horizontal parallax of the Moon. Let us also call the geodetic and declinatory coefficients for the species n ($n = 0, 1, 2$) respectively $g_n(\varphi)$ and $h_n(\delta\ )$. We may then write (2a) in the following more general way :

$$y'_{☾} = K_{☾} \sum_n g_n(\varphi)\,(P_{☾}/\bar{P}_{☾})^3\, h_n(\delta_{☾}) \cos nt_{☾} \qquad (2b)$$

For the Sun we shall obtain a formula similar to (2b) where we replace $\bar{P}$, P , δ , t and K by respectively $\bar{P}_{\odot}$, $P_{\odot}$, $\delta_{\odot}$, $t_{\odot}$ and $K_{\odot}$, and we obtain

$$y'_{\odot} = K_{\odot} \sum_n g_n(\varphi)\,(P_{\odot}/\bar{P}_{\odot})^3\, h_n(\delta_{\odot}) \cos nt_{\odot}$$

but if we designate the sun's mass by S we may write :

$$K_{\odot} = 3Sa\bar{P}_{\odot}^3/4T = (3La\bar{P}_{☾}^3/4T)\,(S\bar{P}_{\odot}^3/L\bar{P}_{☾}^3) = 0.4604\ K_{☾}$$

thus

$$y'_{\odot} = 0.4604\ K_{☾} \sum_n g_n(\varphi)\,(P_{\odot}/\bar{P}_{\odot})^3\, h_n(\delta_{\odot}) \cos nt_{\odot} \qquad (2c)$$

Then by making

$$(P_{☾}/\bar{P}_{☾})^3\, h_n(\delta_{☾}) = B_n^{☾}$$

and

$$0.4604\,(P_{\odot}/\bar{P}_{\odot})^3\, h_n(\delta_{\odot}) = B_n^{\odot} \qquad (2d)$$

we extract from (2b) and (2c) respectively

$$y'_{☾} = \sum_n K_{☾} g_n(\varphi)\, B_n^{☾} \cos nt_{☾} \qquad (2e)$$

and

$$y'_{\odot} = \sum_n K_{\odot} g_n(\varphi)\, B_n^{\odot} \cos nt_{\odot} \qquad (2f)$$

The Laplace prediction method is based on the assumption that the actual tide of species n follows the equilibrium tide with a phase lag of $n\gamma_n$, and and that this tide has as height the height of the equilibrium tide multiplied by a coefficient C_n. Thus for the actual luni-solar tide we are able from (2e) and (2f) to deduce :

$$y = \sum_n K_{☾} g_n(\varphi)\, C_n\,[B_n^{☾} \cos n(t_{☾} - \gamma_n) + B_n^{\odot} \cos n(t_{\odot} - \gamma_n)] \qquad (2g)$$

However the authors of the method have preferred to express y in terms of hour angles with the Greenwich meridian as origin. In these circumstances, for a port that is at λ longitude west of Greenwich we have

$$t_{☾} = t_{GW}^{☾} - \lambda \qquad \text{and} \qquad t_{\odot} = t_{GW}^{\odot} - \lambda$$

and thus by introducing these values into (2g) and developing we obtain :

$$y = \sum_n K_{☾} g_n(\varphi)\, C_n [\cos n(\lambda + \gamma_n)\, (B_n^{☾} \cos nt_{GW}^{☾} + B_n^{\odot} \cos nt_{GW}^{\odot}) + \sin n(\lambda + \gamma_n)\, (B_n^{☾} \sin nt_{GW}^{☾} + B_n^{\odot} \sin nt_{GW}^{\odot})]$$

Now if we put

$$K_{☾} g_n(\varphi)\, C_n \cos n(\lambda + \gamma_n) = u_n$$
$$K_{☾} g_n(\varphi)\, C_n \sin n(\lambda + \gamma_n) = v_n \qquad (2h)$$

$$B_n^{☾} \cos nt_{GW}^{☾} + B_n^{\odot} \cos nt_{GW}^{\odot} = a_n(t)$$
$$B_n^{☾} \sin nt_{GW}^{☾} + B_n^{\odot} \sin nt_{GW}^{\odot} = b_n(t) \qquad (2i)$$

we have as a result :

$$y = \sum_n [u_n\, a_n(t) + v_n\, b_n(t)] \qquad (2j)$$

which supplies the height of the tide at time t by the convolution or weighted sum of the theoretic values $a_n(t)$ and $b_n(t)$. It is obvious that the weights u_n and v_n must be determined by the tidal analysis.

It can be clearly seen that (2j) is a new mathematical representation of the Laplace assumption which served us as our point of departure. We shall see later on how the authors of the method we are considering have generalised the use of (2j).

If we were to carry the development further it would be possible to consider additionally the term of the equilibrium tide that depends on the fourth power of the parallax. However, the development which we have just carried out would not have been altered in any way.

3. — GENERALISING THE METHOD

When establishing formula (2j) we only took into account the coefficient C_n and the phase lag $n\gamma_n$ for species n and for any relative position of the Sun and the Moon. This pair of constants can be converted into a single pair of weights u_n and v_n. If, however, in place of the astronomic coefficients $a_n(t)$ and $b_n(t)$ corresponding to time t we were to take as astronomic coefficients for tidal prediction at time t those that correspond to time $(t - \tau)$, in analysing the tidal observations we would find new values u_n^{τ} and v_n^{τ} which would make it possible to express the height of the tide at time t by the expression

$$y_t = \sum_n [u_n^{\tau} a_n(t - \tau) + v_n b_n(t - \tau)] \qquad (3a)$$

If the values of C_n and $n\gamma_n$ were true constants the difference between these new weight values and u_n and v_n would only depend on interval τ.

However we must consider that in interval τ there is an alteration to the luni-solar tidal wave of species n which is produced by variations in parallax and by the movements of the disturbing bodies in declination altering the astronomic coefficient and the angular frequency of the n-diurnal equilibrium wave. But since there is an alteration of the sea's response to the action of the disturbing bodies which corresponds to an alteration in the characteristics of the equilibrium tide the values of C_n and $n\gamma_n$ will no longer be the same.

By making τ vary and by determining by analysis the pairs of weights u_n and v_n we would be able to determine the height of the tide at time t for each value of τ, but this would be a simple repetition of the original Laplace procedure whose lack of accuracy made LAPLACE himself plan to use harmonic analysis. In actual fact values of y_t would be found which would have important differences which would have to be adjusted through simple averaging. There is, however, a more accurate procedure. This consists of generalising (2*j*) by making $\tau = s\ \Delta t$ where Δt is an interval of time selected beforehand and where $s = 0, \pm 1, \pm 2, \ldots$, and by writing

$$y_t = \sum_n \sum_s [u_n^s a_n(t - s\Delta t) + v_n^s b_n(t - s\Delta t)] \tag{3b}$$

where the weights u_n^s and v_n^s are local constants determined by the least squares method of analysis.

It should be noted that each pair of weights u_n^s and v_n^s determined by this method is part of a set expressing the average conditions in which we may define the correlation between the actual tide at any time t and the equilibrium tide in the interval between time $(t + |S|\Delta t)$ and $(t - |S|\Delta t)$, where $|S| = |s|$ max.

Formula (3*b*) is precisely the one established by MUNK and CARTWRIGHT to whom we are also indebted for the suitable choice of $\Delta t = 2$ days and for the maximum value of ± 3 to be given to s. The justification of this last value is not quite satisfactory enough, a fact which the authors themselves point out in their paper [1].

4. — SOLAR RADIATION

Here we have an interesting innovation. The authors of the method have in fact found a way of introducing the effect of solar radiation into the computations, and thus the oscillations produced by periodic meteorologic phenonema may be fairly logically explained.

It is well known that in the harmonic method as at present practised we take into account several constituents of astronomic origin whose coefficients in the harmonic development are so small that they could be neglected. But the frequencies of these constituents coincide with those of several oscillations of well-defined meteorologic origin and this justifies the retention of these constituents in the practical computations. The constituents S_a and S_{sa} are cases in point. A quick look at the theory of radiation will thus be well justified.

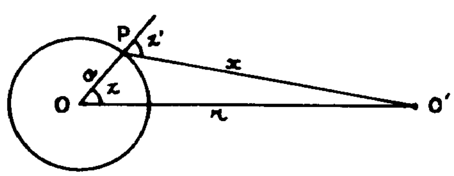
It is not difficult to establish the formula expressing the solar radiation $\mathcal{R}$ corresponding to a point on the surface of the globe. Let k be the constant of solar radiation — equal to 1.946 cal cm^{-2} min^{-1} — and which is the radiation received over a surface of 1 cm^2 during one minute, normal to OO′ (see fig.) and at distance r from the source. If this same surface were situated on P and if it were still normal to O′P the radiation value would be $k(r/x)$. However if this surface is horizontally placed the radiation value will be $k(r/x)\cos z'$ or in practice

$$\mathcal{R} = k(r/x)\cos z \quad \text{for} \quad 0^\circ \leqslant z \leqslant 90^\circ$$
$$= 0 \text{ for any other value of } z$$

Thus we see that the Earth is transparent to gravitation but opaque to radiation. In order to take this very important fact into account the authors have had recourse to a development of $\mathcal{R}$ in spherical harmonics which permits the transformation of the expression just established into a series of continuous terms. Since such a development goes beyond the limits which we have set we shall restrict ourselves to giving the following practical formula where we have made $\bar{r}_\odot/r_\odot \approx P_\odot/\bar{P}_\odot$:

$$\mathcal{R} = k(P_\odot/\bar{P}_\odot)\left[\frac{1}{4} + \frac{1}{2}\cos z + \frac{5}{32}(3\cos^2 z - 1)\right] \tag{4a}$$

In this formula the first term in the square brackets multiplied by the general coefficient gives us the mean radiation since the ratio $(P_\odot/\bar{P}_\odot)$ is very close to unity, and in these circumstances its variation can therefore be ignored. Only the other two terms therefore remain to be considered.



Now we can express the zenithal distance z in function of the latitude, declination and hour angle by the well known formula :

$$\cos z = \sin\varphi \sin\delta_\odot + \cos\varphi \cos\delta_\odot \cos t_\odot$$

thus according to (4a) after several transformations we have :

$$\begin{aligned}\mathcal{R} = k(P_\odot/\bar{P}_\odot)\Big[&\frac{1}{2}\sin\varphi\sin\delta_\odot + \frac{1}{2}\cos\varphi\cos\delta_\odot\cos t_\odot \\ &+ \frac{15}{64}(3\cos^2\varphi - 1)\left(\sin^2\delta_\odot - \frac{1}{3}\right) \\ &+ \frac{15}{64}\sin 2\varphi \sin 2\delta_\odot \cos t_\odot \\ &+ \frac{15}{64}\cos^2\varphi\cos^2\delta_\odot\cos 2t_\odot\Big]\end{aligned} \tag{4b}$$

where the two first terms arise from the development of cos z and the last three from that of ($\cos^2 z - 1$). Comparing (2a) with these three last terms we see that there is an analogy between each of these terms and those in (2a) corresponding to the same species. We also see the terms of (4b) with equal n (0,1) have different geodetic and declinatory coefficients which means that they cannot be grouped together.

In order to make the notation simpler let us express the geodetic and declinatory coefficients of the first two terms of (4b) by respectively $g_n^{\rm I}(\varphi)$ and $h_n^{\rm I}(\delta_\odot)$, and those of the last three terms by $g_n^{\rm II}(\varphi)$ and $h_n^{\rm II}(\delta_\odot)$. Thus

$$\mathcal{R} = k \sum_n g_n^{\rm I}(\varphi)\,(P_\odot/\overline{P}_\odot)\,h_n^{\rm I}(\delta_\odot)\cos nt_\odot + k \sum_n g_n^{\rm II}(\varphi)\,(P_\odot/\overline{P}_\odot)\,h_n^{\rm II}(\delta_\odot)\cos nt_\odot$$

where if we make

$$(P_\odot/\overline{P}_\odot)\,h_n^{\rm I}(\delta_\odot) = B_n^{\rm I} \qquad (4c)$$

and

$$(P_\odot/\overline{P}_\odot)\,h_n^{\rm II}(\delta_\odot) = B_n^{\rm II} \qquad (4d)$$

we shall have :

$$\mathcal{R} = \sum_n k[g_n^{\rm I}(\varphi)\,B_n^{\rm I}\cos nt_\odot + g_n^{\rm II}(\varphi)\,B_n^{\rm II}\cos nt_\odot] \qquad (4e)$$

Still following the Laplace assumption the actual tide corresponding to each term in (4e) will be given by introducing coefficients $C_n^{\rm I}$ and $C_n^{\rm II}$ and phase lags $n\gamma_n^{\rm I}$ and $n\gamma_n^{\rm II}$ into formula (4e). Replacing $t_\odot$ by ($t_{\rm GW}^\odot - \lambda$) we obtain :

$$y_{\mathcal{R}} = \sum_n k[g_n^{\rm I}(\varphi)\,C_n^{\rm I}\,B_n^{\rm I}\cos n(t_{\rm GW}^\odot - \lambda - \gamma_n)$$
$$g_n^{\rm II}(\varphi)\,C_n^{\rm II}\,B_n^{\rm II}\cos n(t_{\rm GW}^\odot - \lambda - \gamma_n^{\rm II})]$$

where by developing and making

$$\left.\begin{aligned} kg_n^{\rm I}(\varphi)\,C_n^{\rm I}\cos n(\lambda + \gamma_n^{\rm I}) &= u_n \\ kg_n^{\rm I}(\varphi)\,C_n^{\rm I}\sin n(\lambda + \gamma_n^{\rm I}) &= v_n \\ B_n^{\rm I}\cos nt_{\rm GW}^\odot &= \alpha(t) \\ B_n^{\rm I}\sin nt_{\rm GW}^\odot &= \beta(t) \end{aligned}\right\} (4f) \qquad \left.\begin{aligned} kg_n^{\rm II}(\varphi)\,C_n^{\rm II}\cos n(\lambda + \gamma_n^{\rm II}) &= u'_n \\ kg_n^{\rm II}(\varphi)\,C_n^{\rm II}\sin n(\lambda + \gamma_n^{\rm II}) &= v'_n \\ B_n^{\rm II}\cos nt_{\rm GW}^\odot &= \alpha'(t) \\ B_n^{\rm II}\sin nt_{\rm GW}^\odot &= \beta'(t) \end{aligned}\right\} (4g)$$

we obtain :

$$y_{\mathcal{R}} = \sum_n [u_n\alpha_n(t) + v_n\,\beta_n(t)] + \sum_n [u'_n\ \alpha'_n(t) + v'_n\,\beta'_n(t)] \qquad (4h)$$

In the case of long period constituents we do not have v_0 to consider because $\beta_0(t) = \beta'_0(t) = 0$, as expressions (4$f$) and (4$g$) show. However in the lists of constants u_n and v_n given by the authors [1] we find values of v_0 corresponding to $\beta_0(t)$. Thus the long period radiational tide for which s is always zero can be expressed by

$$y = u_o\alpha_o(t) + v_o\beta_o(t) + u'_o\alpha'_o(t) + v'_o\beta'_o(t) \qquad (4i)$$

However the authors explain that in this case the values of $\beta_0(t)$ and $\beta'_0(t)$ are given by

$$\beta_o = (365.242/2\pi)\,[\alpha_o(t + \Delta t/4) + \alpha_o(t - \Delta t/4)] \tag{4j}$$

$$\beta'_o = (365.242/2\pi)\,[\alpha'_o(t + \Delta t/4) + \alpha'_o(t - \Delta t/4)] \tag{4k}$$

It should be noted that if the list of values of u_n and v_n for Honolulu and Newlyn given by the authors in their paper [1] is examined we find considerable weights for the semi-diurnal and long period species. For the case of the diurnal tide the values of u_n and v_n are less important but still however fairly significant for the second convolution of (4*h*). On the other hand, the weights for the diurnal part of the first convolution are negligible. Cartwright [2], in the paper which he submitted to the Tidal Symposium held in Monaco in 1967 employed only the second convolution of (4*h*) for $n = 2$. The question of long period tides is not treated in that paper.

5. — THE SHALLOW WATER TERMS

In his latest paper [2] Cartwright has simplified the introduction of the shallow water corrections by starting from the heights of the purely astronomic diurnal and semi-diurnal tides. In order to do this he used two new functions $\tilde{y}'$ and $\tilde{y}'_\odot$ in quadrature with those given by (2*e*) and (2*f*). Let us therefore write

$$\tilde{y}'_{☾} = \sum_n K_{☾} g_n(\varphi)\, B_n \sin nt_{☾}$$

and

$$\tilde{y}'_\odot = \sum_n K_{☾} g_n(\varphi)\, B_n \sin nt_\odot$$

thus

$$\tilde{y} = \tilde{y}'_{☾} + \tilde{y}'_\odot = \sum_n K_{☾} g_n(\varphi)\, (B_n^{☾} \sin nt_{☾} + B_n^{\odot} \sin nt_\odot) \tag{5a}$$

By a development similar to the one which allowed us to find (2*j*) we obtain :

$$\tilde{y} = \sum_n [u_n b_n(t) - v_n a_n(t)] \tag{5b}$$

We then see that $\tilde{y}$ can be computed without any difficulty since the elements figuring in (5*b*) are exactly the same as those in (2*j*). Furthermore (5*b*) is the transformation of (5*a*) when the coefficients C_n and the phase lag $n\gamma_n$ have been introduced. Thus we may write :

$$\tilde{y} = \sum_n K_{☾} g_n(\varphi)\, C_n [B_n^{☾} \sin n(t_{☾} - \gamma_n) + B_n^{\odot} \sin n(t_\odot - \gamma_n)]$$

By putting

$$t_\odot = t_{☾} - \theta$$

we obtain

$$\tilde{y} = \sum_n K_{☾} g_n(\varphi)\, C_n [B_n^{☾} + B_n^{\odot} \cos n(\theta + \gamma_n) \sin nt_{☾} - B_n^{\odot} \sin n(\theta + \gamma_n) \cos nt_\odot]$$

whence by making

$$\begin{aligned} K_{☾} g_n(\varphi)\, C_n [B_n^{☾} + B_n^{☉} \cos n(\theta + \gamma_n)] &= R_n \cos n\beta_n \\ K_{☾} g_n(\varphi)\, C_n B_n \sin n(\theta + \gamma_n) &= R_n \sin n\beta_n \end{aligned} \quad (5c)$$

we have as a result

$$\tilde{y} = \sum_n R_n \sin n(t_{☾} - \beta_n) \quad (5d)$$

Starting from (2e) and (2f) we likewise obtain :

$$y = \sum_n R_n \cos n(t_{☾} - \beta_n) \quad (5e)$$

It is obvious that (5d) is equivalent to (5b) and (5e) to (2j).

Considering now only two oscillations — one belonging to species n and the other to species p — according to (5d) and (5e) we may write :

$$\begin{aligned} y_n &= R_n \cos n(t_{☾} - \beta_n) \qquad & \tilde{y}_n &= R_n \sin n(t_{☾} - \beta_n) \\ y_p &= R_p \cos p(t_{☾} - \beta_p) \qquad & \tilde{y}_p &= R_p \sin p(t_{☾} - \beta_p) \end{aligned} \quad (5f)$$

Let us introduce the two vectors that are defined in the complex plane by

$$\zeta_n = y_n + i\tilde{y}_n \quad (5g)$$

and

$$\zeta_p = y_p + i\tilde{y}_p \quad (5h)$$

and let us replace y_n, $\tilde{y}_n$, y_p and $\tilde{y}_p$ in (5g) and (5h) by their equivalents from the group (5f). Taking the Euler formula

$$\exp(ix) = \cos x + i \sin x \quad (5i)$$

into account we then have the formulae

$$\zeta_n = R_n \exp[\, in(t_{☾} - \beta_n)]$$

$$\zeta_p = R_n \exp[ip(t_{☾} - \beta_p)]$$

whose product is

$$\zeta_n \zeta_p = R_n R_p \exp i[(n + p)t - n\beta_n - p\beta_p] \quad (5j)$$

If we now take the conjugate ζ_p^* of (5h) and also that of (5i) which is equal to exp(—ix) we find :

$$\zeta_n \zeta_p^* = R_n R_p \exp i[(n - p)\, t - n\beta_n + p\beta_p] \quad (5k)$$

According to (5i), from (5j) we then obtain

$$\begin{aligned} \zeta_n \zeta_p = R_n R_p \{ &\cos[(n + p)t - n\beta_n - p\beta_p] \\ &+ i \sin[(n + p)t - n\beta_n - p\beta_p]\} \end{aligned} \quad (5l)$$

and from (5k)

$$\begin{aligned} \zeta_n \zeta_p^* = R_n R_p \{ &\cos[(n - p)t - n\beta_n + p\beta_p] \\ &+ i \sin[(n - p)t - n\beta_n + p\beta_p]\} \end{aligned} \quad (5m)$$

However according to (5c) we have R_n given by the square root of the sum of the squares of these formula, whereas tan β_n is equal to the ratio between the two formula. Consequently R and β_n are functions of B and θ whose variation is slow. Thus the phase angles to be seen in (5*l*) and (5*m*) increase almost proportionally to $(n + p)$ and $(n - p)$ and these define two new species of oscillation that are also proportional to the product of the semi-ranges R_n and R_p. Since this last statement agrees with the theory of tidal propagation in shallow water we can assume that the oscillations actually observed, which have frequencies $(n + p)$ and $(n - p)$, will be respectively proportional to the real terms of (5*l*) and (5*m*). Examining these two formulae we see that from the (5*f*) group and from expressions (5*g*) and (5*h*) we obtain the real parts of (5*l*) and (5*m*) which are equal to $y_{n\pm p}$ whereas their imaginary parts give us $\tilde{y}_{n\pm p}$.

We shall now show that in the convolution (2*j*) it is possible to replace $a_{n\pm p}$ by $y_{n\pm p}$, and $b_{n\pm p}$ by $\tilde{y}_{n\pm p}$. In fact if in (2*h*) and (2*i*) we replace t_{GW} by $t_{GW} - \theta$, after several transformations and also taking (2*d*) into account we find

$$a_n(t) = R'_n \cos n(t^{☾}_{GW} - \beta'_n)$$

and

$$b_n(t) = R'_n \sin n(t^{☾}_{GW} - \beta'_n)$$

where

$$R'_n \cos n\,\beta'_n = B^{☾}_n + B^{\odot}_n \cos n\theta$$

$$R'_n \sin n\,\beta'_n = B^{\odot}_n \sin n\theta$$

Comparing $a_n(t)$ and $b_n(t)$ with the real and the imaginary parts of expressions (5*l*) and (5*m*) we can see that there is a perfect analogy, and this justifies the above-mentioned substitution.

The procedure just described can be repeated in sequence. Thus we are able to introduce into the computations three oscillations belonging to the species m, n, and p, in order to obtain the following species :

$$m + (n + p) \qquad m + (n - p)$$

$$m - (n + p) \qquad m - (n - p)$$

We need only write the new constituents in the complex form

$$\zeta_m = y_m + i\tilde{y}_m \tag{5n}$$

and then to carry out the following multiplications :

$\zeta_m \zeta_n \zeta_p$	$m + n + p$
$\zeta_m (\zeta_n \zeta_p)^*$	$m - n - p$
$\zeta_m \zeta_n \zeta_p^*$	$m + n - p$
$\zeta_m (\zeta_n \zeta_p^*)^*$	$m - n + p$

If, for example, $m=n=p=2$ then the first product represents a sixth-diurnal constituent and the third a semi-diurnal one. On the other hand if $m=n=2$ and $p=1$ these same products will represent respectively a fifth-diurnal and a third-diurnal constituent.

From what we have just said it suffices to remember the following rule. In order to obtain the values of $a(t)$ and $b(t)$ the constituents must be written in their complex form as in (5*g*), (5*h*) and (5*n*); the necessary multiplications must be carried out and the real parts are adopted as values of $a(t)$ and the imaginary part as values of $b(t)$.

In order to be sure of taking into account all the oscillations of about 4 cycles per day, CARTWRIGHT [2] had to consider three quarter-diurnal constituents. First of all he took a combination exactly the same as the one already mentioned, i.e. the product of $\zeta_2 \zeta_2$. Next he considered it necessary also to make the following products :

$$\zeta_2^t \zeta_2^{t-\Delta t}$$
$$\zeta_2^{t-\Delta t} \zeta_2^{t-\Delta t}$$

He then added the indications (0,0), (0,1) and (1,1) to these constituents.

We must now point out that in the method with which we are here dealing the shallow water constituents are considered as a whole. Thus CARTWRIGHT assumes [2] that even for a tide where the distortion due to shallow water is fairly large the number of shallow-water terms will always be smaller in the present method than the number which have to be introduced into the harmonic method in order to obtain a satisfactory prediction. In fact the recent research of ZETLER and CUMMINGS [3] has shown that 114 harmonic constants need to be used in order to represent the tide at Anchorage (Alaska) really satisfactorily. It should be added that it is a remarkable fact that nearly the same number (113) of constituents was quite independently found by LENNON and ROSSITER to be necessary for obtaining a good representation of the River Thames tide [4].

6. — ANALYSIS

In the above section we saw that the shallow water corrections are introduced after the prediction of the basic part. This is why the analysis must be carried out in two stages. In the first stage the authors analyse a curve resulting from a filtering that leaves only the long period, diurnal, semi-diurnal and third diurnal oscillations. The filter used is given in reference [2]. The values of u and v for the above-mentioned frequencies are computed by the least squares method in which we must have

$$[u\ a(t) + v\ b(t) - \overline{y}_t]^2 = \text{minimum}$$

where $\overline{y}_t$ is the height of the filtered curve at time t.

However, since in the filtered curve there are contributions from the semi-diurnal constituents $\zeta_2 \zeta_2 \zeta_2^*$, and the third diurnal constituents $\zeta_2 \zeta_2 \zeta_1^*$, and since the knowledge of $a(t)$ and $b(t)$ for these constituents depends on knowing y_1 and y_2, the values of u and v for the diurnal and semi-diurnal constituents of astronomic origin must be determined by a preliminary analysis in order to obtain the values of y_1 and y_2. Once these values are known the weights u and v in their entirety can be recomputed by the analysis of the difference $[y - (y_1 + y_2)]$.

7. — CONCLUSION

We think that we have given a fairly complete picture of the method just described. We consider in point of fact that CARTWRIGHT's results for six British ports are fairly encouraging and that they merit a more extensive study of their application. We hope that in the near future those who have the means necessary for using this method will let us know the results of their research.

We advise those interested to consult the paper [2] at the end of which are given some particulars which are very useful from the practical point of view.

Bibliography

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