CAMPAIGN FOR DETERMINING THE LONGITUDE OF THE FUNDAMENTAL POINT OF THE ASTRONOMIC OBSERVATORY IN NAPLES

III. THE EQUATION FOR LONGITUDE

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In this article we shall be dealing in more detail with the equation used for determining the longitude of the fundamental point of the Capodimonte Astronomic Observatory, Naples. We shall analyse certain problems well known to specialists but which are not easily found in geodetic astronomy literature. Several new developments never before used in determinations of this kind and that we have deduced during the course of our observations will also be given.

1. — DETERMINATION OF ASTRONOMIC TIME

To determine the *correction* for a clock regulated on sidereal time, it is only necessary to compare time T indicated by the clock at the meridian transit of a star with the right ascension α of this star. If the clock is perfectly regulated the hour read at the instant of the star's meridian transit should be this star's right ascension value.

If the correction to be made to the clock regulated on sidereal time is designated by ΔQ_s we have :

$$\Delta Q_s = \alpha - T \qquad (1)$$

In order to be able to carry out this comparison, the quartz clock must be coupled to a special instrument that makes it possible to find the exact instant of the star's transit at the local meridian. To this end we decided to use the transit instrument.

But a transit instrument, however well preliminarily adjusted, can never be located perfectly in the meridian. Even if at the beginning of operations this was the case, some time later its setting would, for various reasons, lose its alleged perfection. Instead of having continually to correct the instrument to eliminate errors it is easier to determine the amount of these errors at the time of the observation and then to compute their effect on the final result. After the initial adjustment these errors will generally be so small that their square can be considered negligible. Thus it is possible to compute *separately* the influence on the result of each error and finally to total their effect.

In the case of a quickly reversible transit instrument fitted with an impersonal micrometer, the errors having the greatest influence on the observation results are :

a) The error due to the *inclination* of the instrument's secondary axis of rotation — i.e. when this axis is not perfectly parallel to the horizon plane. It should be added that in a transit instrument the axis defined by the two centres $0_1\, 0_2$ of the trunnions is called the secondary axis of rotation — or else simply the *axis of rotation*.

b) The *azimuth* error of this axis, i.e. its deviation in relation to the exact E-W direction.

c) The *collimation* error, i.e. the deviation between the line of collimation and the normal position of the axis. In rapidly reversing instruments, this last error is eliminated, but we shall continue to take it into account because we shall include in this error the effects of both the half-width of the contact in the impersonal micrometer and of the delay due to the play of the screw.

It is obvious that these three errors will cause the telescope position to be out of the meridian and consequently the star transit will be observed not at the true meridian but in its near vicinity, i.e. either before or after the star transit at the local meridian. If we designate by P_0 the time of a star transit observed with an instrument in which the 3 errors mentioned above are present, it will be necessary to correct the observed time P_0 for the effects of the 3 errors in order to obtain the time T of the star's meridian transit at the place of observation, namely :

$$T = P_0 + \Delta\omega_i + \Delta\omega_a + \Delta\omega_c \qquad (2)$$

where $\Delta\omega$ are the hour angles, taken with their sign, of the star at the time of its transit at the central thread of the instrument eyepiece.

The equation for the quartz clock will then become :

$$\Delta Q_s = \alpha - [P_0 + \Delta\omega_i + \Delta\omega_a + \Delta\omega_c] \qquad (3)$$

Let us now determine one by one the $\Delta\omega$ corrections to be made to P_0 to obtain time T on the clock when the star culminates at the meridian.

A) Determination of $\Delta\omega_i$

We saw in the first article of this series (*IH Review*, Vol. XLIV, No. 2, 1967) that the explicit form for this correction is :

$$\Delta\omega_i = i \cos(\varphi - \delta) \sec\delta = i(\cos\varphi + \sin\varphi \operatorname{tg}\delta)$$

Let us assume that the western end of the axis of rotation is higher in relation to the horizontal plane than the eastern end. In this case the zenith of the instrument will be to the east in relation to the zenith of the place of observation, and the star will be observed on the instrument before its meridian transit. For this case the correction $+\Delta\omega_i$ must be added to P_0.

For the case of an observation of a star culminating between the zenith and the south of the observer's horizon, we shall have :

$$\delta < \varphi \qquad \cos(\varphi - \delta) > 0$$

whereas if the star culminates at the upper meridian its meridian transit is between the observer's zenith and the North Pole, and we shall have :

$$\delta > \varphi \qquad \cos(\varphi - \delta) < 0$$

If however in both cases the foregoing formula is to supply directly the correction $+\Delta\omega_i$ to be made to P_0 then i must be positive.

When the eastern end of the axis of rotation in relation to the horizontal plane is higher than the western end, the zenith of the instrument will be to the west in relation to the zenith of the place of observation. In this case a star at its meridian transit (u.t.) will be observed with a delay, i.e. after having passed the meridian, and consequently the correction to be made to P_0 will be $-\Delta\omega_i$.

In the two cases of the upper meridian transit we shall have :

$$\delta \lessgtr \varphi \qquad \cos(\varphi - \delta) > 0$$

and if we wish to obtain correction $\Delta\omega_i$ with the negative sign then i must be negative.

Thus to obtain by means of the general formula the value of the sign of the $\Delta\omega_i$ correction to be made to P_0 to reduce it to the transit time T, it is necessary to consider the value of inclination i as *positive* when the instrument's axis has its western end higher than the eastern end in relation to the horizontal plane and as *negative* when the reverse is the case.

In order to be able to retain this convention for the sign of i even for the lower meridian transits (l.t.) of circumpolar stars, it will be sufficient to consider the $\Delta\omega_i$ angles as negative (figure 1). In fact for the lower meridian transits, it is the opposite to what takes place for the upper meridian transits, i.e. a star is observed later on when the western end is higher and earlier when it is the eastern end which is the highest.

Thus for a star A (fig. 1) we have :

$$\Delta\omega_i = -\, i \sin h \sec \delta$$

but

$$h = \overset{\frown}{AN} = \overset{\frown}{PN} - \overset{\frown}{PA} = \varphi - (90^\circ - \delta) = -\,[90^\circ - (\varphi + \delta)]$$

and consequently :

$$\Delta\omega_i = i \cos(\varphi - \delta) \sec \delta = i(\cos\varphi + \sin\varphi \tan\delta)$$

and with the sign convention adopted for i, we shall still have the correction $\Delta\omega_i$ with the desired sign. The same is also true when the eastern end of the axis of rotation is the higher.

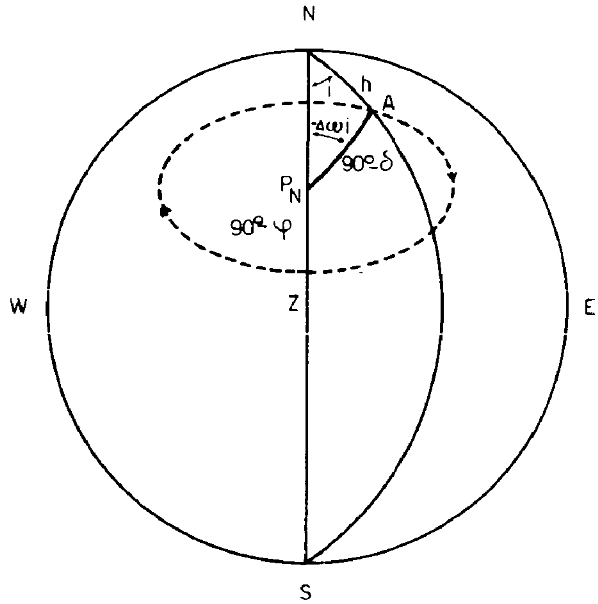

Fig. 1

We can therefore conclude that the correction $\Delta\omega_i$ and its sign for the effect of the inclination of the instrument's axis, to be made to P_0, is given by the following relation.

$$\Delta\omega_i = i\,(\cos\varphi \pm \sin\varphi \tan\delta) \quad \begin{array}{l} + \text{ upper transit} \\ - \text{ lower transit} \end{array} \tag{4}$$

Since the inclination i is found by reading the striding level, let us see what sign must be given to these readings expressed in the level's graduation, *with the sign established by the aforementioned convention.*

Let us take the case of a level whose zero is at the end of the phial as is the case in a transit instrument.

As an example let us take the instance where the western end of the axis is higher in relation to the horizontal plane than the eastern end. The level can then take up two opposite positions on the axis : the zero towards the west or the zero towards the east.

Let us first of all take the instance where the zero is initially to the west (fig. 2a); after having reversed it on its axis (fig. 2b) we shall have $C_W < C_E$ where C_W and C_E are the readings at the centre of the bubble with the zero (of the graduation) first to the west and then to the east.

To achieve the displacement of the bubble centre in such a way that the adopted sign for i is retained (in this case, positive) it is necessary to adopt the readings at the ends of the bubble as positive when the zero is towards the east and the readings made with the zero towards the west as negative.

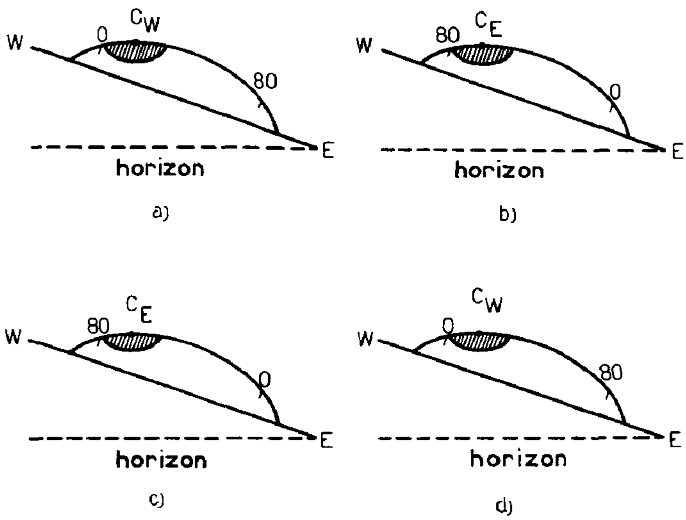

FIG. 2

If the level's zero is first of all to the East (fig. 2c), after the level has been reversed on its supports (fig. 2d) the condition $C_E > C_W$, identical to the one already found, will be obtained, and thus in both cases in order to obtain the value of inclination i with the positive sign it will only be necessary to retain the sign convention already adopted.

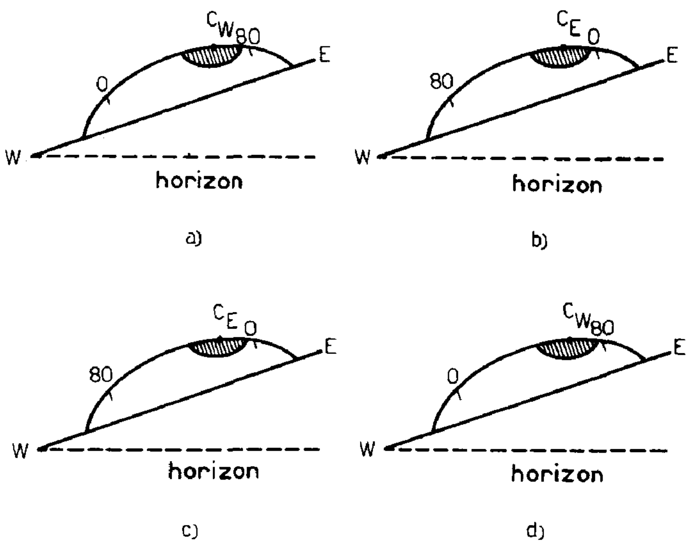

FIG. 3

For the case where the eastern end is the higher ($i < 0$) we shall have the situation shown in figs. 3a, b, c and d.

In this case C_E will be smaller than C_W. Then, following the sign convention, we shall have as we wish the inclination i with a negative sign.

B) Determination of $\Delta\omega_a$

Let us now consider the effect on the determination of the time T of the star's transit of an azimuth error in the instrument.

This error exists when the transit instrument is adjusted on station with its secondary axis of rotation not perfectly set in the E-W direction. The instrument's axis of collimation will then describe a great circle passing through the zenith, i.e. a vertical.

Let us first of all assume that the western end of the instrument axis of rotation is displaced to the south by a small angle a, called the instrumental azimuth (fig. 4a).

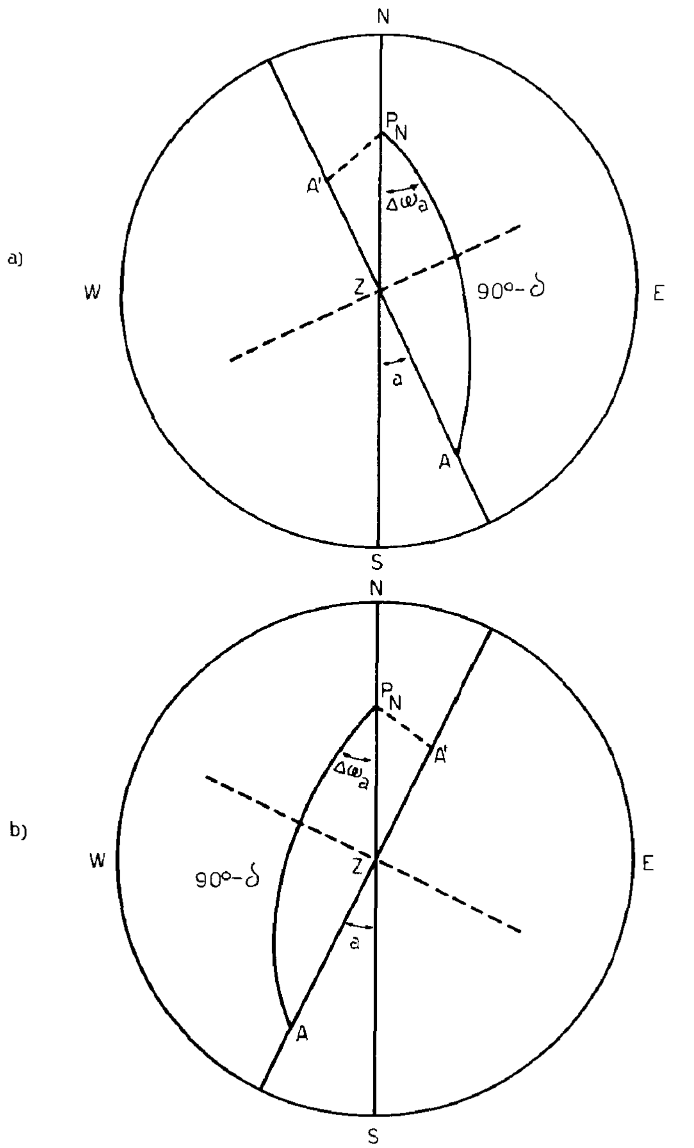

FIG. 4

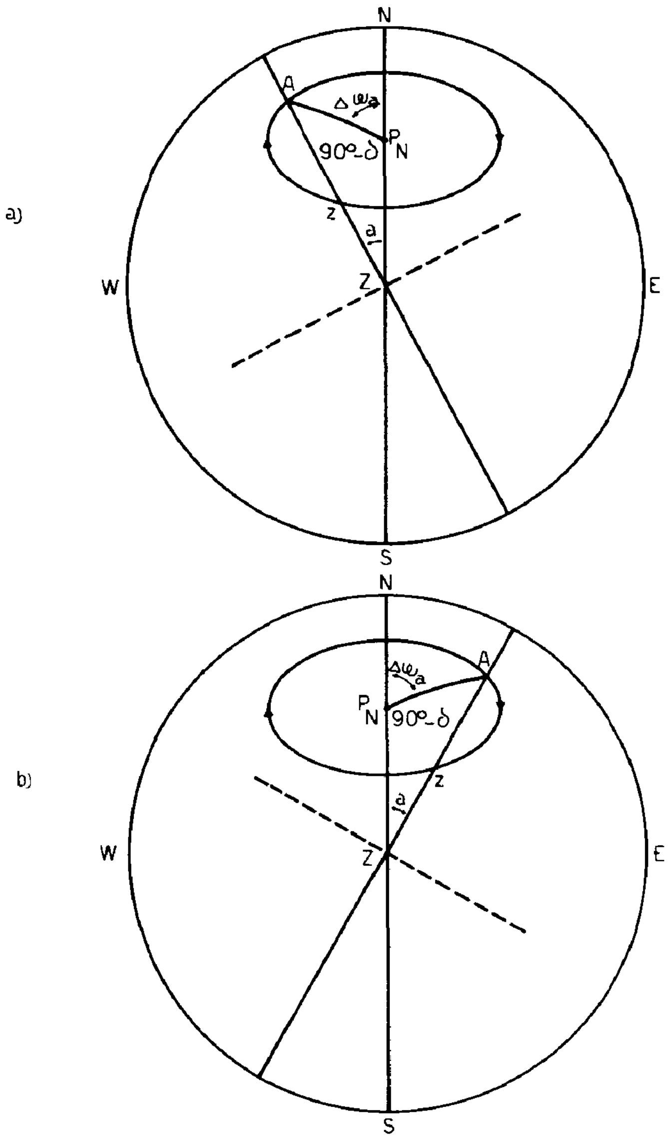

FIG. 5

A star A, having its upper transit to the South of the zenith ($\delta < \varphi$), will be observed before its meridian transit.

Taking the spherical triangle ZPA, we have :

$$\Delta\omega_a = a\,\frac{\sin(\varphi - \delta)}{\cos\delta} = a\,(\sin\varphi - \cos\varphi\tan\delta) \qquad (5)$$

As $\varphi > \delta$, sin ($\varphi - \delta$) will be positive. However the correction $\Delta\omega_a$ to be made to the observed transit P_0 in order to reduce it to time T will have a positive sign because the star was observed beforehand. Thus the sign of a should be positive.

In the case of the star A′ whose upper transit is to the North of the zenith, we have $\delta > \varphi$, and consequently sin ($\varphi - \delta$) < 0.

Considering that the star is observed with a delay, the correction $\Delta\omega_a$ should have the negative sign, and thus the value of a should be taken with the positive sign.

For the case where the eastern extremity of the instrument's secondary axis of rotation is displaced towards the south (fig. 4*b*) formula (5) will still be valid. For when the star culminates south of the zenith ($\delta < \varphi$) this star will be observed in relation to its meridian transit with a delay. The correction $\Delta\omega_a$ must therefore be negative, and thus a negative value of a must be taken.

If the star culminates (u.t.) to the North of the zenith ($\delta > \varphi$) it will be observed in advance; thus the correction $\Delta\omega_a$ must be positive and the sign of a will still be negative.

It can therefore be concluded that in order to obtain the correction $\Delta\omega_a$ with the suitable sign, it is necessary to consider the value a of the instrument's azimuth positive when the western extremity is displaced towards the south and negative when it is the eastern end which is displaced towards the south.

In the case of observations of circumpolar stars making a lower meridian transit (fig. 5*a*, *b*) since $z = 180° - (\varphi + \delta)$ formula (5) becomes:

$$\Delta\omega_a = a \frac{\sin(\varphi + \delta)}{\cos\delta} = a(\sin\varphi + \cos\varphi \tan\delta)$$

and the sign convention for a remains the same.

The general formula is therefore :

$$\Delta\omega_a = a(\sin\varphi \mp \cos\varphi\tan\delta) \quad \begin{matrix} - \text{ lower transit} \\ + \text{ upper transit} \end{matrix} \tag{6}$$

C) **Determination of $\Delta\omega_c$**

Let us finally consider the collimation error c.

This error causes the displacement of the instrument's collimation line which instead of describing the meridian on the celestial sphere will describe a small circle parallel to the meridian.

Let us consider the fig. 6*a* in which the small circle described by the axis of collimation is to the east in relation to the zenith, i.e. the arc of the great circle WZ′ is greater than 90°.

In this case the star will be observed before its meridian transit, and by means of triangle $P_N A\ A_1$ we obtain :

$$\Delta\omega_c = c \sec\delta \tag{7}$$

Considering that the star is observed before its meridian transit, the correction $\Delta\omega_c$ must have the positive sign; therefore the collimation should be taken as positive when the arc WZ′ is greater than 90°.

In the case where the arc WZ′ is smaller than 90° (fig. 6*b*), we shall still have formula (7) with collimation c taking the negative sign.

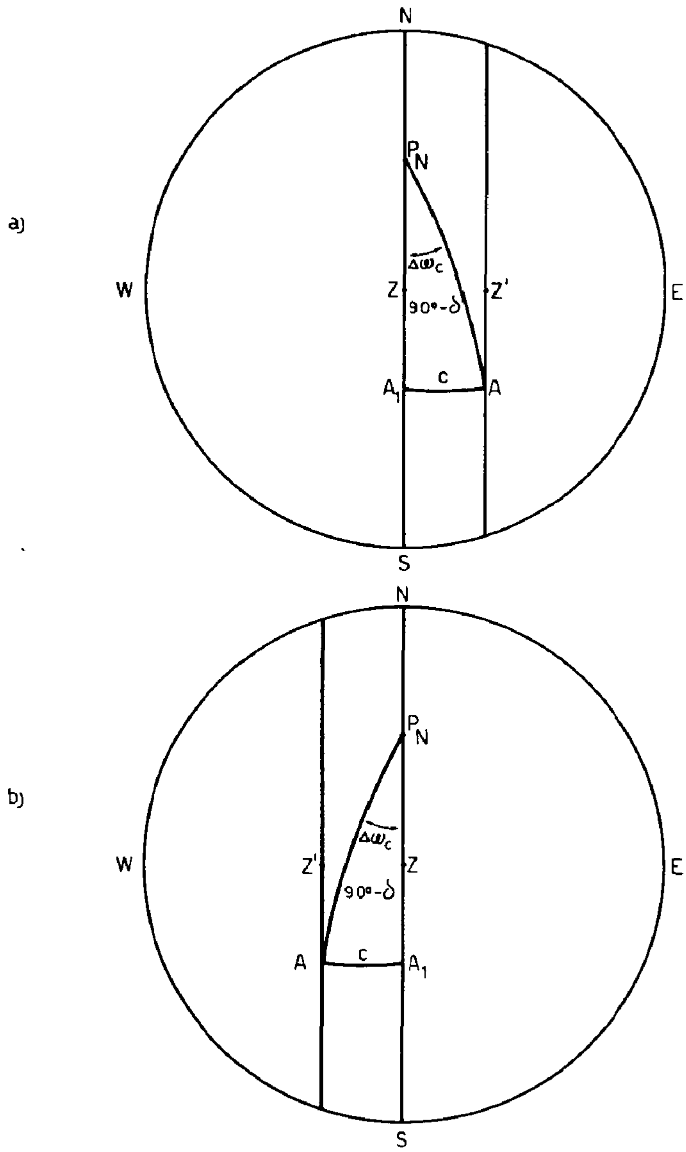

Fig. 6

In lower transits, in order to retain the same sign convention for c so as to be able to find the correction $\Delta\omega_c$ with its sign by means of the formula (7) it is necessary — as we also did for the case of the inclination — to take $-\Delta\omega_c$ instead of $+\Delta\omega_c$ (figure 7).

Therefore we finally have the general formula :

$$\Delta\omega_c = \pm\, c \sec \delta \begin{cases} + \text{ p.s.} \\ - \text{ p.i.} \end{cases} \tag{8}$$

In rapidly reversible instruments such as our AP 100, this error is automatically eliminated in the course of the observation.

In collimation c, however, we include the effects produced by the delays due to the half width of the impersonal micrometer's contact and to the play of the screw.

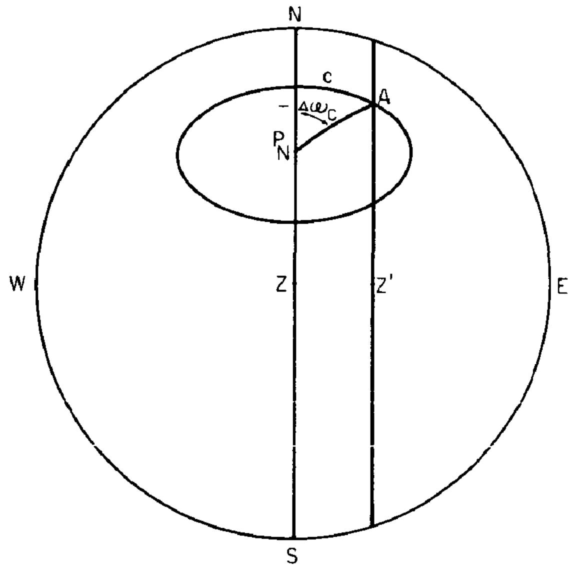

FIG. 7

In our case formula (8) then becomes :

$$\Delta\omega_c = c \sec \delta \tag{9}$$

where the positive sign applies both to upper transit and lower transit.

Finally, taking formulae (4), (6) and (9) into account, formula (2) may be written :

$$T = P_0 + i(\cos\varphi \pm \sin\varphi \tan\delta) + a(\sin\varphi \mp \cos\varphi \tan\delta) + c\sec\delta \begin{cases} \text{u.t.} \\ \text{l.t.} \end{cases}$$

and by making :

$$m = i\cos\varphi + a\sin\varphi \qquad n = i\sin\varphi - a\cos\varphi$$

as Bessel did, we obtain the *Bessel equation* :

$$T = P_0 + m + n\tan\delta + c\sec\delta \begin{cases} \text{u.t.} \\ \text{l.t.} \end{cases}$$

By putting :

$$\Sigma = m \pm n\tan\delta + c\sec\delta \begin{cases} \text{u.t.} \\ \text{l.t.} \end{cases} \tag{10}$$

we finally have :

$$T = P_0 + \Sigma$$

Equation (1) for the determination of astronomic time can therefore be written

$$\Delta Q_s = \alpha - (P_0 + \Sigma) \tag{11}$$

in which all the quantities are expressed in sidereal time.

2. — COMPUTATION OF RIGHT ASCENSIONS AND THE STAR CATALOGUES

We must first of all concern ourselves with the determination of the observed star's right ascension. This is done with the help of star catalogues which in turn are based on particular systems of reference.

A) Fundamental catalogues

Since the equatorial coordinates of the stars vary on account of the phenomenon of annual precession, in order to relate the star positions to a mean epoch which is not that of the catalogue it is necessary to know the effect of the precession on each particular star.

If we examine the formulae expressing the phenomenon we see that the annual precession in right ascension is dependent both on this right ascension and on the star's declination and that the precession is always positive, except for stars of great declination having a right ascension greater than 12 hrs.

But the annual precession varies slowly with time. This is the reason that precession values for each particular star must be also related to a set common epoch which is generally the mean epoch of the catalogue.

Consequently, if the epoch to which we wish to reduce the positions of stars shown in a catalogue differs greatly from the catalogue's mean epoch, it will also be necessary to take into account the variation in the annual precession values in the interval of time. For this reason, modern catalogues show opposite each star position, both for right ascension as well as for declination, the variation of the annual precession for a 100 years, and this is called the *secular variation*. Sometimes, however, when it is a question of very accurate positions, and consequently of fundamental catalogues, the variation in the secular variation is even given, and this is called the *third term of the precession*.

This is not all. By reducing a particular star's positions taken from several catalogues to the same mean epoch, we shall obtain a series of values which will generally include fairly large differences, and which show a progressive trend depending only on the epoch of observations.

These differences are due in part to accidental observational errors and to other systematic causes peculiar to a particular catalogue; but they are due in large part to each star's own proper motion. The annual difference, either positive or negative, and arising in either of the coordinates of the star's real movements, is called the *annual proper motion*. This is deduced from the accurate comparison of the star positions obtained from the various catalogues reduced to a common epoch. It is easy to understand how important the knowledge of the annual proper motion of each star of known position is when drawing up a catalogue. Unfortunately we are far from being able to obtain this result, although for many stars — and in particular all the fundamental stars — the proper motion is already well determined. In many cases we even know the proper motion variation with time.

This is the reason why in very modern catalogues we also find printed beside the right ascension and declination of the stars, the proper motion and the mean observational epoch, in addition to the reduction factors already mentioned, — annual precession, secular variation and in certain cases the third term of the precession.

However this is not sufficient. If we make a comparison between the positions supplied by two of the catalogues we have mentioned, both reduced to the same epoch, we shall see that the positions still differ not only because of unavoidable accidental errors, but also as a result of systematic errors which originate in the considerable differences always present in the instruments and in the methods used for the observation. This is in addition to the personal equation for each observer. Modern techniques have done much to reduce these errors but, for the order of accuracy today required in astro-geodetic measurements, they exist still. We can therefore conclude that for each star there is a slightly different position according to which catalogue is used.

It is to the Heidelberg Astronomic Association that we owe the idea of establishing a *normal system* (giving this term the meaning it has in the Gaussian theory of errors) for star positions supplied by the various catalogues, in order that the fairly important systematic differences existing between these catalogues may be reduced to the minimum.

Thus the *Auwers fundamental system* was born. As the base of this system are to be found the excellent systematic meridian observations in Pulkova, Greenwich, Leipzig, Cambridge and Leyden. These observations authoritatively discussed and coordinated into a simple catalogue have supplied a first homogeneous nucleus of 539 fundamental stars. In 1889 thanks to AUWERS himself 303 other stars were added, constituting a second fundamental catalogue. This catalogue, [2] for which the symbol FC is used, adopts as reference system the position system of the ***First Catalogue of the Astronomische Gesellschaft*** (AGK 1) and uses the Struve constant for the precession.

This first fundamental system of AUWERS was subsequently (1907) revised and perfected by J. PETERS and this work gave rise to a second catalogue in the NFK series, containing the positions of 925 stars in the two hemispheres, and for many years this catalogue was the basis of the Berliner Astronomisches Jahrbuch (B.A.J.).

This last catalogue [3] is the result of the extension of the FC to the southern hemisphere. The system and the individual values of each star position and of the proper motion are based on the observations carried out from 1745 — 1900 and on the Newcomb precession.

In 1934, KOPFF, Director of the Rechen Institut in Berlin, revised this basic catalogue of Auwers and published new positions for 1535 stars referred to 1925 and to 1950. Since 1941, by decision of the International

Astronomic Union, this catalogue [4], (known as FK3 for it is the third of the series) has been the basis for all almanacs.

The quality of the observations on which this system was based from the time of its first edition led almost all the astronomers contemporary with AUWERS to relate their catalogue to this system. AUWERS himself, in an important study (5), gave a resume of all the research he had carried out in order to bring the results of numberless observations into this system.

Side by side with AUWERS' grandiose system are others concerned with still higher standards of accuracy and homogeneity as their aim, among them the works of the American NEWCOMB (1898) and later Boss (1903, 1909). In 1938-39 Newcomb's fundamental system was again worked out by EICHELBERGER. These catalogues are known as G.C. and N. 30. There is a difference, although a small one, between the two systems, but the system F.K. 3 is recognized as the most accurate and homogeneous.

In the meanwhile, observations had accumulated. 72 modern catalogues had already been published when Kopff, at the request of the International Astronomic Union started to revise F.K. 3 to take account of both new observations and the so-called equation of magnitude.

Thus we come to the F.K. 3 R [6] system which was drawn up first, and finally to the present F.K. 4 [7], today considered as being the most accurate and homogeneous system.

For the system for right ascensions (with which we are dealing) from the 4th fundamental catalogue, the compilation sources are the following :

a) α_δ and α_α ($\alpha_\delta/\alpha_\alpha$ catalogues) have been deduced from at least 25 catalogues, to name them :

9 from the Capetown Observatory (Reversible Transit Circle);
1 from Cordoba (Meridian Circle);
4 from Greenwich (Airy = Transit Circle; Cooke = Rev. Transit Circle);
3 from Pulkovo (Transit Instrument);
6 from Washington (4 are *six-inch* and 2 *nine-inch*).

b) α_δ (α_δ catalogues) from 7 catalogues;

3 from Königsberg (two by meridian circle, 1 by Transit Instrument);
1 from Munich (Transit Instrument);
2 from Ottawa (Reversible Transit Circle);
1 from Potsdam (Transit Instrument).

c) α_α (α_α catalogues) from two catalogues, one the Bergerdorf (Transit Instrument) and the other the Greenwich (Airy = Transit Circle).

d) The μ system from 11 catalogues :

3 from Capetown (Rev. Transit Circle);
2 from Greenwich (Airy = Transit Circle);
3 from Pulkovo (Transit Instrument);
1 from Washington (*Six-inch*);
1 from Washington (*Nine-inch*).

This amounts to a total of 45 catalogues. For further particulars about this system, the reader should refer to the study carried by GLIESE [8].

The F.K. 4 is therefore the most accurate catalogue at our disposal today and if there are systematic differences [9] between this catalogue and the G.C. and the N. 30 we are inclined to attribute them, at least in great part, to the errors that these last two catalogues contain.

Following a decision of the International Astronomic Union (Dublin, 1955) the *Astronomischen Rechen-Institut* of Heidelberg publishes yearly the mean and apparent places of the 1535 stars of the F.K. 4 Apparent Places of Fundamental Stars (A P F S) and this is the annual catalogue used for highly accurate geodetic-astronomic determinations like ours.

The right ascension value α deduced from the A P F S represents the most accurate data available at present.

B) The computation of the right ascension value from the A P F S

In the APFS the apparent places of fundamental stars are tabulated at 10-day intervals.

The preface to this volume recommends the use of the second differences when interpolating for the computation of right ascension.

We shall now study the question thoroughly to see exactly what order of accuracy is achieved by considering the second differences in the interpolation but neglecting those of the third order.

Let x_{-2}, x_{-1}, x_0, x_1, x_2, ... be a certain number of equally spaced tabulated values of the independent variable and y_{-2}, y_{-1}, y_0, y_1, y_2 ... the corresponding values of the dependent variable. To illustrate the notations, we give the following table.

$$
\begin{array}{ccccc}
x_{-2} & & & & \\
 & \Delta'_{-2} & & & \\
x_{-1} & & \Delta''_{-1} & & \\
 & \Delta'_{-1} & & \Delta'''_{-1} & \\
x_0 & & \Delta''_0 & & \Delta^{IV}_0 \\
 & \Delta'_1 & & \Delta'''_1 & \\
x_1 & & \Delta''_1 & & \\
 & \Delta'_2 & & & \\
x_2 & & & &
\end{array}
$$

We shall call the difference with subscripts 0 and 1 the central difference for x, namely :

$$\Delta'_1 = y_1 - y_0$$

$$\Delta''_0 = y_1 - 2y_0 + y_{-1}$$

$$\Delta''_1 = y_2 - 2y_1 + y_0$$

The basic interpolation formula which is valid for any distribution of the values of x is the Lagrange formula :

$$f_p(x) = y_0 \frac{(x - x_1)(x - x_2)\dots(x - x_p)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_p)} + y_1 \frac{(x - x_0)(x - x_2)\dots(x - x_p)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_p)} + .$$

where $f_p(x)$ is the p-degree polynomial whose numerical value coincides with the y-value for known values of $x_0, x_1, \dots x_p$.

However, the tabular interval in the ordinary tables is usually chosen so that a polynomial interpolation may always be possible and legitimate. In this case, the problem is solved with the Lagrange formula.

We shall ignore the case in which the function to be interpolated is not a polynomial because in practice this occurs very rarely, and *never* in the astro-geodetic table interpolations.

The problem we have to solve is to determine in the interval (x_0, x_1) a value of x such that $x = x_0 + n\,h$, where h is the tabular interval (constant in our tables) and n an auxiliary variation between 0 and 1.

Let us write the Lagrange polynomials that coincide with y first for x_0 and x_1, then for x_0, x_1 and x_2, and so on, increasing each time by one unit the number of coincidences (the first will obviously arise when $x = x_0$) :

$$f_1 = n(n-1)\left[\frac{y}{n-1} - \frac{y_0}{n}\right] = y_0 + n\,\Delta'_1$$

$$f_2 = \frac{n(n-1)(n-2)}{2!}\left[\frac{y_0}{n-2} - \frac{2y_1}{n-1} + \frac{y_0}{n}\right]$$

. . . .

By expressing the differences $f_p - f_{p-1}$ with the help of suitable differences we obtain :

$$f_2 - f_1 = \frac{n(n-1)}{2!}(y_2 - 2y_1 + y_0) = \frac{n(n-1)}{2!}\Delta''_1$$

$$f_3 - f_2 = \frac{n(n-1)(n-2)}{3!}(y_3 - 3y_2 + 3y_1 - y_0) = \frac{n(n-1)(n-2)}{3!}\Delta'''_2$$

. . . .

The polynomial sought, whose degree can be limited at will, therefore is :

$$f(n) = y_0 + n\,\Delta'_1 + \frac{n(n-1)}{2!}\Delta''_1 + \dots + \frac{n(n-1)\dots(n-2k+1)}{(2k)!}\Delta^{2k}_k + \dots$$

which is the *Newton interpolation formula.*

Limited to its first two terms the f function reduces to a linear expression whose value coincides with the value of y for $n = 0$ and $n = 1$. The linear interpolation is consequently justified when the first differences are constant as in the case of trigonometrical tables, but for our case linear interpolation is not sufficient.

For the APFS interpolation computations, however, the Bessel formula is most suitable and we shall later see the reason.

The use of this formula has many more advantages than Newton's formula.

Let us go back to the expressions given by the central differences, and let us introduce into the Lagrange formula successively the y values contained in these differences. We shall then obtain the following polynomials :

$$f_1 = n(n-1)\left[\frac{y_1}{n-1} - \frac{y_0}{n}\right] = y_0 + n\,\Delta'_1$$

$$f_2 = \frac{(n+1)\,n(n-1)(n-2)}{3\,!}\left[\frac{y_2}{n-2} - \frac{3y_1}{n-1} + \frac{3y_0}{n} - \frac{y_{-1}}{n-1}\right]$$

. . . .

whence :

$$f_2 - f_1 = \frac{n(n-1)}{2(2\,!)}\left[\Delta''_0 + \Delta''_1 + \frac{2n-1}{3}\,\Delta'''_1\right]$$

. . . .

Thus, the *Bessel formula* takes the form :

$$f(n) = y_0 + n\,\Delta'_1 + \frac{n(n-1)}{4}(\Delta''_0 + \Delta''_1) + \frac{n(n-1)(2n-1)}{12}\,\Delta'''_1 + \ldots$$

We see straight away the advantages that the Bessel formula has over the Newton formula in respect of ease of computation.

The Bessel formula uses symmetrically distributed values of the function and furthermore only uses the central differences. We also note that even number coefficients which cancel out for $n = 0$ and $n = 1$ are maximal in absolute value for $n = 0.5$. Moreover, the odd number coefficients cancel out for $n = 0.5$. Let us now compute the maximum value of the coefficient of the third order difference. By development and derivation, we have the following equation : $6\,n^2 - 6\,n - 1 = 0$ which allows both solutions $n' = {}^1/_2 \pm \frac{\sqrt{3}}{6}$.

Consequently, for these n values, the following coefficient becomes maximum :

$$\frac{n'(n'-1)(2n'-1)}{12} = \mp\frac{\sqrt{3}}{216} = \mp\,0.00802$$

Thus if the third difference Δ'''_1 is not higher than 62 in absolute value, by ignoring this difference we only make an error smaller than half a unit in the last significant figure.

Computing the same quantity with the Newton formula, we find $n_{max} = 0.42265$, and the upper limit mentioned above for the third difference must not exceed 7, otherwise we have at least a one unit error over the last figure expressed.

As can be seen, this maximum limit for the third difference of the Bessel formula is 9 times greater than in the Newton formula.

In our APFS tables the third difference for the right ascensions reaches a maximum of 8 and sometimes even 9. Consequently, if we compute the interpolation using the Newton formula we must sometimes take into account the third differences, *whereas by employing the Bessel formula we may always stop at the second differences.*

Finally one last point regarding the suitability of computation with the Bessel formula should be pointed out. Let us consider the following differences :

$$\Delta'_2 = y_2 - y_1 \qquad \Delta''_0 = y_1 - 2y_0 + y_{-1}$$

$$\Delta'_{-1} = y_0 - y_{-1} \qquad \Delta''_1 = y_2 - 2y_1 + y_0$$

that is :

$$\Delta''_0 + \Delta''_1 = \Delta'_2 - \Delta'_{-1}$$

Bessel's formula can thus be written as :

$$f(n) = y_0 + n\,\Delta'_1 - \frac{n(1-n)}{4}(\Delta'_2 - \Delta'_{-1})$$

All the first differences being shown in the APFS, we can then carry out the interpolation without the need to compute the second differences.

Since the interval for x in the APFS is 10 days, and knowing that :

$$\lambda \text{ NAPLES} = 0\text{h}57\text{m} = -0.04\ d$$

the values of $-\dfrac{n(1-n)}{4}$ as a function of d were computed in table 1, d being the difference between the day when the observation was made and the day given in the table.

TABLE 1

d	n_{NA}	$-\dfrac{n(1-n)}{4}$
1	0.096	− 0.022
2	196	39
3	296	52
4	396	60
5	496	62
6	596	60
7	696	53
8	796	40
9	896	23
10	0.996	− 0.001

C) Computation of the $\Delta\alpha$ correction for diurnal aberration

Obviously the local effect of diurnal aberration is not included in the right ascension values of the stars of the APFS.

As is well known of course, there is a difference between the observed star's direction and the initial direction of the light ray, and this is because the speed of celestial bodies in general is not entirely negligible in relation to the velocity of light.

The law of aberration (from the latin *aberratio* = delusion, error) can be simply stated thus : *the apparent direction in which a celestial body is seen, as a result of the effect of light aberration, lies in the plane determined by the true direction and the speed vector of the observer* (the direction of the *apex* of the observer's movement). *This apparent direction is displaced in relation to the true direction towards the speed vector.*

Designating by x and x' the angles formed by respectively the true and the apparent directions with the direction of the observer's speed (positive in the direction of the apex) the $x - x'$ angle of aberration will be given by :

$$x - x' = \frac{v}{c} \sin x = k \sin x$$

where v is the observer's speed, c the velocity of light and k the aberration constant.

In the case of diurnal aberration the observer's speed v will be due to the rotary motion of the Earth. Taking into acount the fact that the earth has the form of a revolving ellipsoid (with Hayford data) a point O on its surface, having geocentric latitude φ' and a vector radius ρ, as a result of earth's rotation will describe during a sidereal day a small circle of radius $r = \rho \cos \varphi'$ in 86164 mean solar time seconds.

Assuming $\rho = a = 6\,378$ km, we shall obtain :

$$v = \frac{2\pi\rho}{86\,164} \cos \varphi' = 0.464 \cos \varphi' \text{ km/sec}$$

from which the following value for the diurnal aberration constant can be taken :

$$k = 206\,265 \frac{v}{c} = 0{,}32'' \cos \varphi' = 0.0213^s \cos \varphi'$$

The angle of diurnal aberration expressed in seconds of time is then given by $0.0213^s \cos \varphi' \sin x$, x being (as we know) the angle formed by the true direction of the celestial body and the speed vector of the observer, that is to say for this case the direction of the *apex of earth's diurnal movement.* Now it is clear that this apex is the *East point.*

Knowing the position of the apex on the celestial sphere, the problem of defining the influence of diurnal aberration on a star's equatorial co-ordinates consists therefore of solving the following elementary problem of spherical astronomy. Assuming that a star's position on the celestial sphere undergoes a displacement from S to S′ such that SS′ can be expressed by SS′ = $k \sin x$, k being a very small constant and x the amplitude of the great circle arc passing through S and S′, and falling between S and its intersection with a fundamental great circle (in our case the celestial equator), determine the variations of the corresponding spherical orthogonal coordinates for S (*).

(*) We must remember, when solving the above-mentioned problem as it applies to our case, that the East point is separated from the zenith by 90°.

For the variation of a star's right ascension due to diurnal aberration we then have : $\Delta\alpha = 0.021^s \cos \varphi' \sec \delta \cos t$.

When the star's declination δ is not too large (and consequently when $\sec \delta$ is not large) we may replace the geocentric latitude φ' by the φ astronomic latitude value.

If we consider the effect of diurnal aberration at the time of a star's transit at the meridian, the fundamental astronomic relation being given by $\Delta\alpha = - \Delta t$, in which t is the star's hour angle, we now observe that at the upper transit (u.t.) we have $t = 0^\circ$, and at the lower transit (l.t.) $t = 180^\circ$, whence designating the star's apparent hour angle by t' we obtain :

$$\left.\begin{array}{ll} \text{u.t.} & t' = 0.0213^s \cos \varphi' \sec \delta \\ \text{l.t.} & t' = 180^\circ + 0.0213^s \cos \varphi' \sec \delta \end{array}\right\} \delta' = \delta$$

It follows that the diurnal aberration does not alter the star's declination at the time of its meridian transit and, because t' is smaller than 0 at the u.t. and larger than 0 at the l.t., the meridian transit is delayed at the u.t. and in advance at the l.t., which is to say that the right ascension appears to be increased at the u.t. and decreased at the l.t.

Thus, if we wish to make this correction to the right ascension computed from the APFS we must use the following formula :

$$\Delta\alpha = \pm\, 0.0213^s \cos \varphi' \sec \delta \left\{\begin{array}{l} \text{u.t.} \\ \text{l.t.} \end{array}\right.$$

If, however, we wish to make this correction to the observed time P_0, we obtain :

$$\Delta t = \pm\, 0.0213^s \cos \varphi' \sec \delta \left\{\begin{array}{l} \text{u.t.} \\ \text{l.t.} \end{array}\right.$$

Now, for a given observatory, the factor $k = 0.0213^s \cos \varphi'$ being constant, we may write the effect of diurnal aberration thus :

$$\Delta t = \mp\, k \sec \delta \left\{\begin{array}{l} \text{u.t.} \\ \text{l.t.} \end{array}\right.$$

which compares with a collimation error.

D) The $\Delta'\alpha$ correction for short period terms of nutation

When, in addition to precession, nutation is also taken into account we say that the positions of the equator, the ecliptic and the equinox are *true* or *instantaneous,* and that *the stars' heliocentric coordinates referred to these positions* are *true.* Thus in order to determine the true coordinates of a star at a given time we must take into account the nutation effect (in longitude and obliquity) on the α and δ coordinates.

Let us examine the question more closely under the more general problem of precession and nutation.

The movement of the vernal point on the ecliptic under the effect of the Sun's and the Moon's attraction (the effect caused by the Moon being predominant) on the bulge of the terrestrial Equator is not uniform, as is easy to understand. In fact, we must first of all take into consideration the fact that the orbits of the Sun (apparent) and the Moon in relation to the Earth are not circular but elliptic, and consequently the distances to the Sun and the Moon from the Earth vary periodically. Also, the Moon moves not in the plane of the ecliptic but in a plane at an angle of about 5°9′ to this plane. To the periodic variation of the distances of the Sun and the Moon there will obviously correspond a variation in the lunisolar disturbing action for which, in the movement of the vernal point, we note the existence of two movements. One is *progressive* (*secular*) and can be expressed by a series of the powers of time t :

$$at + bt^2 + ct^3 + \dots \tag{12}$$

the other is *periodic*, and may be expressed by means of periodic functions of the Sun's longitude Θ and the Moon's longitude Ln :

$$l \sin 2\Theta + m \sin 2\mathrm{Ln} + \dots \tag{13}$$

The progressive movement of the equinox on the fixed ecliptic is called in fact " lunisolar precession in longitude ". The periodic terms constitute the lunisolar nutation in longitude.

Furthermore, from celestial mechanics we know that pole L of the lunar orbit has a periodic motion about the pole of the ecliptic which is completed in 18 2/3 years. This phenomenon introduces two new facts. Firstly, the obliquity ε of the ecliptic cannot remain constant. Secondly, to the periodic terms expressed by the aforementioned relations we must add other periodic terms which are functions of longitude Ω of the lunar orbit's ascending node because a periodic motion on the ecliptic (with the same period of 18 2/3 years) of the Moon's ascending node is correlated to the motion of L in relation to the pole of the ecliptic.

These periodic terms, of the form

$$u \sin \Omega + v \sin 2\Omega \tag{14}$$

constitute the *nutation of the lunar nodes in longitude*. The *total nutation in longitude* $\Delta\Psi$ will thus be obtained from the nutation of the Moon's nodes and from the lunisolar nutation given respectively by formulas (13) and (14).

Hence, substituting numerical values for the constant coefficients in both expressions we obtain :

$$\Delta\psi = -(17.234'' + 0.017''\,\mathrm{T}) \sin \Omega + 0.209'' \sin 2\Omega - 1.272'' \sin 2\Theta_m \tag{15}$$

$$- 0.204'' \sin 2\mathrm{Ln} \dots$$

where the coefficient of sin Ω varies slightly with time T (expressed in centuries from epoch 1900.0 in the formula we have given) and where Θ_m is the mean longitude of the Sun.

As regards the obliquity ε of the ecliptic, its variation $\Delta\varepsilon$ is also of a periodic nature, and its analytic expression is entirely similar to that for

$\Delta\Psi$, except that the sines must be replaced by cosines and the numerical coefficients by other numerical values :

$$\Delta\varepsilon = (9.210'' + 0.001''\,T)\cos\Omega - 0.090''\cos 2\Omega + 0.551''\cos 2\Theta_m \qquad (16)$$

$$+ 0.088''\cos 2Ln \ldots$$

$\Delta\varepsilon$ is called *nutation in obliquity.* The principal term is the term in $\cos\Omega$, and its coefficient 9.210'' is called the *constant of nutation.*

Let us now return to the computation of the true or instantaneous coordinates of the star.

Starting from the coordinates α_0, δ_0 of a star for epoch 1900.0, let us determine the true coordinates for any date $1900 + t + \tau$, where t, expressed in whole tropic years, is the difference between the year in question and 1900, τ being the fraction of the year corresponding to the date. Let α_m, δ_m be the mean coordinates and α_v, δ_v the true coordinates for the star at respectively the epochs $1900 + t$ (beginning of the year) and $1900 + t + \tau$. We already have the identities :

$$\alpha_\nu - \alpha_0 = (\alpha_m - \alpha_0) + (\alpha_\nu - \alpha_m)$$

$$\delta_\nu - \delta_0 = (\delta_m - \delta_0) + (\delta_\nu - \delta_m)$$

which make the determination of the true coordinates at the date $1900 + t + \tau$ depend on the knowledge of the mean coordinates α_0, δ_0 at epoch 1900.0, through the computation of differences. (We shall henceforth consider only the right ascensions).

$$(\alpha_m - \alpha_0) \qquad (17)$$

$$(\alpha_\nu - \alpha_m) \qquad (18)$$

The difference (17) between the mean coordinate at the beginning of $1900 + t$ and at the beginning of 1900 is computed by the process described when we spoke of star catalogues, that is with

$$\alpha_\nu - \alpha_0 = \left(\frac{d\alpha}{dt}\right)_0 t + \frac{1}{2}\left(\frac{d^2\alpha}{dt^2}\right)_0 t^2 + \frac{1}{6}\left(\frac{d^3\alpha}{dt^3}\right)_0 t^3 + \ldots = \text{précession annuelle}$$

$$+ \text{variation séculaire} + \text{troisième terme} + \ldots$$

and furthermore taking the proper motion of the star into account.

The (18) differences, on the contrary, depend on :

a) the effect of precession during time τ;
b) the effect on α of the nutation in longitude $\Delta\Psi$ given by (15);
c) the effect on α of the nutation in obliquity $\Delta\varepsilon$ given by (16);
d) the effect on α of true motion during time τ;

that is :

$$\alpha_\nu - \alpha_m = p_\tau + d\alpha(\psi)\,\Delta\psi + d\alpha(\varepsilon)\,\Delta\varepsilon + \mu_a\,\tau \qquad (19)$$

The first and the fourth terms are deduced from catalogue data. We must, however, dwell on the terms :

$$d\alpha(\psi)\,\Delta\psi + d\alpha(\varepsilon)\,\Delta\varepsilon$$

Let us firstly try to put the expression $d\alpha(\Psi)$ in explicit form. Let us therefore solve the following problem. Given that the longitude λ of a star undergoes a variation $\Delta\lambda$, determine the corresponding variations $d\alpha$ and $d\delta$ of its equatorial coordinates α and δ.

Let us recall that the set of formulas allowing the equatorial coordinates α and δ to be transformed into ecliptic coordinates λ and β is the following :

$$\begin{aligned} \sin\delta &= \sin\beta\cos\varepsilon + \cos\beta\sin\varepsilon\,\sin\lambda \\ \cos\delta\,\cos\alpha &= \cos\beta\cos\lambda \\ \cos\beta\,\sin\lambda &= \sin\delta\,\sin\varepsilon + \cos\delta\,\cos\varepsilon\,\sin\alpha \end{aligned} \tag{20}$$

Considering ε (the obliquity of the ecliptic) and β (the ecliptic latitude) as constants, by differentiating the first relation of the formulas (20) we obtain :

$$\cos\delta\,d\delta = \cos\beta\sin\varepsilon\,\cos\lambda\,d\lambda$$

that is :

$$d\delta = d\lambda\sin\varepsilon\,\frac{\cos\beta\cos\lambda}{\cos\delta} = d\lambda\sin\varepsilon\,\cos\alpha$$

Then, differentiating the second of the formulas in (20) we obtain :

$$\sin\delta\,\cos\alpha\,d\delta + \cos\delta\,\sin\alpha\,d\alpha = \cos\beta\,\sin\lambda\,d\lambda$$

and taking into account the expression just found for $d\delta$:

$$\cos\delta\,\sin\alpha\,d\alpha = d\lambda(\cos\beta\sin\lambda - \sin\delta\,\sin\varepsilon\,\cos^2\alpha)$$

By substituting in this expression the value of $\cos\beta\sin\lambda$ given by the third expression of (20) we shall obtain :

$$\cos\delta\,\sin\alpha\,d\alpha = d\lambda(\sin\delta\,\sin\varepsilon\,\sin^2\alpha + \cos\delta\,\cos\varepsilon\,\sin\alpha)$$

whence :

$$d\alpha = d\lambda(\cos\varepsilon + \sin\varepsilon\sin\alpha\tan\delta)$$

To return to our own case, the variation $d\lambda$ arises from the nutation in longitude $\Delta\Psi$, and consequently we may conclude :

$$d\alpha(\psi) = \cos\varepsilon + \sin\varepsilon\sin\alpha\tan\delta \tag{21}$$

In order to make the expression $d\alpha(\varepsilon)$ explicit, let us solve the following problem. Given that the obliquity of the ecliptic ε increases by $d\varepsilon$, find the corresponding $d\alpha$ and $d\delta$ variations for the equatorial coordinates α and δ.

Differentiating the first expression in (20), and this time considering λ and β as constants, we have :

$$\cos\delta\,d\delta = -\sin\beta\sin\varepsilon\,d\varepsilon + \cos\beta\,\cos\varepsilon\,\sin\lambda\,d\varepsilon$$

Bearing in mind that, by formulas for transformation of coordinates, we also have :

$$\cos\delta\,\sin\alpha = -\sin\beta\,\sin\varepsilon + \cos\beta\cos\varepsilon\sin\lambda$$

we shall immediately obtain : $d\delta = d\varepsilon\,\sin\alpha$.

Differentiating the second expression of (20), and substituting therein the value for $d\delta$ just found, we finally have : $d\alpha = - d\varepsilon \cos\alpha \tan\delta$, which for our case becomes :

$$d\alpha(\varepsilon) = -\cos\alpha \, \tan\delta \tag{22}$$

Thus, we may conclude that the total amount of right ascension nutation is : $\Delta'\alpha = d\alpha(\Psi)\,\Delta\Psi + d\alpha(\varepsilon)\,\Delta\varepsilon$.

However a part of this quantity has already been taken into account in the places computed in the A P F S, but the short period terms given by the small terms in the series (15) and (16) cannot be taken into account in Almanacs giving places for every ten days, the reason being that as they are very small numerically they would not be supplied by the interpolation.

For highly accurate observations such as ours we cannot however neglect these short period terms in the right ascension computation : they must accordingly be taken into account.

The A P F S in fact supplies the values for $d\alpha(\Psi)$ and $d\alpha(\varepsilon)$ expressed in seconds of time for each star — that is by dividing the expressions (21) and (22) by 15, whereas a special table — Table I, Short-period terms of nutation — supplies the values of $d\Psi$ and $d\varepsilon$, i.e. that part of the $\Delta\Psi$ and $\Delta\varepsilon$ terms which cannot be included in the computation of the Apparent places of 10 day stars.

The correction to be made is thus :

$$\Delta'\alpha = d\alpha(\psi)\,d\psi + d\alpha(\varepsilon)\,d\varepsilon$$

It is not at all easy to interpolate these magnitudes. The value of $\Delta'\alpha$ for two consecutive days (to include the day of observation) must be computed, then the interpolation between these values must be made for the time of the transit observed, and this can be estimated (in fractions of a day) with the help of the expression : α — *sidereal time at* 0 *hours* that can be immediately deduced from the U.T. column of the A P F S in view of the values of Italian longitudes.

In this we have computed (or rather have ascertained the way in which we must compute) the Mayer formula term: $\alpha + \Delta\alpha + \Delta'\alpha$ expressed in sidereal time.

However, since the observation is made with a quartz clock regulated on mean time, $\alpha + \Delta\alpha + \Delta'\alpha$ must also be expressed in the same unit. Obviously, for transforming sidereal time into local mean time, the value of the longitude of the place of observation must be precisely known, and it is this value which we have set ourselves to determine.

Let us assume then that we know the approximate value for the longitude of the place of observation, which value we shall call the *conventional longitude* λ_c. The exact longitude λ_v will be given by : $\lambda_v = \lambda_c \pm \Delta\lambda$. Designating the sidereal time at Greenwich for a given day at 0 hours T.U. by T_0h_G, and the correction to be made for transforming an interval of sidereal time into a mean time interval by $\Delta\theta$ we shall have :

$$AR_c = \alpha + \Delta\alpha + \Delta'\alpha + \lambda_c - T_0h_G - \Delta\theta$$

where AR_c is the star's computed right ascension, corrected for the influence of short-period terms of nutation and diurnal aberration, expressed in mean time referred to the conventional meridian λ_c.

As we have seen, it is easy to interpolate with the A P F S and to compute AR_c, but on account of the amount of work which a longitude determination involves we have preferred to carry out the AR_c computation for each day of the years 1967-1968 by an electronic computer. The programme adopted was the EUDOSSO, developed by Professor Aldo KRANIC of Aquila University.

The following are the essential data.

1) A card giving the initial day (IGI), the initial month (IMI), the initial year (IAI), the final day (IGF), the final month (IMF) and the final year (IAF). Then the conventional longitude of the observatory, expressed in hours, minutes and seconds (IØLØN, IMLØN, SECLØN), the latitude expressed in degrees, minutes and seconds of arc (IGLAT, IPLAT, SECLAT) and the observatory's name — over a maximum of 16 columns in all.

2) A card giving the star data (EUCLID programme).

EUDOSSO then supplies the following final values :

1st column : The date;
2nd column : The corresponding day (DJ);
3rd column : AR_c (at transit) for the conventional longitude λ_c;
4th column : The declination.

Data			D.J.	Passaggio			Declinazione		
1	1	1967	39492.139	15^H	20^M	$59^S.576$	42°	8′	59″.75
2	1	1967	39493.137	15	17	3.645	42	8	59.58
3	1	1967	39494.134	15	13	7.716	42	8	59.40
4	1	1967	39495.131	15	9	11.789	42	8	59.23
5	1	1967	39496.128	15	5	15.862	42	8	59.08
6	1	1967	39497.125	15	1	19.936	42	8	58.95
7	1	1967	39498.123	14	57	24.010	42	8	58.83
8	1	1967	39499.120	14	53	28.082	42	8	58.71
9	1	1967	39500.118	14	49	32.154	42	8	58.59
10	1	1967	39501.115	14	45	36.224	42	8	58.45
11	1	1967	39502.112	14	41	40.295	42	8	58.30
12	1	1967	39503.109	14	37	44.365	42	8	58.13
13	1	1967	39504.106	14	33	48.436	42	8	57.93
14	1	1967	39505.104	14	29	52.507	42	8	57.72
15	1	1967	39506.101	14	25	56.580	42	8	57.50
16	1	1967	39507.098	14	22	0.655	42	8	57.27
17	1	1967	39508.096	14	18	4.730	42	8	57.04
18	1	1967	39509.093	14	14	8.807	42	8	56.82
19	1	1967	39510.090	14	10	12.884	42	8	56.62
20	1	1967	39511.087	14	6	16.962	42	8	56.42
21	1	1967	39512.085	14	2	21.041	42	8	56.24
22	1	1967	39513.082	13	58	25.119	42	8	56.07
23	1	1967	39514.079	13	54	29.196	42	8	55.91
24	1	1967	39515.077	13	50	33.272	42	8	55.74
25	1	1967	39516.074	13	46	37.348	42	8	55.57

Data			D.J.	Passaggio			Declinazione		
26	1	1967	39517.071	13	42	41.423	42	8	55.37
27	1	1967	39518.068	13	38	45.498	42	8	55.15
28	1	1967	39519.065	13	34	49.574	42	8	54.91
29	1	1967	39520.063	13	30	53.652	42	8	54.65
30	1	1967	39521.060	13	26	57.732	42	8	54.39
31	1	1967	39522.058	13	23	1.813	42	8	54.14
1	2	1967	39523.055	13	19	5.896	42	8	53.91
2	2	1967	39524.052	13	15	9.980	42	8	53.69
3	2	1967	39525.049	13	11	14.064	42	8	53.49
4	2	1967	39526.046	13	7	18.147	42	8	53.30
5	2	1967	39527.044	13	3	22.229	42	8	53.11
6	2	1967	39528.041	12	59	26.311	42	8	52.91
7	2	1967	39529.038	12	55	30.393	42	8	52.70
8	2	1967	39530.036	12	51	34.474	42	8	52.47
9	2	1967	39531.033	12	47	38.556	42	8	52.22
10	2	1967	39532.030	12	43	42.639	42	8	51.96
11	2	1967	39533.027	12	39	46.723	42	8	51.68
12	2	1967	39534.024	12	35	50.808	42	8	51.40
13	2	1967	39535.022	12	31	54.895	42	8	51.13
14	2	1967	39536.019	12	27	58.984	42	8	50.85
15	2	1967	39537.017	12	24	3.074	42	8	50.59
16	2	1967	39538.014	12	20	7.164	42	8	50.35
17	2	1967	39539.011	12	16	11.255	42	8	50.12
18	2	1967	39540.008	12	12	15.346	42	8	49.90
19	2	1967	39541.005	12	8	19.437	42	8	49.70

The computation example given relates to star No. 869 in Fk 4.

Computation of a star's transit for every day of the year takes 10 seconds.

Further details may be obtained from Bulletins Nos. 1 and 5 of the Ufficio Programmi e dati de l'Aquila.

3. — THE EQUATION FOR LONGITUDE

We must first of all point out that in a highly accurate longitude determination the set of direct observation instruments must always be coupled to a radio receiver set in order to pick up the international time signals. This reception operation, repeated several times a day, will allow us to determine independently of the observation the ΔQ value of the quartz clock correction at the time a given star is observed, if we make exception of small corrections in reception which can be made later on. The correction can therefore be made directly to the observed value P_0 and we shall then be able to consider that this ΔQ value has already been corrected, although for purposes of simplicity we shall continue to denote it by P_0.

The transit instrument observation will therefore supply us with the following equation :

$$\Delta\lambda = AR_c - (P_0 + \Sigma) \tag{23}$$

We should point out that the values for the small terms of the variation in Earth's rotation and for polar displacement are still included in the $\Delta\lambda$ value. These corrections will not be made until later on.

The computation could be terminated at this stage, i.e. the observed time P_0 has been corrected for the Σ quantity. However, proceeding in this way, the accuracy obtained for the final result of $\Delta\lambda$ is of the order of $\pm$ 0.025s, and therefore insufficient for highly accurate determinations.

Let us therefore see how the successive corrections are determined.

A) Determination of Σ_1

It is usual, for a first reduction of the observations, to take the value of the mean inclination i_m of the night's observations as the value for the inclination of the instrument's axis of rotation.

Every T value for each star in the programme will therefore be computed, introducing an error $i - i_m$, where i is the inclination expressed in seconds of time for each star observed.

Let us seek the correction to be made to this error :

$$T = P_0 + (i_m + \Delta i)(\cos\varphi \pm \sin\varphi \tan\delta) + a(\sin\varphi \mp \cos\varphi \tan\delta) + c \sec\delta$$

$$= P_0 + \Sigma + \Delta i(\cos\varphi + \sin\varphi \tan\delta)$$

Consequently the correcting term

$$\Sigma_1 = \Delta i(\cos\varphi \pm \sin\varphi \tan\delta) \quad \begin{matrix}\text{u.t.}\\ \text{l.t.}\end{matrix}$$

i.e.

$$\Sigma_1 = \Delta i \sin\varphi(\text{cotan}\,\varphi \pm \tan\delta) \quad \begin{matrix}\text{u.t.}\\ \text{l.t.}\end{matrix}$$

for each observed star must be added to the equation $\Delta\lambda = AR_c - (P_0 + \Sigma)$.

B) Determination of Σ_2

Further, to obtain the value of i expressed in seconds of time we must first know the value of the level's sensitivity σ with $i^s = i^p \sigma^s$, where i^p is the value of the inclination of the instrument's axis of rotation expressed in graduations of the level.

However, as we saw in the first of this series of articles, the value of σ is a function of temperature t°. Consequently if, during reduction of an observation carried out at a certain temperature t, we take a fixed value of σ_f we then introduce the error $\sigma_t - \sigma_f$ into the result.

To take this influence into account a new correction, Σ_2, must be made.

Denoting the r-th star's inclination value, expressed in graduations of the level, by i_r^p, the correction to be made will be :

$$\Sigma_2 = i_r^p \, \Delta\, \delta \sin \varphi (\cotan \varphi \pm \tan \delta_r) \begin{matrix} \text{u.t.} \\ \text{l.t.} \end{matrix}$$

C) **Determination of Σ_3**

As with the sensitivity of the level, the value of the micrometer's rate varies in function of temperature t. This variation can be expressed for our purposes by simplifying the general formula adopted by the International Latitude Service for reduction of observations, i.e. $R_t = R_0 \, \alpha \, (t - t_0)$ where R_0 is the value of the micrometer's rate at the initial temperature t_0.

This variation has an influence on the value of the collimation c which in the case of a quick reversing instrument includes the half-width s of the impersonal micrometer's contacts and the effect due to the delay caused by the play p of the micrometer screw.

Obviously, the correction is given by $\Delta R = R_t - R_f$ where R_f is the value fixed at the time of the first reduction. As this variation influences only the collimation value, then denoting the collimation expressed in graduations of the impersonal micrometer drum by $c^p = s + p$, we obtain :

$$c^p (R_f + \Delta R) \sec \delta = c^p R_f \sec \delta + c^p \Delta R \sec \delta = c \sec \delta + c^p \Delta R \sec \delta$$

The correction to be made is therefore :

$$\Sigma_3 = c^p \Delta R \sec \delta$$

D) **Determination of Σ_4**

The other correction to be made is called the *residual collimation* correction. Included in the error are all the effects caused by the lateral flexions of the instrument (in function of the observed star's zenithal distance) which make the collimation vary by a quantity Δc when we alter the position of the axis of rotation during the course of the observation from position 1 to the opposite position 2. This collimation Δc is computed by determining the means P_1 and P_2 for the pulses given in the two positions of the eyepiece. The collimation of the middle observing thread respecting the r-th star in the programme will be given by :

$$\frac{(P_2 - P_1)_r}{2 \sec \delta_r} = c_r$$

The mean c_m of these c_r values for each star in the observed programme will give us the mean collimation for a night's programme. The deviations $\Delta c = c_r - c_m$ will represent the corrections to be made to each observed transit P_0. In other words, for each star we shall have the correction $\Sigma_4 = \Delta c \sec \delta$.

E) Determination of Σ_5

Finally, the last correction to be made concerns the error produced by the observer's personal equation.

In the observations carried out with the Bamberg type transit instrument the personal equation arises chiefly as a result of motion. That is, the observer when following the star tends to keep the impersonal micrometer thread continually either before or behind the centre of the star image.

This distance d depends principally on the observer and the instrument used. Denoting this distance, expressed in seconds of time, by e, and relating to an equatorial star observation, for a star whose declination δ is not zero we shall have : $d = \pm\, e \sec \delta$, adopting the apparent motion of the star in the ocular field as the positive direction. This will be $+\, e \sec \delta$ if the observer tends to put the thread before the star image, and $-\, e \sec \delta$ if he puts it after.

To find this correction we shall proceed as follows. Let us write the longitude equation thus :

$$AR_c - P_0 - c \sec \delta - \Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4 - \Delta\lambda - m = n \tan \delta + e \sec \delta$$

The quantity

$$AR_c - P_0 - c \sec \delta - \Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4 = H$$

is a known term, whereas the quantity $-\Delta\lambda - m = K$ is constant for a series of determinations made during one night.

Thus the preceding expression may be written

$$H + K = e \sec \delta \pm n \tan \delta \begin{cases} \text{u.t.} \\ \text{l.t.} \end{cases} \tag{24}$$

Let us now consider separately the equations of expression (24) which concern observations of h stars non-circumpolar at the place of observation and of stars circumpolar at both the upper and the lower transit :

$$H_h + K = e \sec \delta_h + n \tan \delta_h$$

$$H_{ut} + K = e \sec \delta_{ut} + n \tan \delta_{ut}$$

$$H_{lt} + K = e \sec \delta_{lt} - n \tan \delta_{lt}$$

Combining the first equation with respectively the second and then the third expression we obtain :

$$n = \frac{H_{ut} - H_h}{\tan \delta_{ut} - \tan \delta_h} - e\, \frac{\sec \delta_{ut} - \sec \delta_h}{\tan \delta_{ut} - \tan \delta_h} = n_{\text{u.t.}} - eS$$

$$n = \frac{H_h - H_{lt}}{\tan \delta_{lt} - \tan \delta_h} + e\, \frac{\sec \delta_{lt} - \sec \delta_h}{\tan \delta_{lt} + \tan \delta_h} = n_{\text{l.t.}} + e\mathrm{l}$$

where $n_{u.t.}$ and $n_{l.t.}$ are n values determined by stars circumpolar at both the upper and the lower transits.

The values of S and I for the Naples programme combined with pairs of upper and lower transits for circumpolar stars taken from the FK 4 Catalogue are the following : $S = 0.89$ and $I = 0.73$, therefore :

$$n = \frac{n_{ut} + n_{lt}}{2} - \frac{S - I}{2} e = \bar{n} - 0.08\, e$$

$$n_{ut} - n_{lt} = e(S + I) = 1.62\, e \qquad (25)$$

We can see from these expressions that the personal equation influences the determination of n by a maximum of some hundredths of a second of time, whereas there is a systematic difference between the values of n (*and consequently also of the instrument's azimuths*) that are respectively determined by the observation of circumpolar stars at their u.t. and their l.t.

From (25) we take the value of e :

$$e = \frac{n_{ut} - n_{lt}}{S + I} = \frac{n_{ut} - n_{lt}}{1.62}$$

for the Naples Astronomic Observatory.

The correction for the error in the observed time P_0 produced by the personal equation is therefore :

$$\Sigma_5 = \frac{n_{ut} - n_{lt}}{S + I} \sec \delta$$

In practice this correction can be made to the final $\Delta\lambda$, using the formula :

$$\Sigma_5 = e\, \overline{\sec \delta}$$

where $\overline{\sec \delta}$ is the mean value of the $\sec \delta_r$ of the r stars in the observed programme.

The rigorous expression for the Σ_5 correction is the following :

$$\Sigma_5 = e\, \overline{\sec \delta} - \frac{S - I}{2}\left(\frac{\Sigma_r \tan \delta_r}{r} - \tan \varphi\right) e$$

where the subscript r refers to stars in the observed programme. (See N. Stoyko : *Sur la mesure du temps et les problèmes qui s'y rattachent*, 1931).

We have in no way neglected the second term, for in our programme we have proceeded in such a way that :

$$\frac{\Sigma_r \tan \delta_r}{r} - \tan \varphi = 0$$

as we shall see.

In conclusion, the equation for determining a place's longitude with a transit instrument is finally the following :

$$\Delta\lambda + \Sigma_5 = AR_c - (P_0 + \Sigma + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4)$$

with
$$\Sigma = m \pm n \tan \delta_r + c \sec \delta_r \left\{ \begin{matrix} \text{u.t.} \\ \text{l.t.} \end{matrix} \right.$$

$$\Sigma_1 = \Delta i_r \sin \varphi (\text{cotan}\, \varphi \pm \tan \delta_r) \left\{ \begin{matrix} \text{u.t.} \\ \text{l.t.} \end{matrix} \right.$$

$$\Sigma_2 = i_r^p \Delta \sigma \sin \varphi (\text{cotan}\, \varphi \pm \tan \delta_r) \left\{ \begin{matrix} \text{u.t.} \\ \text{l.t.} \end{matrix} \right.$$

$$\Sigma_3 = c^p \Delta \mathrm{R} \sec \delta_r$$

$$\Sigma_4 = \Delta c \sec \delta_r$$

$$\Sigma_5 = e \overline{\sec \delta}$$

If, before starting the campaign, we have made an accurate preliminary study of the instrument we already know the equations for the level's sensitivity and the play of the impersonal micrometer in function of the temperature, as well as the expression for residual collimation in function of zenithal distance. The first reduction can therefore be made taking the Σ, Σ_1, Σ_2, Σ_3, Σ_4 corrections into account, whilst the Σ_5 correction must be determined during the course of the longitude campaign.

4. — COMPUTATION OF P_0, THE TIME OF THE OBSERVED TRANSIT

As we have seen, the whole longitude determination is fundamentally based on the value of P_0, which is the only value experimentally obtained.

The P_0 value for the observed transit time is deduced from observations with an impersonal micrometer linked to a clock, which is in turn quartz-controlled.

The observer follows the star with the movable thread, holding the thread continually on the star over that part of the eyepiece field situated before (in the direction of the star's relative motion in the eyepiece field) the central fixed thread of the eyepiece cross-wires, and he then reverses the eyepiece on its supports, and continues to take a sight on the star (which at that particular time will have a reverse relative motion) on the same eyepiece track as the one used for the first part of the observation.

Let us now see what errors can be committed during this basic measuring operation.

We know that the nearer the star is to the pole the greater is the error in the estimation of the transit time P_0. From the analytic point of view, since this error depends on countless others, it has not yet been possible to lay down a rigorous law for the phenomenon but, after studying a large number of series of star transit observations we have realized that this phenomenon depends on the rate at which the star crosses the eyepiece field. Consequently, its best representation is given by the following expression : $\mu = \varepsilon \sec \delta$, where ε is the error in the estimate of an equatorial star's transit ($\delta = 0°$).

There is much literature on the subject, and for the bibliography the reader is referred to the present author's article published in 1963 [10].

In an unpublished work I have assembled the ε values deduced from about a hundred articles on the subject, and from the most modern and fullest sets of observations (more than 1 000 transits). I have computed the weighted mean, and have obtained the value of : $\varepsilon = \pm 0.03s$ which is already fairly well known, and slightly different from the value determined by observations carried out with older instruments ($\varepsilon = \pm 0.025s$), and furthermore confirmed by computations of the observations of the last World Longitude Campaign during the I.G.Y. (1957-58) conducted with ultra-modern means and techniques.

Table 2 gives the values of $\mu = f(\delta)$

TABLE 2

δ	μ	δ	μ	δ	μ
0°	± 0.025s	65°	± 0.059s	82°	± 0.180s
10	25	70	73	83	205
20	27	72	81	84	239
30	29	74	91	85	287
40	33	76	103	86	358
50	39	78	120	87	478
55	44	80	144	88	0.716s
60	± 0.050s	81	± 0.160s	89	± 1.432s

As can be seen, from 75° in declination the μ value increases to the point that it concerns a second of time. From this we immediately deduce that for the U.T. observation it is generally necessary to exclude stars of large declination [11].

However we must note that it is here a question of averaged sums based on a very large number of values. The individual error can sometimes reach much larger values, even with very experienced observers. (See, for example, the lengthy series of meridian observations at Greenwich or at Paris).

Determination of P_0 is consequently the most delicate operation of the whole observation, and the greatest error in determining longitude belongs exclusively to this particular determination.

However, this is most generally forgotten, and other errors — such as those of the Catalogues — are attentively studied. In the introduction to F K 4 [7] it is the mean error of the system which is in fact given. We wish to mention here, by way of comparison, the error relating to right ascensions (m_α = the deviation error arising from the catalogues used and from the instrument systems; m_μ = the probable error in proper motion) in order to give an idea of its magnitude, and also to be able make an easy comparison with the observational error given in table 2.

TABLE 3

δ	$\varepsilon_{\alpha} \cos \delta$	$\varepsilon_{\mu} \cos \delta$
> + 80°	+ 0.001s	+ 0.010s
70	2	8
50	3	10
20	2	8
0	1	6
− 20	2	12

As we can see, we must therefore pay the greatest attention to the P_0 determination, which must be made according to the recommendations contained in our article on the study of an impersonal micrometer.

When making our star observations we must take the mathematic and statistical principles governing the law of probabilities into account (as well as the considerations of the above mentioned article as regards the number of signals for each micrometer drum rotation) — i.e. take carefully into account the Bernouilli theorem on repeated experiments — and we should observe the star by following it for at least six revolutions of the micrometer drum at the time of the first part of the measurement, and during the same number of revolutions after reversing the instrument's axis.

For the case of our micrometer, each P_0 value for a star is deduced in this way from 156 individual signals from which are deduced 78 values for the star's transit at the thread without collimation.

Before proceeding to the final computation we have to analyse these 78 signals, hence to determine their mean error, and to see if this value is higher than the one given in table 2 for the declination of the observed star. If this value is higher the observation of this star has not been satisfactorily carried out, and the introduction of this higher value into the final computation would be a great blunder. The error must therefore be immediately eliminated. If, on the contrary, the value is either equal or less we must compute the *margin of tolerance* for the error which is, as we know, three times the absolute value of the mean error. All the individual signals (in this case there will be few if any) having deviations from this mean value which are higher, in absolute value, than this margin of tolerance, will be eliminated.

After this analysis the mean is re-computed, and we shall thus obtain a more reliable observed value of P_0.

We should then note that this error in P_0 will be dealt with and reduced as an accidental error in the final computation for the entire observation, but only provided that the series of stars observed is a long one. In practice, 10 stars with declinations uniformly distributed over each observational hour will be sufficient. They will later be reduced in the aggregate. Any pairing of two or more stars observed during an hour's session in order to deduce the final $\Delta\lambda$ is contrary to all the principles of the theory of errors.

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