CAMPAIGN FOR DETERMINING THE LONGITUDE OF THE FUNDAMENTAL POINT OF THE ASTRONOMIC OBSERVATORY IN NAPLES

IV. - THE OPTIMUM CONDITIONS FOR SOLVING THE EQUATION FOR LONGITUDE

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1. THE BARYCENTRIC EQUATION

In order to simplify the exposition we shall consider only the case of stars observed at their upper meridian transit. The equation for solving our problem — that is the determination of $\Delta\lambda$ — can be written :

$$AR_c - [P_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + i\,(\cos\varphi + \sin\varphi \tan\delta) + c \sec\delta]$$
$$= \Delta\lambda + a\,(\cos\varphi \tan\delta - \sin\varphi)$$

Denoting the left member of this equation by N_r (since for each star observed this contains only known quantities), and the number of stars observed by *n*, we shall obtain the following system of condition equations :

$$\left.\begin{array}{c} \Delta\lambda + a \cos\varphi\,(\tan\delta_r - \tan\varphi) = N_r \\ r = 1, 2, 3, \ldots, n \end{array}\right\} \qquad (1)$$

The system supplies an immediate solution for the unknown in the following two cases :

$$a = 0 \qquad \tan\delta_r - \tan\varphi \neq 0$$

$$\tan\delta_r = \tan\varphi \qquad a \neq 0$$

The first is a purely theoretical solution since the azimuth of a transit instrument is never zero, that is the horizontal axis will never coincide exactly with the East-West direction.

On the other hand, the second solution allows us to state an important property of system (1) : namely, *assuming the other instrumental errors to have been perfectly corrected, and that there remains only the azimuthal* *error, we may determine the value for $\Delta\lambda$ independently of the azimuth solely by means of solving the following system :*

$$\left.\begin{aligned} N_r &= \Delta\lambda + a \cos\varphi\,(\tan\delta_r - \tan\varphi) \\ \tan\delta_r &= \tan\varphi \end{aligned}\right\} \qquad (2)$$

Geometrically speaking, the first equation defines a straight line of angular coefficient $a' = a \cos \varphi$, whereas the second one defines a straight line parallel to the y-axis and whose equation is $x = \tan \varphi$.

Logically, there is only one solution to the system (2) : this is the point where these two straight lines intersect. The ordinate of this point directly supplies the value of the unknown $\Delta\lambda$.

These *considerations suggest that the simplest way to solve the* system (2) is the graphic method : i.e. to plot the observed stars with tan δ_r as abscissae, and as ordinates the observational values corrected in the way indicated, i.e. the N_r values : then to join these points to form a straight line, and to read the ordinate of the point on this line which has the value of tan φ as abscissae.

This is the simplest method, but from the analytical viewpoint it is open to many valid criticisms.

The first of these — and the most serious — is the one arising from the fact that the N_r quantities are essentially dependent on experimentally derived data, i.e. from the observation of stars at their meridian transit. The points with coordinates tan δ_r, N_r do not in reality make up a well defined straight line, and it is not at all easy to link them all up. This being so, in order to obtain a result accurate to the thousandth of a second, it would be necessary to employ the successive approximation method of computation, that is to determine a first value for the azimuth by means of a graph, then to substitute this value in the following relations :

$$m = i \cos\varphi + a \sin\varphi$$
$$n = i \sin\varphi - a \cos\varphi$$

which will supply a new set of values for m and n. Then we have to apply to P_0 the entire and re-computed correction Σ, and thus we finally obtain new values for N_r. This operation has to be repeated until we find a nil value for the angular coefficient of the straight line defined by the points with coordinates tan δ_r, N_r, and find $\Delta\lambda$ from the mean of these N_r values supplied by the last approximation.

By proceeding in this way we see that there is then no benefit in employing this graphical method which instead of reducing computation complicates it considerably. It is therefore necessary to find another means of retaining the graphical method, which is always preferable because it gives an instant and complete picture of the way in which the observation has been carried out, i.e. of the observer's accurateness.

The reasoning which we shall now make has two advantages. It is of elementary simplicity, and it adheres strictly to both the logic and the necessities of mathematics.

Let us firstly point out that the straight line joining the points with coordinates $\tan \delta_r$, N_r must necessarily pass through the barycentre B of these points, whose coordinates are

$$\frac{\Sigma_r \tan \delta_r}{n} \quad \text{and} \quad \frac{\Sigma_r N_r}{n} \quad (r = 1, 2, \ldots, n)$$

where n is the number of stars observed. The only uncertainty that the graphical method involves lies exclusively in the accurate determination of the angular coefficient of the straight line, inasmuch as this straight line can pivot, on account of the scattering of the points, by small angles that vary according to the scattering, the pivot point being the barycentre B of the observations.

Hence by taking into account the barycentre B we eliminate with certitude the errors due to the translations of this straight line.

Ordinarily this scattering makes it difficult to distinguish accurately small variations in the true value of the angular coefficient which, as we remember, allows us to determine the value of the instrument's azimuth.

It is easy to see that the further the barycentre B of the observations is from the straight line $x = \tan \varphi$ the greater will be the influence of the uncertainties in the determination of the angular coefficient of the straight line on the value of $\Delta\lambda$. In the ideal case where this barycentre is situated on the straight line $x = \tan \varphi$ the uncertainty in the determination of the angular coefficient no longer affects the determination of $\Delta\lambda$. On the contrary, in this case it would no longer be necessary to plot a graph because $\Delta\lambda$ would be supplied by the ordinate of the barycentre

$$\Delta\lambda = y_B = \frac{\Sigma N_r}{n}$$

and consequently entirely independent of the instrument's azimuth value.

Before being able to adopt this principle we must, however, first demonstrate rigorously the mathematical validity of what we have just deduced intuitively. Furthermore, while developing our reasoning we have tacitly assumed that the azimuth value remains constant during the observation. We must accordingly study this last question from the technical point of view so as to ascertain that during the observation the instrument's azimuth undergoes no variation, or at least that any variation is so minute that it is smaller than the observational errors.

2. THE REDUCED EQUATIONS METHOD

a) The most generally used method for computing the unknowns in a system of equations resulting from observations is the Gauss method based on the least squares principle. This method makes it possible to pass from one system of given equations to another system capable of supplying the most probable values for the unknowns, together with their respective weights. Indeed the greatest mathematical probability of the values of unknowns is always relative to a given system of condition equations and to the results of observations. For this reason these values are dependent on two factors which can be mathematically expressed, and whose determination will give us an idea of the accuracy of the values obtained.

These factors are :

α) *The weights* : these directly affect the equations *inasmuch as these weighted equations give better values for the unknowns*. The weight therefore arises from the coefficients linking the unknowns to the observations.

β) *The mean error per unit* : this affects only the observations and is inherent in these observations.

However, it can be seen that we cannot consider the unknowns as completely determined unless their weights are known.

The solution of a system of condition equations by the Gauss method is numerically somewhat laborious, and often *ad hoc* methods for the determination of the unknowns are preferred. However, with such simplifications the results obtained cannot often be considered as the most probable values for the unknowns.

Frequently in practical astronomy the condition equations contain a certain unknown whose coefficient is 1 in all the equations. In such cases instead of writing the normal equations directly we can previously eliminate the unknowns with coefficient 1 by subtracting from all the condition equations their mean equation, and then computing the normal equations from the set of equations thus obtained. These last we call *reduced equations*.

This method permits a substantial reduction in the numerical work of solving the condition equations, and allows the mathematical determination of the physical conditions essential to the actual determination in order to obtain the best possible results.

Indeed, if the condition equations contain m unknowns the number of coefficients and of known terms to be computed in order to write the normal equations will amount to :

$$N_m = \frac{m(m+3)}{2}$$

whereas the number of coefficients and known terms for the normal equations from the set of reduced equations will be :

$$N_{m-1} = \frac{(m-1)(m+2)}{2}$$

In this way we may eliminate the computation of

$$N_m - N_{m-1} = m + 1$$

coefficients.

For example, for the $(m=2)$ system of condition equations

$$a_r x_1 + b_r x_2 = l_r \qquad (r = 1, 2, 3, \ldots, n)$$

the coefficients and the known terms of the normal equations will be :

$$[aa], [ab], [al], [bb], [bl]$$

that is :

$$N_2 = \frac{m\,(m + 3)}{2} = 5$$

If we can eliminate an unknown by the reduced equations method we shall have to compute only $N_1 = 2$ coefficients, and thus we shall eliminate the computation of

$$N_2 - N_1 = m + 1 = 3$$

coefficients.

We shall demonstrate that by using the "reduced" method for the particular case of equations each having a coefficient of 1 for one and the same unknown the same conclusions may be reached and with the same mathematic rigour as by the Gauss method.

Finally we shall show how to deduce the rules to be followed when choosing the stars to be observed, in order to obtain the best possible system of condition equations.

b) G. Zappa (*) has shown that the following theorem is general. "In condition equations when there is a linear combination of the coefficients of the unknowns that has a constant value for all the equations then the mean equation can be subtracted from each condition equation".

In his article Zappa does not speak of weights. This, however, is justified, since in the particular application with which he is dealing — the reduction of photographic plates — the computation of weights is of minor importance.

Let us now consider the system of n condition equations with m unknowns $(n > m)$:

$$x_o + a_r x_1 + b_r x_2 + \cdots + p_r x_{m-1} = l_r \qquad (r = 1\,, 2\,, \ldots, n) \tag{3}$$

the mean equation will be :

$$x_o + \frac{[a]}{n} x_1 + \frac{[b]}{n} x_2 + \cdots + \frac{[p]}{n} x_{m-1} = \frac{[l]}{n} \tag{4}$$

Subtracting this mean equation from each of the equations (3), and putting :

$$\left.\begin{array}{l} a_r - \dfrac{[a]}{n} = \alpha_r \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ p_r - \dfrac{[p]}{n} = \pi_r \\ l_r - \dfrac{[l]}{n} = \lambda_r \end{array}\right\} \qquad r = 1\,, 2\,, 3\,, \ldots, n \tag{5}$$

we obtain the following reduced equations :

$$\alpha_r x_1 + \beta_r x_2 + \cdots + \pi_r x_{m-1} = \lambda_r \tag{6}$$

(*) G. Zappa : Il calcolo delle costanti delle lastre fotografiche. *Memorie Spettr. Italiani* XL, page 129, Catania 1911.

The normal equations of the (3) and the (6) condition equations will respectively be :

$$\left.\begin{array}{l} n\ x_0 + [a]\ x_1 + [b]\ x_2 + \cdots + [p]\ x_{m-1} = [l] \\ [a]\,x_0 + [aa]\,x_1 + [ab]\,x_2 + \cdots + [ap]\,x_{m-1} = [al] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [p]\,x_0 + [ap]\,x_1 + [bp]\,x_2 + \cdots + [pp]\,x_{m-1} = [pl] \end{array}\right\} \quad (7)$$

and

$$\left.\begin{array}{l} [\alpha\alpha]\,x_1 + [\alpha\beta]\,x_2 + \cdots + [\alpha\pi]\,x_{m-1} = [\alpha\lambda] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [\alpha\pi]\,x_1 + [\beta\pi]\,x_2 + \cdots + [\pi\pi]\,x_{m-1} = [\pi\lambda] \end{array}\right\} \quad (8)$$

Designating by D the determinant of system (7), by D_r that obtained by replacing in D the coefficients of x_r by the known terms, and by Δ and Δx_r the same algorithms for system (8) we have :

$$x_r = \frac{Dx_r}{D} \quad ; \quad x_r = \frac{\Delta x_r}{\Delta} \quad (r = 1, 2, \ldots, m-1) \quad (9)$$

We have to show that these two expressions are identical. Indeed from the relations in (5) we have :

$$\left.\begin{array}{l} [aa] = [\alpha\alpha] + \dfrac{[a]\,[a]}{n} \\ [ab] = [\alpha\beta] + \dfrac{[a]\,[b]}{n} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad (10)$$

Before going further, we shall show that the first of the equalities in (10) holds good.

The first relation in (5) gives us :

$$[\alpha\alpha] = \left[\left(a_r - \frac{[a]}{n}\right)^2\right]$$

and by developing

$$\left[\left(a_r - \frac{[a]}{n}\right)^2\right]$$

we obtain m expressions of the following type :

$$a_1^2 - 2\frac{[a]}{n}\ a_1 + \frac{[a]}{n}\ \frac{[a]}{n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Summing these equations column by column, we shall have :

$$[aa] - 2\frac{[a][a]}{n} + n\frac{[a][a]}{nn}$$

and consequently the first relation in (10).

Substituting the relation (10) into the determinant D, we have

$$D = n \begin{vmatrix} 1 & 0 + \frac{[a]}{n} & \dots & 0 + \frac{[p]}{n} \\ [a] & [\alpha\alpha] + \frac{[a][a]}{n} & \dots & [\alpha\pi] + \frac{[a][p]}{n} \\ \dots & \dots\dots\dots & \dots & \dots\dots\dots \\ \dots & \dots\dots\dots & \dots & \dots\dots\dots \\ [p] & [\alpha\pi] + \frac{[a][p]}{n} & \dots & [\pi\pi] + \frac{[p][p]}{n} \end{vmatrix}$$

a determinant which can be expanded into 2^{m-1} determinants of the same order, $(2^{m-1} - 1)$ of these last determinants being zero because the corresponding elements in two columns at least are proportional. The only one that will differ from zero will be

$$\begin{vmatrix} 1 & 0 & \dots & 0 \\ [a] & [\alpha\alpha] & \dots & [\alpha\pi] \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ [p] & [\alpha\pi] & \dots & [\pi\pi] \end{vmatrix}$$

which is equal to Δ. Consequently we shall have :

$$D = n\Delta \tag{11}$$

We can likewise show that

$$Dx_r = n\Delta x_r \tag{12}$$

and

$$A_{rs} = n'A'_{rs} \tag{13}$$

where A_{rs} and A'_{rs} are the minors complementary to the principal elements of the determinants D and Δ .

Thus the relations (11) and (12) demonstrate that :

$$x_r = \frac{Dx_r}{D} = \frac{\Delta x_r}{\Delta} \qquad (r = 1, 2, 3, \dots, m-1)$$

Finally, *as the weight of the unknowns is defined as the quotient of* *determinant D and of the minor complementary to one of its principal elements* (*), for relations (11) and (13) we have

$$Px_r = \frac{D}{(-1)^{r+s} A_{rs}} = \frac{\Delta}{(-1)^{r+s} A'_{rs}}$$

that is

$$Q_{rs} = Q'_{rs} \qquad {}^{r}_{s} [= 1, 2, 3, \ldots, m-1 \tag{14}$$

and this is what was to be proved.

The value of the unknown x_0 which is eliminated by the subtraction is obtained from the mean equation by the values of $(m - 1)$ other unknowns. It will therefore have the same value as if it had been deduced from the system in (7).

To determine the weight of x_0 we write :

$$Q_{00} = \frac{1}{D} \begin{vmatrix} [\alpha\alpha] + \frac{[a][a]}{n} & [\alpha\beta] + \frac{[a][b]}{n} & \cdots & \cdots \\ [\alpha\beta] + \frac{[a][b]}{n} & [\beta\beta] + \frac{[b][b]}{n} & \cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots \end{vmatrix}$$

Expanding this determinant and introducing the quantities Q_{11}, Q_{12}, ..., we finally have :

$$Q_{00} = \frac{1}{n} + \frac{1}{n^2} \{ [a]^2 Q_{11} + [b]^2 Q_{22} + \cdots + 2 [a] [b] Q_{12} + 2 [a] [c] Q_{13} + \cdots \} \tag{15}$$

a relation which establishes the truth of our statement, since the bracketed polynomial is solely a function of the coefficients of the mean equation and of the $Q_{rs} = Q'_{rs}$ values. [Relation (14)].

c) In the case of a system of the type

$$x_0 + a_r x_1 = l_r \qquad (r = 1, 2, \ldots, n) \tag{16}$$

the solution is very easy.

The mean equation will be

$$x_0 + \frac{[a]}{n} x_1 = \frac{[l]}{n}$$

and by subtraction we obtain the reduced equations

$$\alpha_r x_1 = \lambda_r \qquad (r = 1, 2, \ldots, n)$$

whence the only normal equation

$$[\alpha\alpha] x_1 = [\alpha \lambda]$$

(*) An elegant demonstration of this mathematic proposition has been given by VALENTINER : Handworterbuch der Astronomie, dritter Band, erste Abteilung, Breslau 1899, page 51.

which directly gives

$$x_1 = \frac{[\alpha\lambda]}{[\alpha\alpha]} \quad ; \quad Px_1 = [\alpha\alpha]$$

The mean equation gives us the unknown x_0

$$x_0 = \frac{[l]}{n} - \frac{[a]}{n} x_1$$

and for this case, by putting

$$Q_{11} = \frac{1}{[\alpha\alpha]}$$

we shall have for relation (13) :

$$Q_{00} = \frac{1}{Px_0} = \frac{1}{n} + \frac{[a]^2}{n^2\,[\alpha\alpha]} = \frac{1}{n} + \frac{[a]^2}{n^2\,Px_1}$$

The expression

$$\frac{[a]^2}{n\,Px_1}$$

is a positive quantity and if we designate this by K^2 we shall have

$$Px_0 = \frac{n}{1 + K^2}$$

whence :

$$Px_0 < n$$

The mean errors in the values of the two unknowns will therefore be

$$\mu_{x_0} = \pm \sqrt{\frac{(1 + K^2)\,[vv]}{n\,(n-2)}} \quad ; \quad \mu_{x_1} = \pm \sqrt{\frac{[vv]}{(n-2)\,[\alpha\alpha]}}$$

where v_r are residuals such that they satisfy the condition that the sum $[vv]$ of their squares is minimum (*).

d) In the case where

$$x_0 = \Delta\lambda$$

$$x_1 = a = \text{instrument azimuth}$$

the relations

$$P_a = [\alpha\alpha] \quad ; \quad K^2 = \frac{[a]^2}{np_a} \quad ; \quad P_{\Delta\lambda} = \frac{n}{1 + K^2}$$

(n = the number of stars observed) will allow us to deduce that :

α) the azimuth will be fully determined when the coefficients a_r of

(*) We must take careful note that

$$[\varepsilon\varepsilon] > [vv]$$

where ε_i are the true observational errors. See for instance F.R. HELMERT, Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate, zweite Auflage, 1907, p. 99.

the equations in (16) have *values differing widely* one with the other, so that the differences

$$\alpha_r = a_r - \frac{[a]}{n}$$

will themselves have a high numerical value.

It should be carefully noted that we say a_r values *differing widely.* It will not then be necessary when determining $\Delta\lambda$ to observe stars that are very close to the pole, for these lack accuracy. It will suffice — for our latitudes — if we observe stars whose declination is between — 5° and + 70°. This is what we have adopted for the observational programme for the determination of the longitude of the Astronomical Observatory in Naples.

β) The best determination of $\Delta\lambda$ is for the case where

$$P_{\Delta\lambda} = n \quad ; \quad K^2 = 0 \tag{17}$$

that is to say for :

$$[a] = 0 \tag{18}$$

In another publication (*) we have demonstrated this proposition for the case of the Mayer and the Döllen methods which have been adopted by the International Latitude Service (ILS). We shall now show the optimum condition for the Bessel method.

The Bessel equation can be written as follows

$$\Delta\lambda + (\cos\varphi \tan\delta_r - \sin\varphi)\, a = N_r$$

where φ is the latitude of the observatory, and N_r are the known expressions.

In this case

$$a_r = (\cos\varphi \tan\delta_r - \sin\varphi)$$

whence the optimum condition

$$[a] = [\cos\varphi \tan\delta_r - \sin\varphi] = 0 \qquad (r = 1, 2, 3, \ldots, n)$$

that is to say :

$$\overline{\tan\delta_r} = \frac{\Sigma \tan\delta_r}{n} = \tan\varphi$$

3. THE SUPPORTING PILLAR AND ITS STABILITY

The technical principles governing the installation of a transit instrument and its sighting marks are more or less the same as for all highly accurate meridian instruments.

The most suitable proportions and shapes for the supporting pillars are established following the general rules of construction, and in function

(*) E. FICHERA : Sulla determinazione di Tempo con osservazioni di passaggi in meridiano. Memorie S.A.I., XXVIII-I, 1957, page 19.

of the materials used, the nature of the ground on which the pillar rests, and the pillar's height.

The detailed study of this aspect of the installation is an engineering matter necessitating a knowledge of the basic principles of both flexion and torsion, and of the stability factor.

By way of explanation we shall give some general indications on the computation method in which three numerical quantities a, b, and c are used. a and b are in fact complex functions of all the factors concerned in the actual construction — such as the weight the stand has to support (this factor also takes into account the weight of the pillar itself, as this will be used in the computation of the foundation), the nature of the ground, the material used, etc., whereas c, called the coefficient of stability, is a factor relating to the whole installation.

This coefficient makes it possible to establish the ratio between the area of the pillar's supporting base in the ground and the area of the instrument stand.

It is easy to see that to obtain maximum stability, from the astronomic point of view, it is necessary that this support should be built in the ground itself, and that an existing construction should not be used.

At the present time the preferred shape for the pillar is the frustum of a square based pyramid, preferably of cemented construction.

On account of this, and in view of the weight of a transit instrument (including a safety factor) the coefficient of stability is of the order of $c = 0.38$, accurate allowance being made for the effects of torsion and flexion which can arise on the layers of building material of which the pillar is made, and for both perpendicular and horizontal pressures.

The other two numerical data depend principally on the geometric shape of the stand and, once determined, serve for working out the ratios between the different dimensions of the pillar.

We shall now give a method of approximate computation which is in actual practice sufficient for our particular case.

We take the rounded-off integer values adopted for a and b. Knowing from the theory of construction that the functions expressing these two quantities are such that a will always be greater than b, we establish the following equations :

$$x = a^2 - b^2$$

$$y = 2ab \qquad a > b \text{ (integers)}$$

$$z = a^2 + b^2$$

so that :

$$x^2 + y^2 = z^2$$

x being the base's apothem (half the side),
y, the pyramid's height
z, the pyramid's apothem.

The computation is made in centimetres.

Thus we are first of all able to compute a pyramid having the same base as the frustum of the desired pyramid. The coefficient of stability makes it possible to find the last missing element for the frustum construction with the help of the following relation :

$$\frac{\text{minor base apothem}}{\text{major base apothem}} = 0.19$$

Thus we have all the elements for constructing the pillar.

In our case the 5-metre high frustum of the square pyramid is buried in such a way that the major base, 2.30 m square, is supported by the ground and the minor base, 0.65 m square, is at ground level. The 1-metre high parallelepiped support on which the transit instrument rests is constructed on this minor base.

The pillar is independent of both the floor and the observation cupola.