FUNDAMENTALS OF POWER SPECTRAL ANALYSIS AS APPLIED TO DISCRETE OBSERVATIONS

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1. — INTRODUCTION

It is well known that Fourier's expansion of a time function $\Psi(t)$ over a limited period $0 \leqslant t \leqslant T$ gives a representation of that function in terms of a series of harmonics of a fundamental frequency $F_{min.} = 1/T$. Hence such an expansion allows the determination of the amplitude H(F) for discrete (step by step) values of F, as well as the phase of each harmonic term. Since $F = n \cdot F_{min.}$ where $n = 0, 1, 2, \ldots$, the difference between two consecutive values of F will be equal to $F_{min.}$. Now if we take a system of orthogonal axes and we plot the step by step values of F on the abcissae axis from each of these points we can draw segments parallel to the ordinates' axis and equal to the amplitude corresponding to each frequency. This is the usual representation of the *line* spectrum of frequencies.

If we wish to represent $\Psi(t)$ for $0 \leqslant t \leqslant \infty$ then we do not fix the upper limit of T which will also be considered as a variable. Thus, if T tends to infinity $F_{min.} = 1/T$ will become infinitesimal. In this case another kind of expansion is necessary : the Fourier integral.

The Fourier integral permits us to express $\Psi(t)$ as a sum of an infinity of harmonic terms with amplitudes H(F), so that H(F) is in this case a continuous function of F and is the continuous spectrum of frequencies.

Now, we know from the sine water-wave theory that the wave energy flux per period is given by $E = 1/2\, \mu g H^2$, where μ is the water density, g the acceleration of gravity, and H the amplitude of the sine wave. Hence if we express E in units of μg, then we can write $E = 1/2\, H^2$ for the energy of the wave with H cm amplitude, and E will be expressed in cm^2 per period.

If fig. 1A represents the continuous curve of $1/2\, H^2(F)$ as a function of F, the total energy between ordinates $1/2\, H^2(F)$ and $1/2\, H^2(F + dF)$ can be taken as the area between these limits, and this is equal to

$$dE = \frac{1}{2} H^2 (F)\, dF$$

$$\text{then } dE/dF = \frac{1}{2} H^2(F)$$

is the energy rate per unit of frequency variation, and $1/2\,H^2(F)$ measures this rate, named *power spectral density*, in cm^2/c.p.d. (cycles per day). Thus power spectral analysis deals with the determination of the function $1/2\,H^2(F)$.

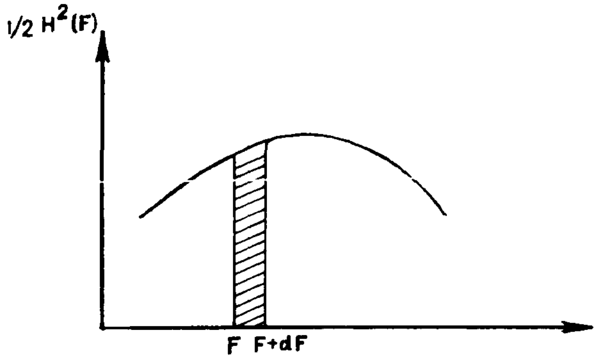

FIG. 1A. — Representation of the wave energy flux per period as a continuous function of frequency.

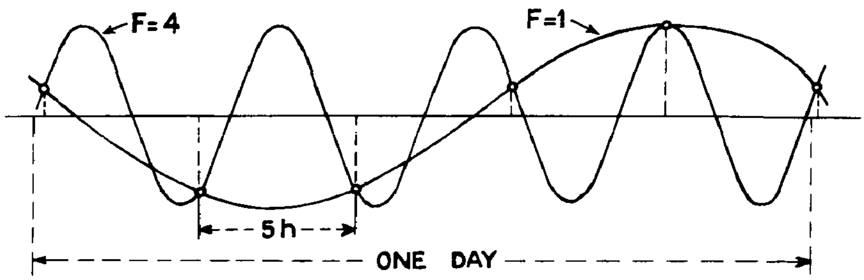

FIG. 1B. — An example of "aliasing" or "folding".

Now, it is easy to understand that power spectral analysis is a mathematical procedure to evaluate the quantitative importance in terms of energy of every oscillation of any frequency contained in function $\Psi(t)$.

The amount of computation involved in this kind of analysis was too laborious to be considered practicable until electronic computers became readily available. But recently this mathematical tool has been successfully applied to the study of sea waves and tsunamis and it is now used in tidal work. Let us explain some of its interesting tidal applications.

If one of the conventional harmonic analyses of hourly tidal heights is made, it is possible to predict the tidal curve. The differences between the ordinates of the two curves are residuals which can be studied by power spectral analysis. What can we conclude from the power spectral analysis of residuals ? We can detect oscillations which are not considered in conventional analysis, and can deduce the efficiency of the filtering. In addition, if several methods of analysis are applied to the same set of tidal data, it is possible to compare the accuracy of these methods [1].

Another application of power spectral analysis is that of determining the "noise" level of a tidal curve. The word "noise" has the same meaning here as in the electronic vocabulary. It is well known that when a radio receiver is syntonized to a transmitting station, we hear almost nothing but "modulated" sound which is superimposed on the carrier wave. However if we move the dial out of syntonization we hear a continuous noise known as "white noise" as well as some sporadic noise bursts. The former and a great part of the latter are *filtered* by syntonizing the oscillating circuit. The sporadic noise can be compared to "surges" which disturb astronomical tides. A continuous noise exists in all geophysical phenomena.

We can say that it is the noise level which limits the minimum length of tidal records for useful analysis. MUNK and HASSELMANN [2] give the condition necessary to separate two oscillations of frequencies F_1 and F_2 :

$$F_2 - F_1 = T \quad [\text{signal/noise level}]^{-1/2}$$

where T is the length of time the series covers, and the signal/noise level ratio is the ratio of the amplitude of tidal oscillation to that of noise oscillation. This formula shows that, to separate two oscillations of close frequencies, the noise level must be very low. Every hydrographer is well acquainted with the fact that when short series of hourly heights are analyzed, constituents of close speeds, as for example (S_2, K_2 and T_2), are treated as one single constituent. We can now understand why formerly it was accepted empirically that one month of hourly observations was the shortest span which could give *good* harmonic constants.

In order to emphasize the efficiency of power spectral analysis we may observe that for the port of Anchorage (Alaska) where shallow water plays an important part, ZETLER and CUMMINGS [3] detected by power spectral analysis large contributions from oscillations with frequencies not usually considered in classical harmonic analysis. Then they sought compound constituents with frequencies they had found and effected a harmonic analysis by considering 114 constituents, including fifth-diurnals. Their conclusion was that "Anchorage predictions using the additional constituents matched the observations better in range, shape of curve, and lunitidal intervals". The same kind of research was carried out independently and almost simultaneously by LENNON and ROSSITER [4] for the Thames Estuary and a similar result was found.

We will give now a summary of several terms and definitions usually employed in connection with power spectral analysis.

Sometimes it is interesting to express frequencies in another unit. In fact if we have to analyze a curve $y = \Psi(t)$ and we select ordinates at intervals Δt, it may be convenient to express frequencies in terms of the unit frequency $\frac{1}{2\,\Delta t}$, known as the Nyquist unit. Let us imagine that we have a diurnal oscillation which means that its frequency is 1 c.p.d. If we take the ordinates at 1 hour intervals we have $\Delta t = 1/24$ of the basic interval (one day) and the Nyquist unit of frequency will correspond to 12 c.p.d.. Hence the diurnal tide will have a frequency of $1/12 = 0.167$ Nyquist unit.

Let us now explain the most important phenomenon called *aliasing* or *folding*. Fig. 1B shows an oscillation whose frequency is four cycles per period T. If such an oscillation is superimposed on others and we analyze the whole curve by sampling at a fixed interval greater than T/8, then the four cycles-per-period oscillation would be completely masked by a non-existent oscillation of lower frequency (fig. 1B). Such a drawback can be avoided by sampling at intervals inferior to one half the period of the highest frequency oscillation to be detected. If, for instance, we are interested in detecting oscillations having 12 c.p.d., sampling at one hour intervals, i.e. half the period of the oscillation, will be the limit. We should remember that 12 c.p.d. is equivalent to 1 Nyquist unit when sampling at 1 hour intervals. Thus we see that it is the connection between the Nyquist unit of frequency and the sampling interval that avoids aliasing. When the aliasing is known beforehand it is possible to use it as a means of reducing computations. MYAZAKI's method of harmonic analysis is one example of such use. In point of fact he succeeded in analyzing one year of tidal observations by sampling at 35.5 hour intervals.

Finally, we define the term "resolution" met very often in power spectral analysis and whose meaning is easy to grasp. We can say that if our analysis permits the separation of oscillations with neighbouring frequencies F_1 and F_2, the difference ΔF measures the "resolution" of our analysis.

2. — ANALYSIS THROUGH AUTOCORRELATION

Let us take an oscillation with two harmonic terms :

$$y(t) = H \cos (qt - r) + H' \cos (q't - r')$$

where q and q' are the angular frequencies equal to 2π multiplied respectively by F and F′ which are multiples of the fundamental frequency. For time $(t - \theta)$, where θ is a time *lag* counted from t, we have :

$$y(t - \theta) = H [\cos q (t - \theta) - r] + H' [\cos q' (t - \theta) - r']$$

The product of these two expressions gives :

$$\begin{aligned} y(t)\, y(t - \theta) = {} & H^2 \cos (qt - r) \cos [q(t - \theta) - r] \\ & + H'^2 \cos (q't - r') \cos [q' (t - \theta) - r'] \\ & + HH' \cos (qt - r) \cos [q' (t - \theta) - r'] \\ & + HH' \cos (q't - r') \cos [q (t - \theta) - r] \end{aligned}$$

since

$$\cos a \cos b = \frac{1}{2} [\cos (a + b) + \cos (a - b)]$$

one can write

$$y(t)\,y(t-\theta) = \frac{1}{2}H^2 \cos q\theta + \frac{1}{2}H'^2 \cos q'\theta$$

$$+ \Sigma C \cos(\alpha t - \beta)$$

where

$$C = H^2/2\,,\ H'^2/2\,,\ HH'/2$$

$$\alpha = 2q\,,\ 2q'\,,\ q+q',\ q-q' \qquad (2a)$$

and β is formed by a combination of q, q', θ, r and r'.

The mean value of $y(t)y(t-\theta)$ over the interval $-T/2$ to $T/2$ is called the *autocorrelation* function and is expressed by

$$\langle y(t)\,y(t-\theta)\rangle = \frac{1}{2}H^2 \cos q\theta + \frac{1}{2}H'^2 \cos q'\theta$$

$$+ \Sigma \frac{C}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\alpha t - \beta)\, dt \qquad (2b)$$

but

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\alpha t - \beta)\, dt = \frac{2}{\alpha}\cos\beta \sin\left(\alpha \frac{T}{2}\right) \qquad (2c)$$

and according to (2a) we have

$$\alpha = 2\pi s F_0$$

s being a whole number, and F_0 the fundamental frequency equal to $1/T$. Thus, since

$$\alpha = 2\pi s/T$$

expression (2c) vanishes. Therefore we can write (2b) as:

$$A(\theta) = \langle y(t)\,y(t-\theta)\rangle = \frac{1}{2}H^2 \cos q\theta + \frac{1}{2}H'^2 \cos q'\theta \qquad (2d)$$

This expression, which can be generalized for any number of harmonic terms, is also a satisfactory approximation when finding the mean of discrete values of $y(t)\,y(t-\theta)$ when T is large. The generalized form of (2d) is

$$A(\theta) = \frac{1}{2}\sum_q H^2(q)\cos q\theta \qquad (2e)$$

This formula was found by writing the oscillations in the usual tidal form, that is to say, by taking t in hours and q as the hourly speed. This form of representation is very convenient for sampling observations at hourly intervals. However it may be preferable in power spectral analysis to sample at shorter intervals, and a more general definition of q is therefore necessary.

Let us put as usual:

$$q = 2\pi F \qquad (2f)$$

Then, if F_0 is the fundamental frequency we have $F = nF_0$, $(n = 0, 1, 2, ...)$ and we can write

$$q = 2\pi n F_0$$

but $F_0 = 1/T$ and $T = (N-1)\Delta t$, where Δt is the sampling interval and N the number of such intervals. Therefore

$$q = 2\pi n/(N - 1)\,\Delta t$$

is the phase variation per unit interval. In this expression n is the frequency in units of the fundamental frequency.

Now according to the needs of the analysis a series of lags is used in such a way that $\theta_p - \theta_{p-1} = \Delta t$. If we designate the total number of lags by $(m+1)$ and

$$\theta = \tau \Delta t \qquad (\tau = 0, 1, 2, \dots m)$$

we obtain from (2e)

$$A(\tau) = \frac{1}{2} \sum_n H^2(n) \cos 2\pi n\tau/(N - 1) \tag{2g}$$

or, if a factor k is introduced such that

$$n = k\,(N - 1)/2m \tag{2h}$$

we have as a result

$$A(\tau) = \frac{1}{2} \sum_k H^2(k) \cos (k\pi\tau/m) \tag{2i}$$

Since we have $(m + 1)$ values of $A(\tau)$ a further analysis of these values is possible in order to separate the values of $1/2\,H^2(k)$. In fact, as m is a parameter established beforehand, if k is taken as a series of integers up to m, then $1/m$ is a constant playing exactly the same role as a frequency whose value is twice the fundamental frequency F_0 in the Fourier expansion. Moreover, if we have $(m+1)$ lags we can obtain $1/2\,H^2(k)$ for each value of k in the same way as we find the coefficients of cosines in a Fourier series. However $1/2\,H^2(k)$ is a *density* which can only be found if $A(\tau)$ is a continuous function of τ. Thus the Fourier analysis of discrete values of $A(\tau)$ gives only an *estimate* $X(k)$ of the spectral density $1/2\,H^2(k)$. Hence a Fourier analysis of (2i) gives

$$X(k_p) = \frac{\delta_k}{m + 1} \sum_{\tau=0}^{m} A(\tau) \cos (k_p \pi\tau/m)\ ; \quad \delta_k = \begin{cases} \frac{1}{2} \text{ if } k_p = 0 \text{ or } m \\ 1 \text{ for any other value of } k_p \end{cases} \tag{2j}$$

For such whole values of k the corresponding values of n will not be integers, and vice-versa.

For the ideal separation of energy centered at k from energies corresponding to $k-1$ and $k+1$, k should correspond to the total energy between $k - 1/2$ and $k + 1/2$. But it will be shown later that the usual filtering only allows the separation of frequency bands that correspond to the interval between the limits $k-2$ and $k+2$.

3. — ANALYSIS THROUGH AMPLITUDES

Let us now study the residual interference of an oscillation whose frequency is proportional to k on an oscillation whose frequency is proportional to k_p by considering $A(\tau)$ reduced to two terms only. Thus the energy centered at k_p which is affected by energies from oscillations of neighbouring frequencies, is obtained from (2i) and (2j):

$$X(k_p) = \frac{k}{m+1} \sum_{\tau=0}^{m} \left[\frac{1}{2} H^2(k_p) \cos(k_p \pi\tau/m) \cos(k_p \pi\tau/m) + \frac{1}{2} H^2(k) \cos(k\pi\tau/m) \cos(k_p\pi\tau/m) \right] \tag{3a}$$

In the above expression the interference is represented by the k terms. Thus we may write:

$$\Delta X(k) = \frac{1}{2} H^2(k) \frac{1}{m+1} \delta_k \sum_{\tau=0}^{m} \left[\frac{1}{2} \cos(k_p + k) \pi\tau/m + \frac{1}{2} \cos(k_p - k) \pi\tau/m \right]$$

If m is made as large as possible, then the fundamental ratio $1/m$ is small enough to allow the replacement of the averages of discrete values over the interval $\tau=0$ to $\tau=m$ by integrals of the form

$$\text{average} = \frac{1}{m} \int_0^m \cos(\alpha\tau)\, d\tau = \frac{1}{\alpha m} \sin(\alpha m)$$

where α is any of the coefficients of τ in the previous expression. Thus we have

$$\Delta X(k) = \frac{\delta_k}{2} H^2(k) \left[\frac{\sin\pi(k_p - k)}{2\pi(k_p - k)} + \frac{\sin\pi(k_p + k)}{2\pi(k_p + k)} \right] \tag{3b}$$

Now if we examine a table of values of $\sin \pi u/\pi u$ we see that this function of u has zero values for $u = 1, 2, 3, \ldots$ and that there will be consecutive maxima and minima for $u = 0$, 1.45, 2.46, etc. Again we see that these maxima and minima have decreasing absolute values and that the maximum corresponding to $u = 2.46$ is only 13% of the maximum at $u = 0$. However the absolute values of the maxima and minima that follow the maximum at $u = 2.46$ do not die away very quickly. Thus the term with (k_p+k) in (3b) cannot be omitted, for $(k_p+k) < 4$. But when m is large only a few low harmonics corresponding to small values of k_p will be affected by this term. Hence if we put

$$\frac{\sin\pi(k_p - k)}{2\pi(k_p - k)} = \varphi(k_p - k) \tag{3c}$$

we obtain from (3b)

$$\Delta X(k) = \frac{\delta_k}{2} H^2(k)\, \varphi(k_p - k) \tag{3d}$$

The dashed line of Fig. 3A shows the function $\varphi(k_p-k)$ given by (3c).

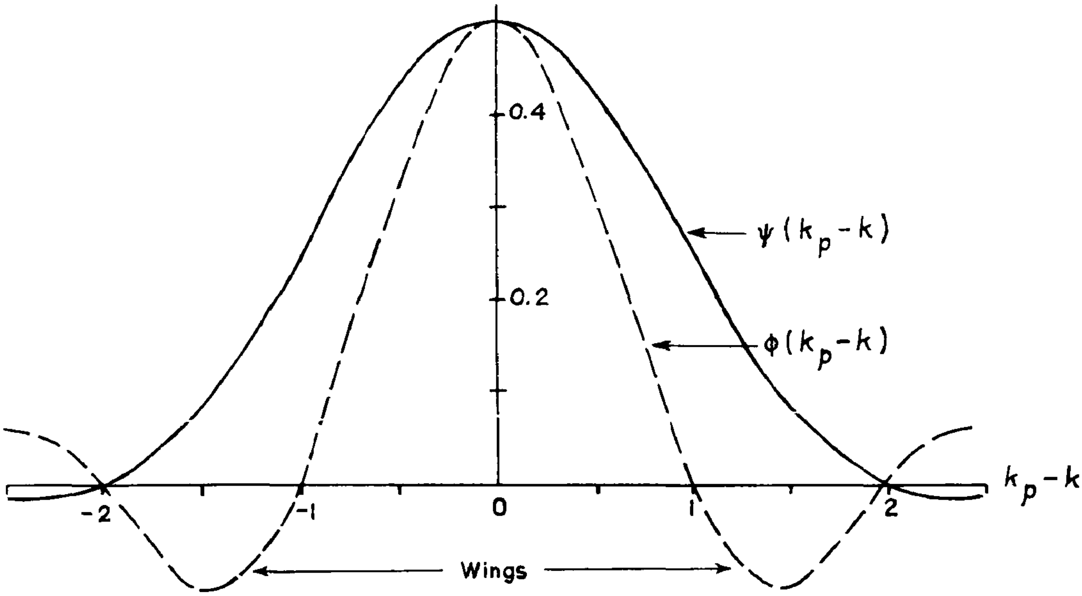

FIG. 3A. — Response Functions.

It is possible to reduce the side wings of the dashed line in Fig. 3A by filtering $A(\tau)$. In fact, if we use $A(\tau)(1+\cos \pi\tau/m)$ instead of $A(\tau)$ then the fading function $(1 + \cos \pi\tau/m)$, devised by TUKEY, will be a new factor in (3a). A similar development of the expression resulting from introducing that new factor into (3a) will give

$$\Delta X(k) = \frac{\delta_k}{2} H^2(k)\, \psi(k_p - k) \tag{3e}$$

where

$$\psi(k_p - k) = \frac{\sin \pi\, (k_p - k)}{2\pi\, (k_p - k)\, [1 - (k_p - k)^2]} \tag{3f}$$

The solid line of Fig. 3A represents the function $\psi(k_p - k)$ as given by (3f). We see that $\psi = 0$ for $|k_p - k| = 2$, and that $\psi = 0$ for $|k_p - k| > 2$. Since φ and ψ define the "response" $A(\tau)$ to filtering they are named "response functions".

Now, if we compare the first term of (3a) with the second we see that through a development of the first term which is similar to that of the second, a formula analogous to (3d) will be found. The coefficient $H(k)$ will then be $H(k_p)$ and function φ will be found by making $k = k_p$ in (3c). Hence (3a) can be replaced by the general expression

$$X(k_p) = \Sigma \Delta X = \delta_k \sum_{k=0}^{m} \frac{1}{2} H^2\, \varphi\, (k_p - k) \tag{3g}$$

or, if the Tukey filter is used,

$$X(k_p) = \delta_k \sum_{k=0}^{m} \frac{1}{2} H^2(k)\, \psi\, (k_p - k) \tag{3h}$$

If we consider φ or ψ as weight functions, then we see that (3g) and (3h) are weighted sums of the values of $1/2\,H^2(k)$ from $k=0$ to $k=m$.

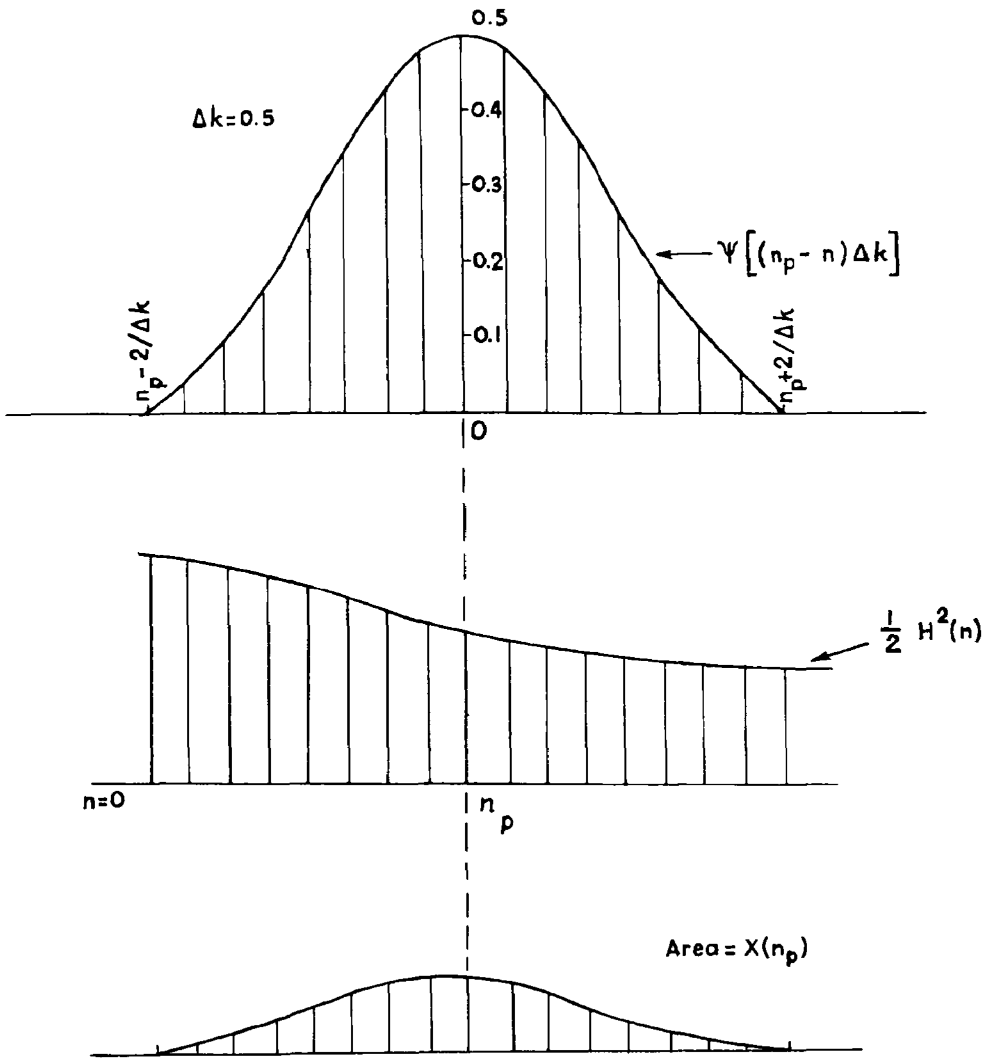

FIG. 3B. — Graphical Representation of $X(n_p)$.

We are now in a position to draw a most important conclusion: the energy centered at k_p can be found by the weighted sum of the values of $1/2\,H^2(k)$, the weights being the corresponding values of $\varphi(k_p-k)$ or $\psi(k_p-k)$, as the case may be. Consequently, if we compute the amplitudes of a series of harmonic terms through a Fourier's analysis, then we do not need to pass through autocorrelation to compute the energy. However the Fourier's analysis gives the amplitudes $H(n)$ for $n = 0, 1, 2, \ldots (N-1)/2$. Thus it will be necessary to modify expressions (3g) and (3h) in order to use $H(n)$ instead of $H(k)$. Since φ and ψ are computed for the difference (k_p-k), k_p may correspond to any central frequency. Thus all we need in order to use these functions in connection with frequencies n is to change the scale of the axis of abscissae in Fig. 3A taking $(n_p - n)$ instead of $(k_p - k)$. We derive from (2h)

$$\Delta k = k/n = 2m/(N - 1) \tag{3i}$$

which is the variation of k per unit of n. Hence, since Δk is a parameter depending on m and N, we can write

$$\varphi(k_p - k) = \varphi[(n_p - n)\,\Delta k]$$

and (3j)

$$\psi(k_p - k) = \psi[(n_p - n)\,\Delta k]$$

The limits of the summation over k must now be replaced by those corresponding to the summation over n. We saw that $\psi = 0$ for $|k_p - k| = 2$ and $\psi = 0$ for $|k_p - k| > 2$ when the Tukey filter is used. Thus we can reduce the above-mentioned limits to

$$(n_p - n)\,\Delta k = \pm\, 2$$

hence

$$n_p - n = \pm\, 2/\Delta k$$

expression (3h) can therefore be written as follows:

$$X(n_p) = \delta_k \sum_{n=n'}^{n''} \frac{1}{2} H^2(n)\, \psi\,[(n_p - n)\,\Delta k] \tag{3k}$$

where

$$n' = n_p - 2/\Delta k$$

and (3l)

$$n'' = n_p + 2/\Delta k$$

Since the limits of summation are narrower when ψ is used, we shall always adopt this function.

The interval of summation being reduced to $4/\Delta k$, this ratio is equal to the number of harmonics covered by the weighted sum; that is

$$s = 4/\Delta k$$

or, according to (3i)

$$s = 2(N - 1)/m \tag{3m}$$

Since m is the largest lag used when the energy is found through autocorrelation it is easy to grasp that $m = (N-1)/2$ is also the largest lag which can be used in computations. Thus the least possible value of s will be 4. In addition, since any value less than $(N-1)/2$ can be assigned to m, the value of s can be fixed at will.

Some words on the choice of s remain to be said. Since $F = n/T = n/(N-1)\,\Delta t$, if the sampling interval is 1 hour and we desire to express F in cycles per day, we must express Δt as a fraction of one day, i.e., $\Delta t = 1/24$. Thus,

$$F = 24\,n/(N - 1) \text{ c.p.d.} \tag{3n}$$

and

$$\Delta F = 24\,\Delta n/(N - 1) \tag{3o}$$

Now, by examining a table of angular frequencies of the usual tidal constituents we see that the highest frequency corresponding to one species is equal to the lowest angular frequency of the higher species less about 11 degrees per hour. Thus, in order to have a filter width that never mixes with neighbouring species, we must have $\Delta F_{max.} = 11°/15° = 0.013$ c.p.d. Hence we obtain from (3o)

$$\Delta n = 0.003\,(N - 1) = s - 1 \tag{3p}$$

In order to effect the weighted sum so that the central ordinate of the curve $\psi(u)$ is centered on the whole value n_p it will be necessary to make an approximation. By so doing s will always be an odd number. Hence it is possible to write (3p) as follows:

$$(s - 1)/2 = 0.0015\,(N - 1) = L \tag{3q}$$

where an integer value of L will be found by truncation. Consequently we may write

$$n_p - n = I - L \qquad (I = 0, 1, 2, 3, \ldots .\, 2\,L)$$

which is a convenient formula for programming electronic computation.

It has been shown that the least possible value of s is 4. But, since s must be an odd number, we have $s_{min.} = 5$. By replacing this value in (3q) we obtain N = 1334 which is the least number of points to be analysed in order to obtain accurate results.

If we desire to know the value of m corresponding to the approximation employed, we obtain from (3m) and (3q)

$$m = 2(N - 1)/(2L + 1) \tag{3r}$$

We close this section with a graphical interpretation of the weighted sum. Fig. 3B is in three parts: the upper part is the functin $\psi[(n_p - n)\Delta k]$; the central part is the function $1/2\,H^2(n)$ between the limits of $n_p - 2/\Delta k$ and $n_p + 2/\Delta k$. The ordinates of the lower curve are equal to the products of the corresponding ordinates of the upper and central curves. The area of the lower curve is equal to $X(n_p)$. Thus, $X(n_p)$ is not a function of n_p but expresses the area centered at n_p. It is called *functional* of $1/2 \cdot H^2(n_p)$.

It is obvious that the area which gives $X(n_p)$ must be determined by adding the small strips of areas having heights $1/2\,H^2(n)\,\psi\,[(n_p - n)\Delta k]$. Hence the distance between two consecutive ordinates used in computations depends on the accuracy with which it is desired to obtain $X(n_p)$. In fact, in many cases we do not need to use all the harmonics.

4. — THE COOLEY-TUKEY ALGORITHM

Spectral analysis effected according to (3k) is not so convenient as the excellent algorithm devised by Cooley and Tukey [10]. This algorithm, called "Fast Fourier Transform" allows us to reduce the computation of a Fourier analysis down to 100 times when $N = 2^{10}$. In fact, such an economy is based on always using N as a power of 2.

In order to explain the algorithm let us start with the expressions for determining the coefficients of a Fourier analysis when a series of discrete observations is given:

$$a(n) = \frac{2}{N} \sum_{t=0}^{N-1} y(t) \cos 2\pi nt/N$$

$$b(n) = \frac{2}{N} \sum_{t=0}^{N-1} y(t) \sin 2\pi nt/N$$

By putting

$$c(n) = \frac{N}{2} [a(n) - ib(n)] \tag{4a}$$

where $i = \sqrt{-1}$, it follows from the preceding formulas that

$$c(n) = \sum_{t=0}^{N-1} y(t)\ e^{-2i\pi nt/N} \tag{4b}$$

If we now put

$$e^{-2i\pi/N} = W \tag{4c}$$

we can change (4b) into

$$c(n) = \sum_{t=0}^{N-1} y(t)\ W^{nt}$$

It is well known that the maximum value of n with physical meaning in a Fourier analysis is N/2. However, in order to simplify further calculations the upper limit of n can be extended to N−1. By so doing N/2 fictitious values of $c(n)$ will appear, and a square matrix can be constructed with the values of W^{nt}. We can then write the last expression under the following matrix form:

$$\{c(n)\} = \| W^{nt} \| \{y(t)\}$$

Let us now write this expression for N=8:

$$\begin{pmatrix} c(0) \\ c(1) \\ c(2) \\ c(3) \\ c(4) \\ c(5) \\ c(6) \\ c(7) \end{pmatrix} = \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\ 1 & W^2 & W^4 & W^6 & W^8 & W^{10} & W^{12} & W^{14} \\ 1 & W^3 & W^6 & W^9 & W^{12} & W^{15} & W^{18} & W^{21} \\ 1 & W^4 & W^8 & W^{12} & W^{16} & W^{20} & W^{24} & W^{28} \\ 1 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} & W^{30} & W^{35} \\ 1 & W^6 & W^{12} & W^{18} & W^{24} & W^{30} & W^{36} & W^{42} \\ 1 & W^7 & W^{14} & W^{21} & W^{28} & W^{35} & W^{42} & W^{49} \end{Vmatrix} \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{pmatrix} \tag{4d}$$

But from (4c) we can take:

$$W^{nt} = e^{-i2\pi nt/N}$$

where

$$nt/N = \alpha + \beta/N$$

α being the integer quotient of the division and β the remainder. Thus

$$W^{nt} = e^{-i2\pi\alpha}\, e^{-i2\pi\beta/N}$$

Therefore, since α is an integer,

$$W^{nt} = W^{\beta}$$

and we may transform (4d) into:

$$\begin{pmatrix} c(0) \\ c(1) \\ c(2) \\ c(3) \\ c(4) \\ c(5) \\ c(6) \\ c(7) \end{pmatrix} = \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\ 1 & W^2 & W^4 & W^6 & W^0 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^1 & W^4 & W^7 & W^2 & W^5 \\ 1 & W^4 & W^0 & W^4 & W^0 & W^4 & W^0 & W^4 \\ 1 & W^5 & W^2 & W^7 & W^4 & W^1 & W^6 & W^3 \\ 1 & W^6 & W^4 & W^2 & W^0 & W^6 & W^4 & W^2 \\ 1 & W^7 & W^6 & W^5 & W^4 & W^3 & W^2 & W^1 \end{Vmatrix} \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{pmatrix} \qquad (4e)$$

It is interesting to note that if we ignore the first row in the above square matrix, then the values of W^{β} symmetrical with respect to the row corresponding to $n = N/2$ are tied by the relation $W^{\beta'} = W^{N-\beta}$. But expression (4c) shows that $W^N = 1$. Hence, since $W^{\beta'} = W^{-\beta}$, W^{β} and $W^{\beta'}$ are complex conjugates. Consequently it will be the same with values of $c(n)$ symmetrical with respect to $c(N/2)$.

Now if we interchange the rows of the square matrix in order to obtain $c(n)$ with n corresponding to flipping the bits which represent $n = 0, 1, 2, 3, \ldots 7$, then the column vector of the first member can be rearranged as follows:

$$\left.\begin{matrix} (0)\ 000 \\ (1)\ 001 \\ (2)\ 010 \\ (3)\ 011 \end{matrix}\right\} \text{flips into} \left\{\begin{matrix} 000\ (0) \\ 100\ (4) \\ 010\ (2) \\ 110\ (6) \end{matrix}\right. \qquad \left.\begin{matrix} (4)\ 100 \\ (5)\ 101 \\ (6)\ 110 \\ (7)\ 111 \end{matrix}\right\} \text{flips into} \left\{\begin{matrix} 001\ (1) \\ 101\ (5) \\ 011\ (3) \\ 111\ (7) \end{matrix}\right.$$

and (4e) can be changed into:

$$\{c'(n)\} = \begin{pmatrix} c(0) \\ c(4) \\ c(2) \\ c(6) \\ c(1) \\ c(5) \\ c(3) \\ c(7) \end{pmatrix} = \left\Vert \begin{array}{cc:cc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^4 & W^0 & W^4 & W^0 & W^4 & W^0 & W^4 \\ \hdashline 1 & W^2 & W^4 & W^6 & W^0 & W^2 & W^4 & W^6 \\ 1 & W^6 & W^4 & W^2 & W^0 & W^6 & W^4 & W^2 \\ \hline 1 & W^1 & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\ 1 & W^5 & W^2 & W^7 & W^4 & W^1 & W^6 & W^3 \\ \hdashline 1 & W^3 & W^6 & W^1 & W^4 & W^7 & W^2 & W^5 \\ 1 & W^7 & W^6 & W^5 & W^4 & W^3 & W^2 & W^1 \end{array} \right\Vert \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{pmatrix} \qquad (4f)$$

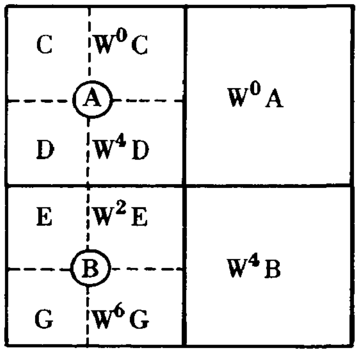

FIG. 4A. — Parts of the whole matrix.

Examining the square matrix of the above expression, we can see that it may be split according to Fig. 4A. Thus, if we designate the upper half of vector $\{y(t)\}$ by V_1, and the lower half of the same vector by V_2 we can write (4f) as follows:

$$\{c'(n)\} = \begin{Vmatrix} A & W^0A \\ B & W^4B \end{Vmatrix} \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix} \begin{Vmatrix} U & W^0U \\ U & W^4U \end{Vmatrix} \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} \tag{4g}$$

where U are unit matrices. Hence we have the first reduction:

$$\{y_1(t)\} = \begin{Bmatrix} V_1 + W^0V_2 \\ V_1 + W^4V_2 \end{Bmatrix} \tag{4h}$$

If we replace the vectors V_1 and V_2 by their respective elements, we then obtain:

$$\begin{aligned} y_1(0) &= y(0) + W^0y(4) & y_1(4) &= y(0) + W^4y(4) \\ y_1(1) &= y(1) + W^0y(5) & y_1(5) &= y(1) + W^4y(5) \\ y_1(2) &= y(2) + W^0y(6) & y_1(6) &= y(2) + W^4y(6) \\ y_1(3) &= y(3) + W^0y(7) & y_1(7) &= y(3) + W^4y(7) \end{aligned} \tag{4i}$$

From (4g) and (4h) we find:

$$\{c'(n)\} = \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix} \{y_1(t)\}$$

Now if we split $y_1(t)$ into four vectors with two elements each, and replace A and B by their respective matrices as shown in Fig. 4A we arrive at

$$\{c'(n)\} = \left\| \begin{matrix} C & W^0C & 0 & 0 \\ D & W^4D & 0 & 0 \\ 0 & 0 & E & W^2E \\ 0 & 0 & G & W^6G \end{matrix} \right\| \begin{Bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{Bmatrix}$$

or

$$\{c'(n)\} = \left\| \begin{matrix} C & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & G \end{matrix} \right\| \times \left\| \begin{matrix} U & W^0U & 0 & 0 \\ U & W^4U & 0 & 0 \\ 0 & 0 & U & W^2U \\ 0 & 0 & U & W^6U \end{matrix} \right\| \begin{Bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{Bmatrix} \tag{4j}$$

we then have the second reduction:

$$\{y_2(t)\} = \begin{Bmatrix} V_1' + W^0V_2' \\ V_1' + W^4V_2' \\ V_3' + W^2V_4' \\ V_3' + W^6V_4' \end{Bmatrix} \tag{4k}$$

Since

$$V_1' = \begin{Bmatrix} y_1(0) \\ y_1(1) \end{Bmatrix}, V_2' = \begin{Bmatrix} y_1(2) \\ y_1(3) \end{Bmatrix}, V_3' = \begin{Bmatrix} y_1(4) \\ y_1(5) \end{Bmatrix} \text{ and } V_4' = \begin{Bmatrix} y_1(6) \\ y_1(7) \end{Bmatrix}$$

we can write the second reduction in the following explicit form:

$$\begin{aligned} y_2(0) &= y_1(0) + W^0y_1(2) & y_2(4) &= y_1(4) + W^2y_1(6) \\ y_2(1) &= y_1(1) + W^0y_1(3) & y_2(5) &= y_1(5) + W^2y_1(7) \\ y_2(2) &= y_1(0) + W^4y_1(2) & y_2(4) &= y_1(4) + W^6y_1(6) \\ y_2(3) &= y_1(1) + W^4y_1(3) & y_2(5) &= y_1(5) + W^6y_1(7) \end{aligned} \tag{4l}$$

we then obtain from (4f), (4j), (4k) and (4l):

$$\{c'(n)\} = \begin{pmatrix} y_3(0) \\ y_3(1) \\ y_3(2) \\ y_3(3) \\ y_3(4) \\ y_3(5) \\ y_3(6) \\ y_3(7) \end{pmatrix} \left\| \begin{matrix} 1 & 1 & & & & & & \\ 1 & W^4 & & & & & & \\ & & 1 & W^2 & & & & \\ & & 1 & W^6 & & & & \\ & & & & 1 & W^1 & & \\ & & & & 1 & W^5 & & \\ & & & & & & 1 & W^3 \\ & & & & & & 1 & W^7 \end{matrix} \right\| \begin{pmatrix} y_2(0) \\ y_2(1) \\ y_2(2) \\ y_2(3) \\ y_2(4) \\ y_2(5) \\ y_2(6) \\ y_2(7) \end{pmatrix}$$

which gives the third and last reduction:

$$\begin{aligned} y_3(0) &= y_2(0) + W^0y_2(1) & y_3(4) &= y_2(4) + W^1y_2(5) \\ y_3(1) &= y_2(0) + W^4y_2(1) & y_3(5) &= y_2(4) + W^5y_2(5) \\ y_3(2) &= y_2(2) + W^2y_2(3) & y_3(6) &= y_2(6) + W^3y_2(6) \\ y_3(3) &= y_2(2) + W^6y_2(3) & y_3(7) &= y_2(6) + W^7y_2(6) \end{aligned} \tag{4m}$$

Further simplifications are still possible. In fact, expression (4c) gives:

$$W^{\beta+N/2} = e^{-2i\pi\beta/N} \cdot e^{-i\pi} = -1 \cdot W^{\beta}$$

Thus for $N=2^3$ we only need to know the values of W^0, W^2, W^1 and W^3. For $N=2^\gamma$ we have to know $2^{\gamma-1}$ values of W^β.

Expression (4m) shows that successive values of β are equal to n and appear in the computation in the same order as n does in vector $\{c'(n)\}$ of (4f). Hence all values of β can be obtained by flipping the binary bits representing the integers in their natural order, from 0 to $N-1$. However, this is not the best way of finding β if BCD (binary coded decimal) computers are used. In fact, since it has been pointed out that we do not need all the values of β for computation, it will be preferable to obtain the values of β by using the recurrence expressions worked out by Eng. E. BERGAMINI.

Let us write the natural numbers from 0 to $2^{\gamma-1}$ in the binary system. By flipping the bits and writing the corresponding results as decimal figures we can form the complete sequence of values of β in due order. If we repeat such an operation for $\gamma = 2, 3, 4, 5$ and select the *alternate* even values of β, then we can construct the following table:

γ \ k	0	1	2	3	4	5	6	7
2	0							
3	0	2						
4	0	4	2	6				
5	0	8	4	12	2	10	6	14

From this table it is easy to grasp that each sequence is equal to the double of its precedent *continued* by another sequence obtained by summing 2 to each element of this double. We will express this fact by the general expression:

$$S'_\gamma = 2S_{\gamma-1} \, , \, 2S_{\gamma-1} + 2 \tag{4n}$$

To obtain the odd values of β necessary for the computations all we need to do is to add 1 to each element of S'_γ. Hence we have the final sequence of β values used in the computations expressed by:

$$S_\gamma = S'_\gamma \, , \, S'_\gamma + 1 \tag{4o}$$

We can then express vector $\{W^\beta\}$ by the general formula

$$\{W^{\beta k}\} = \{\cos 2\pi\beta_k/N - i \sin 2\pi\beta_k/N\}$$

with

$$k = 0, 1, 2, \ldots\ldots\ldots\ldots 2^{\gamma-1} - 1 \tag{4p}$$

In order to obtain the recurrence formulas expressing the successive reductions (4i), (4l) and (4m), let us write these expressions in the following form:

$$l = 1 : \begin{cases} y_1(m) = y(m) + W^0 . y(m + N/2) \\ y_1(m + N/2) = y(m) - W^0 . y(m + N/2) \end{cases} \tag{4q}$$

with

$$m = 0, 1, 2, \ldots\ldots\ldots\ldots N/2 - 1$$

$$l = 2 : \begin{cases} y_2(m) = y_1(m) + W^0 . y_1(m + N/4) \\ y_2(m + N/4) = y_1(m) - W^0 . y_1(m + N/4) \\ y_2(m + 2N/4) = y_1(m + 2N/4) + W^1 . y_1(m + 3N/4) \\ y_2(m + 3N/4) = y_1(m + 2N/4) - W^1 . y_1(m + 3N/4) \end{cases}$$

with

$$m = 0, 1, 2, \ldots\ldots\ldots\ldots N/4 - 1$$

and

$$l = 3 : \begin{cases} y_3(m) = y_2(m) + W^0 . y_2(m + N/8) \\ y_3(m + N/8) = y_2(m) - W^0 . y_2(m + N/8) \\ y_3(m + 2N/8) = y_2(m + 2N/8) + W^1 . y_2(m + 3N/8) \\ y_3(m + 3N/8) = y_2(m + 2N/8) - W^1 . y_2(m + 3N/8) \\ y_3(m + 4N/8) = y_2(m + 4N/8) + W^2 . y_2(m + 5N/8) \\ y_3(m + 5N/8) = y_2(m + 4N/8) - W^2 . y_2(m + 5N/8) \\ y_3(m + 6N/8) = y_2(m + 6N/8) + W^3 . y_2(m + 7N/8) \\ y_3(m + 7/N8) = y_2(m + 6N/8) - W^3 . y_2(m + 7N/8) \end{cases}$$

with

$$m = 0, 1, \ldots N/8 - 1$$

The values of y_3 given by the last reduction are the elements of the column vector $\{ c'(n) \}$. To obtain the elements of the column vector $\{ c(n) \}$ all we need to do is to re-arrange the elements of $\{ c'(n) \}$ according to the increasing orders of n. Since $N = 2^\gamma$ we can generalize the above reduction by representing any couple of expressions by the following couple of recurrence formulas:

$$\begin{aligned} y_l(m + (2k) . 2^{\gamma-l}) &= y_{l-1}(m + (2k) . 2^{\gamma-l}) + W^{\beta k}\, y_{l-1}(m + (2k + 1) . 2^{\gamma-l}) \\ y_l(m + (2k + 1) . 2^{\gamma-l}) &= y_l(m + (2k) . 2^{\gamma-l}) - 2W^{\beta k}\, y_{l-1}(m + (2k + 1) . 2^{\gamma-l}) \end{aligned} \tag{4r}$$

$$l = 1, 2, \ldots\ldots\ldots\ldots\ldots \gamma$$

$$k = 0, 1, 2, \ldots\ldots\ldots\ldots 2^{l-1} - 1$$

$$m = 0, 1, 2, \ldots\ldots\ldots\ldots 2^{\gamma-l} - 1$$

Let us now take a glance at the economy obtained by using this algorithm. If we count the operations indicated in reductions (4i), (4l) and (4m) we find 24 sums and 24 products. But if we note that $24 = 3 \times 2^3$ for $N = 2^3$ we may generalize this calculation for $N = 2^\gamma$ by writing

$$T = \gamma 2^\gamma$$

however we saw above that since $W^{\beta+N/2} = -W^\beta$, the number of products can be halved, thus for the number of products we have:

$$T = \gamma . 2^{\gamma-1}$$

Now if (4c) is used in the classical way, each element of $\{c(n)\}$ is obtained by N products. Hence $N \times N$ is the number of products necessary to compute all the elements of $\{c(n)\}$. Since $N = 2^\gamma$ we have

$$T' = 2^{2\gamma}$$

for the total number of products. Consequently the economy will be represented by the ratio

$$T'/T = 2^{\gamma+1}/\gamma$$

which gives 128 fewer multiplications when $\gamma = 13$. This figure corresponds to a series with 8 192 points.

We have seen that the last reduction gives the elements $y_\gamma(t)$ which are equal to the elements of $\{c'(n)\}$. In addition we saw that the values of β_k used in the computations are also the alternate values of n. Thus, since the sequence S_γ given by (4o) is the sequence of values of β_k, we can write:

$$S_\gamma = \beta_k = n_{2k} \quad k = 0, 1, 2, \ldots\ldots 2^{\gamma-1} - 1 \tag{4s}$$

Consequently the complete sequence of values of n will be

$$S''_\gamma = n_{k'} \quad k' = 0, 1, 2, \ldots\ldots 2\gamma - 1 \tag{4t}$$

where for $k' = 2k + 1$ we have:

$$n_{k'} = n_{2k} + 2^{\gamma-1} \tag{4u}$$

The Fortran IV programs for using both this algorithm and formula (3k) are given here under.

```
?OLL
FILE    9=FRANCO,UNIT=DISK,LOCK,AREA=2000,RECORD=192
FILE    6=NOFILE,UNIT=PRINTER,UNLABELED
C       CNAE ** SECA * PADILHA - 10/9/68
C       FAST  FOURIER  TRANSFORM
C       YA(I) AND THE EXPONENT OF 2, DESIGNED GAMA, ARE THE DATA.
        COMPLEX YA(512),WBETA(256),AUX
        INTEGER GAMA,COEF1,COEF2,BETA(256)
        DPI=6.2831852
        READ(5,1000) GAMA
        I=GAMA-2
        J=0
        BETA(1)=0
        DO 200 M=1,I
        J=J+1
        NL=2**J
        L=NL/2
        II=L+1
        DO 100 JJ=1,L
  100   BETA(JJ)=2*BETA(JJ)
        DO 200 JJ=II,NL
  200   BETA(JJ)=BETA(JJ-L)+2
        DO 300 JJ=1,NL
        KK=JJ+NL
  300   BETA(KK)=BETA(JJ)+1
        LL=2*NL
        N=2**GAMA
        PN=N
        I=N/2
        DO 400 K=1,I
        PBETA=BETA(K)
```

```
  400 WBETA(K)=CMPLX(COS(DPI*PBETA/PN),SIN(DPI*PBETA/PN))
      READ(5,1200)(YA(J),J=1,N)
      DO 500 L=1,GAMA
      LGAMA=GAMA-L
      MM=2**LGAMA
      KK=2**(L-1)
      DO 500 K=1,KK
      DO 500 M=1,MM
      COEF1=M+(K-1)*2*MM
      COEF2=M+(2*K-1)*MM
      YA(COEF2)=YA(COEF2)*WBETA(K)
      YA(COEF1)=YA(COEF1)+YA(COEF2)
  500 YA(COEF2)=YA(COEF1)-2.*YA(COEF2)
      M=0
      XI=1
      NM=I+1
      I=XI
      DO 600 K=1,I
      MM=BETA(K)+1
      GO TO 700
  550 MM=MM+I
      GO TO 700
  600 CONTINUE
      WRITE(6,1300) (YA(I),I=1,N)
      WRITE(9)(YA(I),I=1,256)
      STOP
  700 M=M+1
      IF(M .GT. MM) GO TO 800
      AUX=YA(MM)/XI
      YA(MM)=YA(M)/XI
      YA(M)=AUX
      IF(M .EQ. 1 .OR. M .EQ. NM) YA(M)=YA(M)/2.
  800 IF(MOD(M,2) .EQ. 0) GO TO 600
      GO TO 550
 1000 FORMAT(I5)
 1100 FORMAT(26X,16I5)
 1200 FORMAT(200(24F3.0/))
 1300 FORMAT(9X,2E44.8)
      END
HOLL
FILE  6=NOFILE,UNIT=PRINTER,UNLABELED
FILE  9=FRANCO,UNIT=DISK,LOCK,AREA=2000,RECORD=192
C     CNAE  ** SECA ** FRANCO   25/11/68
C     SPECTRAL  ANALYSIS
C     PN = NUMBER OF POINTS OF THE FOURIER ANALYSIS AND AM = NUMBER
C     OF LAGS USED IN AN EQUIVALENT POWER SPECTRAL ANALYSIS THROUGH
C     AUTOCORRELATION.
      DIMENSION PEPSI(100),Z(512)
      READ(5,101) PN,AM
      PI=3.141592
      IM=1
      AI=0.
      IF(PN-1334.)2,2,1
    1 L=(PN-1.)*0.0015
      GO TO 3
    2 L=(PN-1.)/AM
    3 AL=L
      DELKA=4./(2.*AL)
    5 IF(AI.NE.AL) GO TO 9
      PEPSI(IM)=0.25
      GO TO 6
```

```
    9 U=(AI-AL)*DELKA
      IF(U.NE.1.) GO TO 10
      PEPSI(IM)=0.125
      GO TO 6
   10 G=PI*U
      D=G*(1.-U**2)*4.
      PEPSI(IM)=SIN(G)/D
      IF(AI-2.*AL)6,6,70
    6 IM=IM+1
      AI=AI+1.
      GOTO 5
   70 N=PN
      READ(9)(Z(J),J=1,N)
      DO 200 J=1,N,2
      M=(J+1)/2
  200 Z(M)=Z(J)**2+Z(J+1)**2
      Z(1)=0.
      LL=2*L
      NI=N/2
      DO 300 M=1,NI
      KK=NI-M+1
  300 Z(KK+L)=Z(KK)
      DO 350 J=1,L
      Z(J)=0.
  350 Z(NI+J+L)=0.
      DO 400 K=1,NI
      IL=LL+1
      A=0.
      DO 370 IM=1,IL
      N=IM+K-1
  370 A=A+Z(N)*PEPSI(IM)
  400 Z(N-LL)=A
      WRITE(6,1700) (Z(I),I=1,NI)
 1700 FORMAT(9X,2E44.8)
  101 FORMAT(2F4.0)
      END
```

5. — CONCLUSION

It is of interest to point out that a real link has been established between the classical power spectral analysis and the analysis carried out through the Fourier method. In addition the law for resolving the matrix into factors was well established by starting from rearranging the rows of the original matrix according to the flipping of binary bits.

Another interesting feature of this procedure is the fact that expressions (3g) and (3h) are discrete forms of the *convolutions* of $1/2\ H^2(k)$ with $\Phi(k)$ or $\Psi(k)$, as the case may be. In fact, since $H(k) = 0$ for $k < 0$ and $k > m$, the convolution for continuous values of $1/2\ H^2(k)$ is:

$$X(k_p) = \int_0^m \frac{1}{2} H^2(k)\ \Psi(k_p - k)\ dk$$

When the Tukey filter is used Δk does not appear in the discrete forms because $\Delta k = 1$ in such cases.

We are now in a position to foresee a new development of this subject so far as tidal analysis is concerned. In actual fact, it is not usual to take

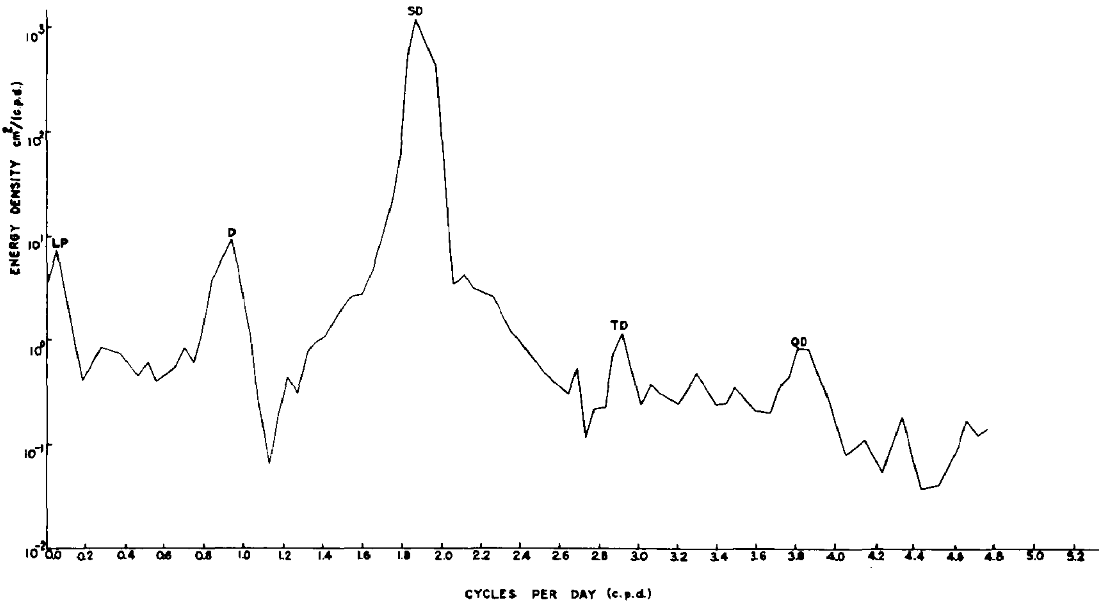

FIG. 5A. — Power Spectrum — Aratū Bay.

advantage of the Fourier analysis used to obtain the power spectrum to compute tidal harmonic constants. However, we believe that either the Myazaki or the Cartwright-Catton method may be used to "adjust" the Fourier analysis to the angular frequency of the astronomical constituents in order to find these constants. If so, we will be able to "fish" the needed harmonic terms from among those given by the Fourier analysis. The only objection is that the span is tied to a power of 2 and not to a classical multiple of one lunation. We know, however, that some least square analyses have been effected with no regard paid to the conventional spans and that the results were shown to be correct. Hence we hope to find an economical solution for avoiding heavy supplementary computations in order to arrive at the harmonic constants from the Fourier analysis itself, such as it is used to obtain the power spectrum.

It has been shown that the least possible number of hours to be analysed to obtain accurate results is 1 334. However, some useful information can be extracted from shorter records. In fact, we effected the analysis of a very short record ($512 = 2^9$ points) taken at Aratū Bay (Brazil) by using the filter at its narrowest possible width. The result (see fig. 5A) showed very clearly the maxima corresponding to 0, 1, 2, 3 and 4 c.p.d. The relative importance of the clusters is about the same as that obtained with the conventional harmonic analysis by the Tidal Institute method for a 32-day span. It is interesting to note that two spikes can be seen, corresponding respectively to the third and fourth diurnal species, which are particularly small (about 2 cm).

It should be pointed out that the resolution for such a short span is too poor to separate adequately numerous constituents of the same species. But the species themselves are very well enhanced. In many cases this is very useful information. In order to obtain a good separation of the constituents it is necessary to analyse a six-month record (4 096 points), especially if we desire to identify new shallow water constituents [3].

ACKNOWLEDGEMENTS

I am very much indebted to Mr. Bernard ZETLER who guided my first steps in power spectrum analysis and who has provided the great majority of papers on which this article is based.

I must also express my gratitude to Engs. Eduardo W. BERGAMINI and Fred Z. PADILHA for their valuable assistance in solving some difficult points on this subject. Eng. PADILHA is also the author of the computer program which elegantly completes my modest work.

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