BOOK REVIEW

MATHEMATICAL GEODESY

Martin HOTINE, ESSA Monograph 2; U.S. Department of Commerce, Environmental Science Services Administration, Washington, D.C.; xvi + 416 pages, 1969; $ 5.50.

Mathematical Geodesy is probably the outstanding mathematical contribution to the geodetic literature of this century and is the culmination of the efforts of Martin HOTINE to "free geodesy from its centuries-long bondage in two (curvilinear) dimensions". Since Cartesian coordinates in three dimensional space are not suitable for all geodetic processes, one must necessarily introduce more general curvilinear systems which, in treating their associated differential geometry, require recourse to the tensor calculus including vector calculus in index notation. (The basic mathematical framework involves Riemannian space with its intrinsic curvature tensor.)

The importance of the tensor analysis, quite apart from the generality of the index notation and the application of the concepts of curved space in n-dimensions to three dimensional space, is that a tensor equation, true in one coordinate system, is true in any coordinate system. "The fact is of fundamental importance in all applications of the subject, particularly the physical applications, because a physical law must, from its very nature, be independent of a man-chosen coordinate system and so is best expressed in tensor form".

Thus the emphasis throughout the text is the application of the tensor calculus to the problems of mathematical geodesy, particularly to the differential geometry of surfaces and curves on surfaces such as the surfaces used in approximating the shape of the earth and the associated geopotential field. Perhaps a more descriptive title for HOTINES's book would read : "Applications of the tensor calculus to mathematical geodesy, differential geometry, and potential theory". *But Hotine always had in mind the applications to mathematical geodesy.* In one of his letters (to the reviewer dated January 26, 1965) he said, "I have now proved conclusively that any family of surfaces generated by a continuous scalar can be members of a triple orthogonal system and I shall be producing a paper on this and other matters for the IAG Symposium on Mathematical Geodesy in Turin in April. Yet, A.R. Forsyth and indeed all the standard texts say this is not so. The effects are quite considerable." In fact Hotine's book includes, in refined and condensed form, his principal precursive research papers which were : "Metric Properties of the Earth's Gravitational Field", "Geodetic Coordinate Systems", these two papers presented to the IAG, Toronto 1957; "A Primer of

Non-Classical Geodesy", presented to the IAG Symposium, Venice 1959; "The Third Dimension in Geodesy", presented to the IAG, Helsinki, 1960; "Triply Orthogonal Coordinate Systems", "Geodetic Applications of Conformal Transformations in Three Dimensions", both papers presented to the IAG Symposium on mathematical geodesy, Turin, 1965.

Part I

HOTINE's *Mathematical Geodesy* is arranged in three parts with an extensive preface, an appended index of symbols, a 45-page summary of formulas, and a general index. *Part I* (*Chapters 1-11*, 66 pages) is actually a condensed introduction of vector analysis in index notation and tensor calculus "for present and foreseeable future geodetic applications". The treatment is characterized by clear exposition of useful formulas, applicable to any coordinate system, but with intentionally less emphasis on the rigorous proofs since "the geodesist, who has to keep up to date in many other areas, is prepared to take much on trust, and is able to do so because he deals only with such well-behaved functions as Newtonian potentials in free space or with very regular functions suitable as coordinates".

Chapter 1 gives a brief introduction to general Cartesian vectors with definitions of covariant and contravariant components relative to a non-rectangular coordinate system, the summation convention for a repeated index, the scalar product, vectors in curvilinear coordinates, and transformation of vectors.

In *Chapter 2* the tensor is defined and general rules for operation with tensors are given (such as contraction) with a discussion of tests for tensor character. Also presented are the associated metric tensor, the permutation symbols in two and three dimensions, the generalized Kronecker deltas and vector products.

Chapter 3 is concerned with the covariant differentiation of a tensor which involves the two sets of differentials of the metric tensor known as the Christoffel symbols of first and second kinds. Covariant derivatives are defined and the rules given for covariant differentiation. Differential invariants in generalized coordinates are discussed, such as the Laplacian which for a general vector is the divergence. The generalized equation for the curl of a vector is also given.

A discussion of the intrinsic properties of curves in two and three dimensions is given in *Chapter 4*. Expressions are included for properties such as vector curvature, first curvature, second curvature (torsion) and the Frenet-Serret formulas.

Chapter 5 deals with the intrinsic curvature of space and defines the curvature tensor, locally Cartesian systems, special forms of the curvature tensor, Riemannian curvature, and curvature in two dimensions.

The extrinsic properties of surfaces is the topic of *Chapter 6*, i.e., the forms of surface equations, the metric equations, the metric tensors, surface vectors, the unit normal, surface covariant derivatives, the Gauss equations and Gaussian curvature, the Weingarten equations, and the Mainardi-Codazzi equations.

In *Chapter 7* we find extrinsic properties of surface curves such as tangent vectors, curvature, torsion, curvature invariants, and the principal curvatures.

Chapter 8 discusses additional extrinsic properties of curves and surfaces including the contravariant fundamental forms, covariant derivatives of the fundamental forms, relation between surface and space tensors with the extension to curved space.

Chapter 9 presents the elements of area and volume, surface and contour integrals, volume and surface integrals.

Chapter 10 describes the conformal transformation of space presenting the metrical relations, the curvature tensor in three dimensions, transformation of tensors, curvature and torsion of corresponding lines, transformation of surface normals, transformation of surfaces, geodesic curvatures, extrinsic properties of corresponding surfaces in conformal space and the Gauss-Bonnet theorem.

Spherical representation is defined in *Chapter 11*, with the fundamental forms of the surfaces, corresponding surface vectors, the principal directions, scale factor and directions referred to the principal directions, Christoffel symbols, and the representation of a family of surfaces.

Comments on part I

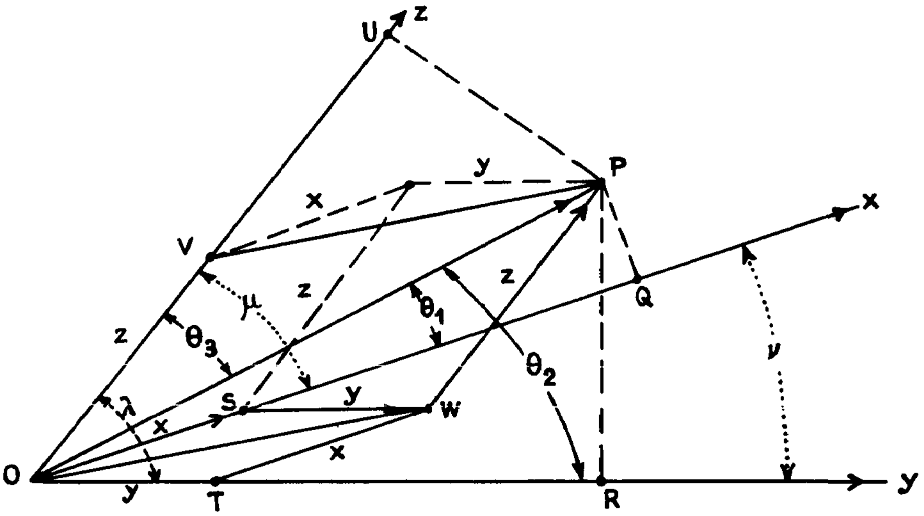
The breadth of the coverage in Part I is clearly exhibited from the above account by chapter of the contents, and 315 numbered equations (sets of equations) are included. I found it useful, both for the purpose of this review, and for compiling a set of lecture notes, to read each of the 11 chapters of Part I rapidly at first, underlining Hotine's phrases of emphasis. For instance, paragraph 39, page 15 : "The equations in Equations 2-32 are of particular importance"; paragraph 7, page 44 : "The main advantage of the formulas in this section is"; paragraph 3, page 63 : "In later applications we shall find this important", etc. On second reading these underlined phrases and paragraphs provided a thread of formulas through each chapter, and these formulas were checked by making the substitutions and performing the operations as directed by Hotine. No errors were noted although only a few of the 315 equations of Part I were actually checked.

Certainly Hotine has fully anticipated in Part I the "present and foreseeable future geodetic applications" of the tensor calculus.

In the discussion of vectors, Part I, Chapter 1, paragraph 3, we find the statement "Alternatively we could specify the vector completely by taking *the differences* in coordinates OS, OT as components, which we shall call the contravariant components". Since these are scalar components, the *contravariant components are actually the scalar coordinates OS, OT of the point P* in the oblique reference frame (see Hotine's Figure 1, page 3).

In paragraph 5 of Chapter 1, we find that "The reader with an inclination for spherical trigonometry can verify that equation 1.04 holds equally well in three dimensions". As mentioned before, in preparing this review it appeared useful to compile some elaborative notes which, based on Hotine's text, could form the basis for a course in the subject. Although Hotine's elaboration is certainly technically adequate, as he states (Preface) "The treatment in Part I, necessarily compressed in a book which is required to cover in outline the entire ground of theoretical geodesy, may prove too difficult for the beginner". It seemed desirable to arrive at the three dimensional form of equations 1.01-1.05 first and then to show the planar model as a limiting case without explicitly introducing spherical trigonometry. *This alternative was certainly considered by Hotine and probably rejected because of space limitations.* It is presented here as being perhaps useful to others who may be constructing a course based on Hotine's text. Hotine's notation has been followed as closely as possible in the following three dimensional development of equations 1.01-1.05.

The following figure is the three dimensional analog of Hotine's Figure 1, page 3.



$$\lambda = \langle zOy,\ \mu = \langle zOx,\ \mu = \langle xOy$$

$$\theta_1 = \langle POx,\ \theta_2 = \langle POy,\ \theta_3 = \langle POz$$

Let $\mu = \lambda = \pi/2$. Then $z = OV = PW$ is perpendicular to the xy-plane. Now let $z = 0$, then $\theta_3 = \pi/2$, P will coincide with W, and $\nu = \theta_1 + \theta_2$. We have then Figure 1, Hotine, page 3.

From the figure we have :

$$\begin{aligned} x &= l^1 = OS \\ y &= l^2 = OT \\ z &= l^3 = OV \end{aligned} \tag{1}$$

These are the contravariant scalar components of vector $\overrightarrow{OP}$, the coordinates of P referred to the oblique triaxial reference frame.

$$\begin{aligned} l_1 &= OQ = OP \cos\theta_1 = OS + SQ = l^1 + SQ \\ l_2 &= OR = OP \cos\theta_2 = OT + TR = l^2 + TR \\ l_3 &= OU = OP \cos\theta_3 = OV + VU = l^3 + VU \end{aligned} \tag{2}$$

These are the covariant scalar components of the vector $\overrightarrow{OP}$, the orthogonal projections of OP upon the coordinate axes.

Now $\overrightarrow{OP} = \overrightarrow{x} + \overrightarrow{y} + \overrightarrow{z}$. If we consider the projection of OP on Ox, Oy, Oz, OP — it is equal, in each case, to the sum of the projections of OS, SW, WP as projected on Ox, Oy, Oz, OP. We get then the following four scalar equations:

$$\begin{aligned} OQ &= OP \cos\theta_1 = x + y\cos\nu + z\cos\mu \\ OR &= OP \cos\theta_2 = x\cos\nu + y + z\cos\lambda \\ OU &= OP \cos\theta_3 = x\cos\mu + y\cos\lambda + z \\ OP &= |\overrightarrow{OP}| = x\cos\theta_1 + y\cos\theta_2 + z\cos\theta_3 \end{aligned} \tag{3}$$

$$\begin{aligned} l_1 &= \text{OP} \cos\theta_1 = l^1 + l^2 \cos\nu + l^3 \cos\mu \\ l_2 &= \text{OP} \cos\theta_2 = l^1 \cos\nu + l^2 + l^3 \cos\lambda \\ l_3 &= \text{OP} \cos\theta_3 = l^1 \cos\mu + l^2 \cos\lambda + l^3 \\ \text{OP} &= |\overrightarrow{\text{OP}}| = l^1 \cos\theta_1 + l^2 \cos\theta_2 + l^3 \cos\theta_3 \end{aligned} \tag{4}$$

From the first three of equations (4) we have :

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{pmatrix} \begin{pmatrix} l^1 \\ l^2 \\ l^3 \end{pmatrix} \quad \text{or} \quad l_i = a_{ij}\, l^j \;\; i, j = 1, \ldots 3 \tag{4a}$$

and where

$$a_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{pmatrix}$$

$$\text{whence } a_{ij} = a_{ji} \text{ (symmetric)}, a_{ii} = a_{jj} = 1, \text{ and } D = |a_{ij}|. \tag{5}$$

By multiplying both sides of the last of equations (4) by OP, we have :

$$\begin{aligned} \text{OP}^2 &= l^1 (\text{OP} \cos\theta_1) + l^2 (\text{OP} \cos\theta_2) + l^3 (\text{OP} \cos\theta_3) \\ &= l^1 l_1 + l^2 l_2 + l^3 l_3 = l^r l_r,\; r = 1, \ldots 3. \end{aligned} \tag{5a}$$

We now eliminate x, y, z, OP among the four equations of (3) to obtain *the relation which must be satisfied by the direction cosines of any vector in the oblique reference system.* The eliminant is merely the determinant of fourth order, in the coefficients of x, y, z, OP in equations (3), placed equal to zero :

$$\overbrace{\quad\quad}^{D}$$

$$\begin{vmatrix} 1 & \cos\nu & \cos\mu & \cos\theta_1 \\ \cos\nu & 1 & \cos\lambda & \cos\theta_2 \\ \cos\mu & \cos\lambda & 1 & \cos\theta_3 \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 & 1 \end{vmatrix} = 0. \tag{6}$$

Expanding (6), making use of elementary operations on determinants such as interchanging rows and columns without changing their relative order, we may write (6) as :

$$\cos\theta_1 \begin{vmatrix} \cos\theta_1 & \cos\nu & \cos\mu \\ \cos\theta_1 & 1 & \cos\lambda \\ \cos\theta_3 & \cos\lambda & 1 \end{vmatrix} + \cos\theta_2 \begin{vmatrix} 1 & \cos\theta_1 & \cos\mu \\ \cos\nu & \cos\theta_2 & \cos\mu \\ \cos\mu & \cos\theta_3 & 1 \end{vmatrix} + \cos\theta_3 \begin{vmatrix} 1 & \cos\nu & \cos\theta_1 \\ \cos\nu & 1 & \cos\theta_2 \\ \cos\mu & \cos\lambda & \cos\theta_3 \end{vmatrix} = D$$

or

$$D_1 \cos\theta_1 + D_2 \cos\theta_2 + D_3 \cos\theta_3 = D \tag{7}$$

where D is the symmetric determinant given by (5), indicated internally in (6), and the meanings of D_1, D_2, D_3 are apparent.

If we multiply both sides of (7) by OP^2/D the equation may be written :

$$(OP\cos\theta_1)(OP\,D_1/D) + (OP\cos\theta_2)(OP\,D_2/D) + (OP\cos\theta_3)(OP\,D_3/D) = OP^2 \qquad (8)$$

and comparison of (8) with (1) and (5a) shows that the *contravariant vector scalar components* are :

$$l^1 = x = OS = OP\,D_1/D,\ l^2 = y = OT = OP\,D_2/D,\ l^3 = z = OV = OP\,D_3/D \qquad (9)$$

The same result could have been obtained by solving directly for l^1, l^2, l^3 in the first three of equations (4).

If we have a second vector $\overrightarrow{OL}$, with direction cosines $\cos\bar\theta_1$, $\cos\bar\theta_2$, $\cos\bar\theta_3$, which makes an angle θ with the vector $\overrightarrow{OP}$ we can, in the same coordinate system, write directly the covariant and contravariant scalar components from (4) and (9) :

$$\begin{aligned} m_1 &= OL\cos\bar\theta_1 = m^1 + m^2\cos\nu + m^3\cos\mu \\ m_2 &= OL\cos\bar\theta_2 = m^1\cos\nu + m^2 + m^3\cos\lambda \\ m_3 &= OL\cos\bar\theta_3 = m^1\cos\mu + m^2\cos\lambda + m^3 \end{aligned} \qquad (10)$$

$$m^1 = OL\,\bar D_1/D,\ m^2 = OL\,\bar D_2/D,\ m^3 = OL\,\bar D_3/D, \qquad (11)$$

where D is the symmetric determinant as given in (5), and $\bar D_1$, $\bar D_2$, $\bar D_3$ are the same as D_1, D_2, D_3 of (7) except for the direction cosines ($\cos\bar\theta_1$, $\cos\bar\theta_2$, $\cos\bar\theta_3$) of OL.

If we project OP on OL we have :

$$OP\cos\theta = x\cos\bar\theta_1 + y\cos\bar\theta_2 + z\cos\bar\theta_3. \qquad (12)$$

If we now eliminate x, y, z, OP among the four equations consisting of the first three of (3) and (12), which again is done by placing the fourth order determinant of the coefficients equal to zero, we have :

$$\overbrace{\begin{vmatrix} 1 & \cos\nu & \cos\mu & \cos\theta_1 \\ \cos\nu & 1 & \cos\lambda & \cos\theta_2 \\ \cos\mu & \cos\lambda & 1 & \cos\theta_3 \\ \cos\theta_1 & \cos\bar\theta_2 & \cos\bar\theta_3 & \cos\bar\theta \end{vmatrix}}^{D} = 0. \qquad (13)$$

Note that (13) will become (6) when $\theta = 0$, and $\bar\theta_i = \theta_i$, $i = 1,..3$, as would be expected. Expanding (13) by elements of the last row and then by elements of the last column, we may write :

$$\begin{aligned} D\cos\theta &= D_1\cos\bar\theta_1 + D_2\cos\bar\theta_2 + D_3\cos\bar\theta_3 \\ &= \bar D_1\cos\theta_1 + \bar D_2\cos\theta_2 + \bar D_3\cos\theta_3. \end{aligned} \qquad (14)$$

Now multiply both sides of (14) by OL.OP/D to get :

$$\begin{aligned} OL\,OP\cos\theta &= (OL\cos\bar\theta_1)(OP\,D_1/D) + (OL\cos\bar\theta_2)(OP\,D_2/D) + (OL\cos\bar\theta_3)(OP\,D_3/D) \qquad (15) \\ &= (OP\cos\theta_1)(OL\,\bar D_1/D) + (OP\cos\theta_2)(OL\,\bar D_2/D) + (OP\cos\theta_3)(OL\,\bar D_3/D). \end{aligned}$$

From (9), (10), ; (2), (11) we have for (15) that

$$OL\ OP \cos\theta = l^1 m_1 + l^2 m_2 + l^3 m_3 = l_1 m^1 + l_2 m^2 + l_3 m^3 = l^r m_r \qquad r = 1,..3 \qquad (16)$$

which is *Hotine's equation 1.05, page 4, i.e. the scalar product* of the vectors $\overrightarrow{OL}$, $\overrightarrow{OP}$.

From (4a), when the three coordinate axes are mutually orthogonal, $\cos\mu = \cos\nu = \cos\lambda = 0$, and the covariant and contravariant scalar components are identical. For the planar case, we have, $\mu = \lambda = \theta_3 = \pi/2$, $z = l^3 = 0$,

$$z = l^3 = 0,$$

$$\nu = \theta_1 + \theta_2, \ D = \sin^2(\theta_1 + \theta_2), \ D_1 = \cos\theta_1 - \cos\theta_2 \cos(\theta_1 + \theta_2), \ D_3 = 0$$

$$D_2 = \cos\theta_2 - \cos\theta_1 \cos(\theta_1 + \theta_2), \ D_1/D = \sin\theta_2 / \sin(\theta_1 + \theta_2),$$

$$D_2/D = \sin\theta_1 / \sin(\theta_1 + \theta_2),$$

and equations (2), (9), (5a), (16) become respectively Hotine's equations 1.01-1.04.

Part II

Part II (Chapters 12-19, 74 pages) *is concerned "with coordinate systems of special interest in geodesy".*

Chapter 12, the properties of a general class of three dimensional systems are developed from a single-valued continuous and differentiable scalar function of position in three dimensional space which serves as one coordinate while the other two coordinates are defined by the magnitude and direction of the gradient.

If N is a scalar function, then points having a particular value of N, say C, will lie on the surface N = C and as C varies a family of such surfaces is generated. Since N is specified throughout some region of space, the magnitude n, and direction V_r of its gradient N_r also specified since N_r, n, V_r satisfy the basic gradient equation $N_r = nV_r$. But the direction of V_r, in relation to three fixed Cartesian axes in flat space, will define two independent scalars, which can take the form of longitude ω, and latitude φ. (Every sufficiently small portion of a Riemann space is flat and is Euclidean if the metric, or linear element, is positive definite which is the assumption, see Hotine, page 5. This flat space may then be represented by a Cartesian coordinate system. Assuming the N-systems share a common Cartesian system with an axis parallel to the physical axis of the earth's rotation ensures that the space remains flat during coordinate transformations between the N-systems, see Hotine, page 131). The coordinate system is then actually (ω, φ, N), but generated by the scalar function N, the magnitude and direction of its section (n and V_r). Thus the position of a point in space is the intersection of three surfaces; one from each of the ω, φ, and N families, over which each of the three coordinates has an assigned value. But unlike the fixing of points in space by the intersection of three planes, as in Cartesian three dimensional coordinates, the coordinate surfaces ω, φ, and N being curved, their lines of intersection are curved, and hence the coordinates are curved. By treating the subject in this way, *Chapter 12* also provides the geometry of the gravitational field so that the scalar function N will have a physical meaning (for instance in the treatment of the gravitational potential as found in Chapter 20 where the N surfaces become the equipotential surfaces, and n is the gravitational force "g").

A general discussion is given in *Chapters 13 and 14* concerning transfer of point functions from a point in space to a particular surface provided by the scalar function N (the rigorous counterpart of several operations of classical geodesy). The methods of transfer along the isozenithals (the scalar coordinate lines) and along the normals to the scalar surfaces are given for each coordinate system introduced. The process, intimately connected with the Gaussian spherical representation as discussed in Chapter 11, is developed in this context and extended to nonspherical representation in Chapter 13. In *Chapters 15 through 18*, the scalar function is restricted to provide simpler systems whose properties can then be derived at once from the general results of Chapter 12. *Chapter 19* deals with transformations between members of the general class for different values of the scalar function N.

Again the breadth of treatment in Part II is overwhelming with its 510 numbered equations (sets of equations), only a few of which were checked. No errors were detected.

Part III

Part III (*Chapters 20-30*, 208 pages) is concerned "with the main geodetic application of the mathematics in Parts I and II".

In *Chapter 20* is shown how the geometry of the Newtonian gravity field can be treated as a special case of a (ω, φ, N) coordinate system, as described in Part II, where N is the potential, the N-surfaces are equipotentials, and the form of N is restricted by the Newtonian law of gravitation. The discussion includes a summary of mechanical principles involved, the Poisson equation, the flux of gravitional force, and measurement of the parameters.

Chapter 21 discusses the expansion of the potential in spherical harmonics and considers generalized harmonic functions, the Newtonian potential at distant, near, and internal points; rotation of the earth, analytic continuation, alternative expression of the external potential, Maxwell's theory of poles, representation of gravity, curvatures of the field, determination of the potential in spherical harmonics, the magnetic analogy.

Chapter 22 treats the expansion of the potential in spheroidal harmonics and discusses the coordinate system, the meridian ellipse, spheroidal coordinates, the potential in spheroidal coordinates, the mass distribution, convergence, relations between spherical and spheroidal coefficients, the potential at near and internal points, the differential form of the potential.

In *Chapter 23*, gravity field models are presented and a discussion given of symmetrical models, the spheroidal model, the standard potential in spheroidal harmonics and in spherical harmonics, standard gravity on the equipotential spheroid and in space, standard gravity in spherical harmonics, curvatures of the field, and the gravity field in geodetic coordinates.

Chapter 24 is concerned with the geometrical corrections for atmospheric refraction and the discussion includes the laws of refraction, the differential equation of the refracted ray, the spherically symmetrical medium, the geometry of flat curves, arc-to-chord corrections, the geodetic model atmosphere, arc-to-chord corrections — geodetic model, velocity correction, the equation of state, index of refraction (optical and micro wave lengths), measurement of refractive index, curvature, lapse rates, astronomical refraction, measurement of refraction.

In *Chapter 25*, a discussion is given of the use of the line of observation in three dimensional space to replace the several operations of classical geodesy such as the corrections of observed directions for "geoidal tilt" and elevation of the station sighted; the replacement of the two curves of normal section by a spheroidal geodesic and the solution of geodesic triangles on the spheroid of reference. The general equations of the line of observation are given and its representation in geodetic coordinates; Taylor expansion along the line, expansion of the gravitational potential, expansion of geodetic heights, expansion of latitude and longitude, astro-geodetic leveling, deflections by torsion balance measurements.

Chapter 26 is concerned with the internal adjustment of networks. The triangle in space is discussed with variation of position in geodetic coordinates, with observation equations in geodetic coordinates and in Cartesion coordinates. The formation of differential observation equations for most of the usual systems of geodetic measurement is given, including, in some cases, the derivation of finite formulas that may be necessary to provide computed values, differentiation then giving the observation equations. The treatment includes the observations generated by flare triangulation, stellar triangulation, satellite triangulation, lunar observations, and line crossing techniques (such as Hiran).

In *Chapter 27* the external adjustment of networks is discussed and hence the change of spheroid, change of origin, changes of Cartesian axes, change of scale and orientation, extension to astronomical coordinates, adjustment procedure, and figure of the earth.

Chapter 28 is devoted to a rather elaborate treatment of dynamic satellite geodesy as evidenced by the range of subjects treated, namely; the equations of motion (inertial axes and moving axes), first integrals (inertial axes and moving axes), the Lagrangian, the canonical equations, the Kepler ellipse, perturbed orbits, variation of the elements, the Gauss equations, derivatives with respect to the elements, the Lagrange planetary equations, curvature and torsion of the orbit, the Delaunay variables, first integrals of the equations of motion — further general considerations, integration of the Gauss, Lagrange, and canonical equations; differential observation equations — direction and range, range rate; the variational method.

Chapter 29 discusses the integration of gravity anomalies from the Poisson-Stokes approach and covers the following individual topics : surface integrals of spherical harmonics, series expansions, introduction of the standard field, the spherical standard field, Poisson's integral, Stokes' integral, deflection of the vertical, gravity and deflection from Poisson's integral, extension to a spheroidal base surface, Bjerhammar's method, the equivalent spherical layer.

Chapter 30 presents the integration of gravity anomalies, from the Green-Molodenskii approach, where after some general remarks, a discussion is given of: the S-surface in (ω, φ, h) coordinates (the S-surface is either the smoothed topographic surface or the model earth and h is the distance along the normal to coordinate surfaces from the base coordinate surface to the S-surface); application of Green's theorem, potential and attraction of a single layer, potential and attraction of a double layer, the equivalent surface layers, the basic integral equations in geodetic coordinates, the equivalent single layer.

There are 939 numbered equations (sets) in Part III. No errors were detected in the limited number actually checked.

GENERAL REMARKS

Part III dramatically illustrates the power of the tensor calculus, reducing to almost trivial the results of other geodesists who had in their possession, at the time of their studies, much less serviceable mathematical tools. Even so, the magnitude of Hotine's accomplishments is astonishing and while some may regret the omission of, say, the formation of normal equations from the observation equations in adjustment procedures, instrumentation, and statistical analysis of data, the omissions were deliberate since "these matters (formation of normal equations) are not peculiar to geodesy and are best studied in the standard literature". "*An attempt has been made to cover the basic mathematical discipline of geodesy*, excluding such specialized matters as routine computer programs and including only such references to instrumentation and field (or laboratory) procedures as may be necessary to a full appreciation of the underlying theory. The book accordingly bears much the same relation to the whole of geodesy as numerous books entitled "Mathematical Physics" do to the whole of physics". Hotine's categorization of this text is true with *the very large exception* that his *Mathematical Geodesy is unique.*

As a friend of the author, I unashamedly shed some tears in reviewing his last great work. I felt his presence, his genius in every page, his warm, modest, humble, unassuming, energetic, forceful, brilliant personality always open minded to new ideas. As he states (Preface) "It is difficult to make adequate acknowledgement covering a lifetime of study, discussion, and collaboration. The author's main source of inspiration in the subject of this book has been Professor Antonio Marussi of the University of Trieste, not only for range and originality of his ideas but also for continual advice and encouragement." Again (Preface) "To offset what must seem like cavalier treatment, no priority is claimed for any results, although it is believed that some are new, either in content or in presentation." His never failing sense of humor such as (Preface) "Most geodesists shy at the notion of covariance and contravariance, which seems to be the counterpart of the Euclidean pons asinorum..."

It is always the hope of a reviewer to find some minuscule errors in formulas, if only in sign, but a tribute to the meticulous Hotine and to the obviously thorough checking and editing provided by ESSA, none were discovered, although only a small part of the total 1764 numbered equations (sets of equations) included in Parts, I, II and III were actually checked in detail and none of the 1272 numbered equations (sets) printed in the Summary of Formulas was checked. The only typographical errors detected occur on : the last page of the Preface, where a *q* appears for *g* in *distinguished*; page 156, where *of* appears at the end and beginning of the second and third lines following equation 21.012; page 315, Figure 39, where the distance PQ should be indicated as *l*, to conform with the text, and the line PT is supposed to be tangent to the circle at T; page 340, Figure 46, where the minus sign was omitted before the term $m_r dm$.

The book is beautifully bound, the printing superb, and the cost minimal ($ 5.50).

Anyone interested in or involved in theorical geodesy or in potential theory relative to any field of geophysics should read this book, with or without prior knowledge of the tensor calculus. One will gain a knowledge of the "new treatment" even if additional reading may be required to master the details.

Surely no degree in geodesy, particularly no advanced degree in this field, will be awarded henceforth by any university in any nation without a knowledge of the contents of this treatise. It should provide inspiration in the constant quest for dissertation topics since Hotine has indicated, particularly in the applications, several areas of needed research. For instance in analytic continuation (pages 172, 173), Maxwell's theory of poles (pages 178), determination of the potential in spherical harmonics (pages 184), the lapse rates for vapor pressure and temperature (page 221), convergence of the iterative procedure where the gravity disturbance equation is used with boundary equation 29.55 (page 312), topographical smoothing in reduction of gravity and potential measurements to the Model Earth (page 328).

Hotine's Mathematical Geodesy is the great heritage of this century to the world geodetic community, and is clearly destined to become the authoritative mathematical source in geodesy.

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