DETERMINATION OF SHORE AND SHIP POSITIONS FROM VISUAL AND ELECTRONIC MEASUREMENTS AT SEA

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GENERAL

The approach to be taken in this problem utilizes the principles and procedures for the ordinary coastal hydrographic survey. However, it varies considerably in that portions of ordinarily known information and available data are now considered as unobtainable for various reasons. With these limitations imposed, the remaining tools at the disposal of the hydrographer, including high-speed computer processing, are utilized to fill in the missing information and arrive at the desired positional results for the survey.

With this lack of certain usually available data, the accuracy of the ultimately determined positions will necessarily be somewhat less than had all of the supporting information been present. However, in operational situations it often becomes imperative to sacrifice a certain degree of accuracy in order to allow the required mission to be accomplished in an expeditious manner.

The situation to be dealt with in this instance is best applied along a stretch of inaccessible coastline. In this area a requirement exists for a rapid and fairly accurate determination of the positions of prominent objects ashore and water depths along the coast. As an example of such a situation the following military operation and the associated hydrographic requirements are outlined. Water depth information and landmark locations are needed along the coast of a hostile nation so that combatant naval vessels may navigate safely and fix their positions visually in order to provide accurate offshore gunfire support for destruction of enemy fortifications. Accurate inshore soundings are also required to facilitate and insure the navigational safety of anticipated amphibious landings.

The only available nautical charts of the area are old and incomplete. They do not contain sufficient information to enable the navigator to fix his position visually and are practically devoid of sounding information. They include only a sparse few topographic features which could be used as navigation landmarks. The location of all charted information is questionable owing to the poor accuracies of the original surveys. An additional restriction imposed in this situation is that the area cannot be overflown for low level reconnaissance photography for use in planning and executing the survey.

Adjustment computation procedures will be applied to the observed data in order to arrive at the most likely values for the unknown quantities in the problem. Estimations of accuracies for all adjusted values will also be determined.

With sufficient shore control, the mechanical solution using a three-arm protractor normally provides adequate positions for the placement of sounding data on nautical charts. However, with little or no known terrestrial position information this conventional procedure cannot be applied. Also, even with satisfactory visual control ashore, the conventional sextant resection method may not yield results accurate enough to fulfill the most stringent offshore positioning requirements. For these reasons the adjustment procedure which follows has been developed for application to coastal survey situations. Although the use of adjustment computations has seldom been applied in coastal hydrography, procedures have been devised for certain aspects of ocean surveying [1] and geodetic positioning at sea away from the coast [2].

A planimetric approach is taken to this problem since all visual observations, both sextant angles and compass bearings, are normally measured between objects in approximately the same horizontal plane as the observer aboard the survey vessel. When it becomes necessary to measure a sextant angle between two points with a significant difference in elevation, the angle is corrected before being used in plotting or computations by a graphical method based on the formula :

$$\cos \alpha = \frac{\cos O}{\cos h} \tag{1}$$

In this expression α is the horizontal or computed angle, O is the observed inclined angle and h is the angular elevation of the elevated object from the point of observation. The angular elevation h is normally determined by a sextant observation from the survey vessel.

PROBLEM OBSERVATION DATA

It is assumed that an electronic positioning system has been established so as to give suitable coverage in the coastal area to be surveyed. Transmitter sites for the electronic system are established outside the inaccessible area, either in neighboring friendly territory or, if this is not feasible, aboard floating platforms moored at offshore locations as described by ATWOOD [3].

The survey ship, as it proceeds along the coast at a distance of two to five miles from shore, obtains the usual sextant angles to prominent terrestrial objects. If the shore points are at elevations considerably above the height of the observer aboard the ship, the angles are reduced with equation (1) before being entered into the adjustment computation. This preliminary reduction of inclined angles, when necessary, allows the problem to be dealt with planimetrically in all cases.

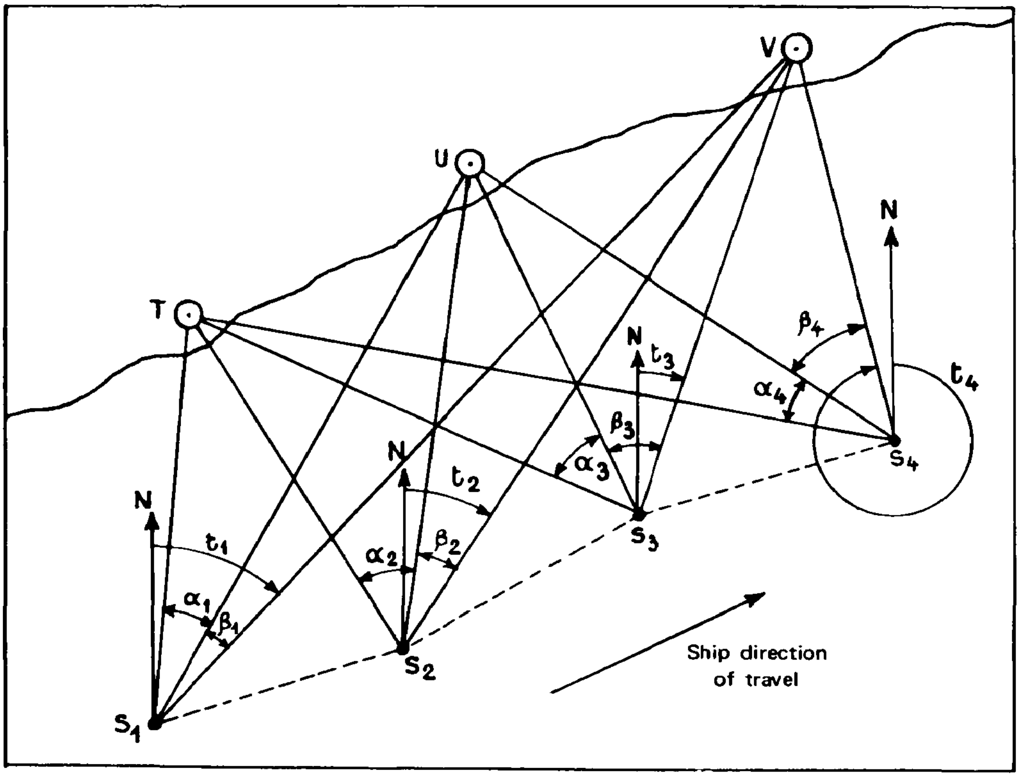

FIG. 1. — Coastal hydrographic survey network for ship observations.

Each time sextant observations are made, a gyro compass bearing is taken to the extreme object in the direction closest to the ship's track. This particular point is chosen since, in most instances, its bearing from the ship will be changing at a slower rate of speed than will bearings of the other shore points. A principal reason that only one such direction is observed for each ship fix position is that normally only a single gyro compass repeater is available for making such a reading.

As the visual data is being obtained, the electronic positioning system coordinates of the ship are recorded at each designated fix time. Depth information is recorded continuously during the operation. Any other desired information such as gravimetric determinations are also made simultaneously with the visual observations. Times are noted for all data readings to permit later correlation of the information.

Referring to figure 1, the points S_1 through S_4 are successive positions of the survey vessel proceeding along the coast. The points T, U and V represent prominent objects along the shoreline such as towers, buildings and natural features between which the sextant angles α_i and β_i are observed. All compass bearings t_i are taken to terrestrial object V.

ADJUSTMENT PROCEDURE

The generalized least squares adjustment method with parameters treated as observables is utilized for the solution of the problem. The matrix notations and adjustment procedure are according to UOTILA [4]. Owing to the linear relationships which exist among the observed quantities and the unknown parameters, and the method used for computing approximations for the parameters, a satisfactory solution should normally be achieved without the necessity of recycling the adjustment. For completeness of the derivation, however, at the end of the general adjustment procedure described in this section and in the detailed equation formations of the following section, the modifications necessary for recycling the adjustment are explained.

Sextant angles and compass bearings taken on the ship compose one set of observations, L_{1_b}. These observations are considered as belonging to the first mathematical structure F_1. The adjusted values of the observable quantities are designated L_{1_a} and the adjusted values of the unknown parameters, the ship and shore positions, are X_a. There is only one observable quantity, a sextant angle or a compass direction, in any F_1 equation.

A second set of observations, L_{2_b}, consists of the successive positions of the survey ship, taken as belonging to the mathematical structure F_2. In this structure, L_{2_a} are the adjusted values of the ship positions, which are also some of the unknown parameters X_a.

The two mathematical structures may be written as :

$$F_1 (L_{1_a}, X_a) = 0 \qquad F_2 (L_{2_a}, X_a) = 0 \tag{2}$$

Through the Taylor series development, equation (2) can be expressed as follows :

$$\frac{\partial F_1}{\partial L_{1_a}} V_1 + \frac{\partial F_1}{\partial X_a} X + F_1 (L_{1_b}, X_0) = 0 \tag{3}$$

$$\frac{\partial F_2}{\partial L_{2_a}} V_2 + \frac{\partial F_2}{\partial X_a} X + F_2 (L_{2_b}, X_0) = 0 \tag{4}$$

where V_1 and V_2 are residual vectors, X_0 denotes the approximations to the parameters and X represents the alterations which will be applied to the approximations to obtain the adjusted values X_a. Using the notations :

$$B_1 = \frac{\partial F_1}{\partial L_{1_a}} \qquad B_2 = \frac{\partial F_2}{\partial L_{2_a}}$$

$$A_1 = \frac{\partial F_1}{\partial X_a} \qquad A_2 = \frac{\partial F_2}{\partial X_a}$$

$$W_1 = F_1 (L_{1_b}, X_0) \qquad W_2 = F_2 (L_{2_b}, X_0)$$

the differential forms (3) and (4) become :

$$B_1 V_1 + A_1 X + W_1 = 0 \tag{5}$$

$$B_2 V_2 + A_2 X + W_2 = 0 \tag{6}$$

Since there is only one observable quantity in any equation of the form F_1, B_1 is an identity matrix. Hence (5) is reduced to :

$$V_1 + A_1 X + W_1 = 0 \tag{7}$$

Further, W_2 for the first cycle of the adjustment will a be a null vector because L_{2_b}, the observed values of the ship coordinates, and the approximate values of the ship coordinates, are the same. Also since there is only one observed quantity, an x or y coordinate of a ship position in any F_2 equation, B_2 is an identity matrix. Therefore (6) reduces to :

$$V_2 + A_2 X = 0 \tag{8}$$

The function which must be minimized in order to fulfill the principle of least squares requirement is thus, using the Lagrange multipliers method :

$$\phi = V_1^T P_1 V_1 + V_2^T P_2 V_2 - 2 K_1^T (V_1 + A_1 X + W_1) - 2 K_2^T (V_2 + A_2 X) \tag{9}$$

K_1 and K_2 are column vectors composed of Lagrange multipliers. P_1 is the weight matrix for sextant angle and compass direction observations, and P_2 is the weight matrix for the parameters treated as observations. Taking the partial derivatives of ϕ with respect to V_1, V_2 and X and setting them equal to zero, the three sets of condition equations which must be fulfilled for all observations and parameters are :

$$\frac{1}{2} \frac{\partial \phi}{\partial V_1} = P_1 V_1 - K_1 = 0 \tag{10}$$

$$\frac{1}{2} \frac{\partial \phi}{\partial V_2} = P_2 V_2 - K_2 = 0 \tag{11}$$

$$\frac{1}{2} \frac{\partial \phi}{\partial X} = - A_1^T K_1 - A_2^T K_2 = 0 \tag{12}$$

The expressions (7), (8) and (10) through (12) are a system of five sets of equations having five unknown vectors : V_1, V_2, K_1, K_2 and X. Since the solution of this problem consists of determining the X vector, the five set system is first reduced to a system of two sets of equations having only two unknown vectors.

The procedure for this reduction is begun by first solving (11) for V_2 and substituting the resulting expression into (8) to give :

$$P_2^{-1} K_2 + A_2 X = 0 \tag{13}$$

Next, (10) is solved for V_1 and the result substituted into (7) giving :

$$P_1^{-1} K_1 + A_1 X = - W_1 \tag{14}$$

Each term in (14) is then multiplied by $A_1^T P_1 W_1$ to obtain :

$$A_1^T K_1 + A_1^T P_1 A_1 X = - A_1^T P_1 W_1 \tag{15}$$

Finally, (12) is solved for $A_1^T K_1$ and the result inserted into (15) yielding :

$$A_1^T P_1 A_1 X - A_2^T K_2 = - A_1^T P_1 W_1 \tag{16}$$

Thus (13) and (16) are the desired system of two sets of equations with unknown vectors K_2 and X.

In (16) the term $A_1^T P_1 A_1$ is the normal equation coefficients array which will be designated N_1. Also, the term $A_1^T P_1 W_1$ is the constant vector of the normal equations, to be designated U. Using these notations (13) and (16) become :

$$P_2^{-1} K_2 + A_2 X = 0 \tag{17}$$

$$N_1 X - A_2^T K_2 = - U \tag{18}$$

Solving (17) for K_2,

$$K_2 = - P_2 A_2 X \tag{19}$$

which, when substituted into (18) gives :

$$N_1 X + A_2^T P_2 A_2 X = - U \tag{20}$$

Factoring out X,

$$(N_1 + A_2^T P_2 A_2) X = - U \tag{21}$$

But each F_2 equation contains only one unknown parameter, an x or y ship position coordinate. Hence the only non-zero elements of A_2 are -1's. Further, by virtue of the F_2 structure, A_2 is a square array with non-zero values occurring only in the main diagonal elements corresponding to ship parameters (see Direction coefficients — Second structure, below). Therefore $A_2^T P_2 A_2$ is merely P_2 with non-zero elements only on the main diagonal at positions where dependence is estimated between ship position coordinates as explained under Weight matrix — Second structure below. In matrix notation then :

$$A_2^T P_2 A_2 = P_2 \tag{22}$$

Substituting (22) into (21) gives :

$$(N_1 + P_2)X = - U \tag{23}$$

The factor $(N_1 + P_2)$ in (23) is known as the generalized normals matrix. Denoting it as N_{gen}, (23) can be rewritten as :

$$N_{gen} X = - U \tag{24}$$

The approximate values X_0 of the shore point coordinates and the ship station coordinates are then combined with the alterations determined from (24) to obtain the adjusted values of both the terrestrial points and the ship station coordinates.

$$X_a = X_0 + X \tag{25}$$

If the magnitudes of the final alterations determined by (24) indicate that the first approximations X_0 are not sufficiently accurate, the adjustment would be recycled. In this situation, the X_a computed from (25) for the first cycle would become the X_0 for the second cycle of the adjustment. The adjustment would then proceed as outlined above with certain modifications.

The W_2 misclosures vector would no longer be null but would instead consist of the differences between the initially computed adjusted x and y ship coordinates and the originally observed values. With this addition, the W_2 vector would be included in equations (8), (9) and (13). Thus the constant vector U would become

$$U = A_1^T P_1 W_1 - P_2 W_2 \tag{26}$$

With the value of U from (26) substituted into (24) the problem is solved as previously explained to obtain the new alterations X. In (25) the finally adjusted parameters computed from the second cycle would consist of the parameters X plus the approximations X_0 which are now the adjusted values computed from the first cycle.

Should an additional recycling be deemed necessary, the adjusted values from the second cycle would be utilized as the approximations for the third cycle and the operation repeated as described above.

FORMATION OF EQUATIONS

The quantities necessary to accomplish the adjustment presented in the previous section are now described, and the expressions which are required in order to perform the indicated computations are developed.

The network parameter notations to be employed are as shown in figure 2. The cartesian coordinates of each ship position S_i are denoted as (x_i, y_i). For each shore point the coordinates correspond to the point concerned. For example, for terrestrial object V the coordinates are $(x_V,$

y_V). For the purposes of this general development it is assumed that there are three terrestrial points and n ship positions.

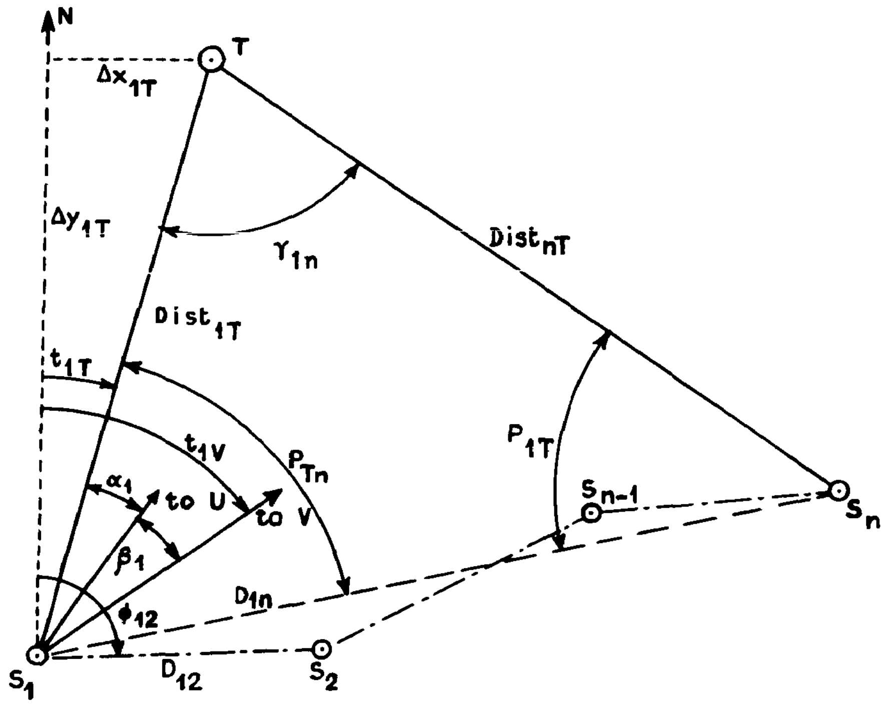

FIG. 2. — Geometric relationships for computation of approximate coordinates of a shore point.

Observed quantities — First structure

The quantities L_{1_b} are the directly observed sextant angles

$$\alpha_{1_b},\ \beta_{1_b},\ \alpha_{2_b},\ \beta_{2_b},\ \ldots\ \alpha_{n_b},\ \beta_{n_b}$$

and the directly observed compass bearings

$$t_{1_b},\ t_{2_b},\ \ldots\ t_{n_b}$$

to the terrestrial point closest to the direction of ship's travel.

Observed quantities — Second structure

The quantities L_{2_b} are the ship position coordinates as determined by an electronic positioning system.

$$(x_{1_b}, y_{1_b}),\ (x_{2_b}, y_{2_b}),\ \ldots\ (x_{n_b}, y_{n_b})$$

Approximate values of parameters — First structure

These parameters are the approximate coordinates of the shore points. They are computed in the following manner. With the observed coordinates of ship positions S_1 and S_n, the coordinates of point T are determined to illustrate the procedure (see figure 2). Using plane geometric and trigonometric relationships, the approximations for the distances between ship positions and for the angles P and γ are calculated. With these values, the approximate distance from the ship positions to T are computed according to the law of sines. The distance from S_1 to T is thus given by :

$$\text{Dist}_{1T} = \frac{\sin P_{1T} \times D_{1n}}{\sin \gamma_{1n}} \qquad (27)$$

With the observed compass bearing t_{1V} to terrestrial point V and the observed sextant angles α_1 and β_1, the approximate azimuth t_{1T} is computed.

$$t_{1T} = t_{1V} - (\alpha_1 + \beta_1) \qquad (28)$$

the horizontal and vertical increments Δx_{1T} and Δy_{1T} are then written as :

$$\begin{aligned} \Delta x_{1T} &= \text{Dist}_{1T} \times \sin t_{1T} \\ \Delta y_{1T} &= \text{Dist}_{1T} \times \cos t_{1T} \end{aligned} \qquad (29)$$

The approximate coordinates of point T are thus expressed as :

$$\begin{aligned} x_{T_0} &= x_{1_0} + \Delta x_{1T} \\ y_{T_0} &= y_{1_0} + \Delta y_{1T} \end{aligned} \qquad (30)$$

In similar manner the approximate coordinates of points U and V are determined.

Approximate values of Parameters — Second structure

For ship coordinates, these parameters are the observed quantities L_{2_b} described above.

Computed values of sextant angles

The general expression for a computed sextant angle formed between shore points J and K, subtended at ship position S_i, is given by :

$$\alpha_i = \tan^{-1}\left[\frac{x_J - x_i}{y_J - y_i}\right] - \tan^{-1}\left[\frac{x_K - x_i}{y_K - y_i}\right] \qquad (31)$$

Computed values of compass bearings

The general expression for a computed compass bearing from ship position S_i to shore point J is :

$$t_i = \tan^{-1}\left[\frac{x_J - x_i}{y_J - y_i}\right] \tag{32}$$

First mathematical structure

This is the model of the form $F_1(L_{1_a}, X_a) = 0$. An expression is developed for each sextant angle and compass bearing.

$$\begin{aligned}
\tan^{-1}\left[\frac{x_{U_a} - x_{1_a}}{y_{V_a} - y_{1_a}}\right] - \tan^{-1}\left[\frac{x_{T_a} - x_{1_a}}{y_{T_a} - y_{1_a}}\right] - \alpha_{1_a} &= 0 \\
\tan^{-1}\left[\frac{x_{V_a} - x_{1_a}}{y_{V_a} - y_{1_a}}\right] - \tan^{-1}\left[\frac{x_{U_a} - x_{1_a}}{y_{U_a} - y_{1_a}}\right] - \beta_{1_a} &= 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
\tan^{-1}\left[\frac{x_{U_a} - x_{n_a}}{y_{U_a} - y_{n_a}}\right] - \tan^{-1}\left[\frac{x_{T_a} - x_{n_a}}{y_{T_a} - y_{n_a}}\right] - \alpha_{n_a} &= 0 \\
\tan^{-1}\left[\frac{x_{V_a} - x_{n_a}}{y_{V_a} - y_{n_a}}\right] - \tan^{-1}\left[\frac{x_{U_a} - x_{n_a}}{y_{U_a} - y_{n_a}}\right] - \beta_{n_a} &= 0 \\
\tan^{-1}\left[\frac{x_{V_a} - x_{1_a}}{y_{V_a} - y_{1_a}}\right] - t_{1_a} &= 0 \\
\vdots \qquad\qquad & \vdots \\
\tan^{-1}\left[\frac{x_{V_a} - x_{n_a}}{y_{V_a} - y_{n_a}}\right] - t_{n_a} &= 0
\end{aligned} \tag{33}$$

Second mathematical structure

This model, the parameters treated as observations, is expressed as $F_2(L_{2_a}, X_a) = 0$. The shore point parameters are only treated as observations in the sense of being observations with zero weight. Thus shore coordinate functions do not enter this structure. To present the ship parameter portion of the structure in a more meaningful manner, the relationships $L_{2_a} = L_{2_b} + V_2$ and (25) are substituted permitting the observation equations to be written. They take the following form :

$$
\begin{aligned}
x_{1_b} + v_{2_1} - \Delta x_1 &= 0 \\
y_{1_b} + v_{2_2} - \Delta y_1 &= 0 \\
x_{2_b} + v_{2_3} - \Delta x_2 &= 0 \\
y_{2_b} + v_{2_4} - \Delta y_2 &= 0 \\
\vdots \quad & \\
x_{n_b} + v_{2_{2n-1}} - \Delta x_n &= 0 \\
y_{n_b} + v_{2_{2n}} - \Delta y_n &= 0
\end{aligned}
\tag{34}
$$

Misclosure vector — First structure

The misclosures W_1 are obtained by evaluating the mathematical model F_1 with L_{1_b} and X_0. To illustrate the procedure, the expressions for the first sextant angle and the first compass direction are shown below. Since there are three observations at each of the n ship positions, W_1 consists of a total of $3n$ elements.

$$\rho' \tan^{-1}\left[\frac{x_{U_o} - x_{1_o}}{y_{U_o} - y_{1_o}}\right] - \rho' \tan^{-1}\left[\frac{x_{T_o} - x_{1_o}}{y_{T_o} - y_{1_o}}\right] - \alpha'_{1_b} = w_{1_1} \tag{35}$$

$$\rho' \tan^{-1}\left[\frac{x_{V_o} - x_{1_o}}{y_{V_o} - y_{1_o}}\right] - t'_{1_b} = w_{1_{2n+1}} \tag{36}$$

Misclosure vector — Second structure

As explained previously under Adjustment Procedure, the W_2 matrix is null for the first cycle since L_{2_b} and X_0 are the same quantities. However, for the recycled adjustment for ship parameters, (x_{i_0}, y_{i_0}) are the adjusted coordinates computed from the first cycle and (x_{i_b}, y_{i_b}) are the observed values from the electronic positioning system. In this case, that portion of W_2 corresponding to ship coordinates takes the following form :

$$
\begin{aligned}
x_{1_o} - x_{1_b} &= w_{2_1} \\
y_{1_o} - y_{1_b} &= w_{2_2}
\end{aligned}
$$

(Continued on the following page)

$$
\begin{aligned}
x_{2_o} - x_{2_b} &= w_{2_3} \\
y_{2_o} - y_{2_b} &= w_{2_4} \\
\cdot \quad & \cdot \quad \cdot \\
\cdot \quad & \cdot \quad \cdot \\
\cdot \quad & \cdot \quad \cdot \\
x_{n_o} - x_{n_b} &= w_{2_{2n-1}} \\
y_{n_o} - y_{n_b} &= w_{2_{2n}}
\end{aligned}
\tag{37}
$$

Direction coefficients — First structure

This array, of the general form $A_1 = \partial F_1/\partial X_a$, is obtained by taking the partial derivatives of the structure expressions with respect to each of the unknown parameters. For ease in computation and presentation, the A_1 array is partitioned into two parts, $\mathring{A}$ representing the partials with respect to the terrestrial object parameters and $\bar{A}$ designating the partials with respect to ship position parameters.

$$A_1 = [\mathring{A} \mid \bar{A}] \tag{38}$$

Developing first the $\mathring{A}$ array, the partial derivatives for α_1 and t_1 are presented to illustrate the types of expressions which make up the elements of the matrix.

$$
\begin{aligned}
\frac{\partial \alpha_1}{\partial x_T} &= \frac{-\rho'(y_T - y_1)}{(x_T - x_1)^2 + (y_T - y_1)^2} \\
\frac{\partial \alpha_1}{\partial y_T} &= \frac{+\rho'(x_T - x_1)}{(x_T - x_1)^2 + (y_T - y_1)^2} \\
\frac{\partial \alpha_1}{\partial x_U} &= \frac{+\rho'(y_U - y_1)}{(x_U - x_1)^2 + (y_U - y_1)^2} \\
\frac{\partial \alpha_1}{\partial y_U} &= \frac{-\rho'(x_U - x_1)}{(x_U - x_1)^2 + (y_U - y_1)^2} \\
\\
\frac{\partial t_1}{\partial x_V} &= \frac{-\rho'(y_V - y_1)}{(x_V - x_1)^2 + (y_V - y_1)^2} \\
\frac{\partial t_1}{\partial y_V} &= \frac{+\rho'(x_V - x_1)}{(x_V - x_1)^2 + (y_V - y_1)^2}
\end{aligned}
\tag{39}
$$

Since partials are taken with respect to the x and y coordinates of each of three shore points, the $\mathring{A}$ matrix has a column dimension of six. Because there are three observations, two sextant angles and a compass bearing, at each of the n positions of the ship, the $\mathring{A}$ matrix has a row dimension of $3n$. The arrangement of the partial derivative elements is shown in the following schematic array :

$$
{}_{3n}\mathring{A}_6 = \begin{bmatrix}
\frac{\partial\alpha_1}{\partial x_T} & \frac{\partial\alpha_1}{\partial y_T} & \frac{\partial\alpha_1}{\partial x_U} & \frac{\partial\alpha_1}{\partial y_U} & 0 & 0 \\
0 & 0 & \frac{\partial\beta_1}{\partial x_U} & \frac{\partial\beta_1}{\partial y_U} & \frac{\partial\beta_1}{\partial x_V} & \frac{\partial\beta_1}{\partial y_V} \\
\frac{\partial\alpha_2}{\partial x_T} & \frac{\partial\alpha_2}{\partial y_T} & \frac{\partial\alpha_2}{\partial x_U} & \frac{\partial\alpha_2}{\partial y_U} & 0 & 0 \\
0 & 0 & \frac{\partial\beta_2}{\partial x_U} & \frac{\partial\beta_2}{\partial y_U} & \frac{\partial\beta_2}{\partial x_V} & \frac{\partial\beta_2}{\partial y_V} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{\partial\alpha_n}{\partial x_T} & \frac{\partial\alpha_n}{\partial y_T} & \frac{\partial\alpha_n}{\partial x_U} & \frac{\partial\alpha_n}{\partial y_U} & 0 & 0 \\
0 & 0 & \frac{\partial\beta_n}{\partial x_U} & \frac{\partial\beta_n}{\partial y_U} & \frac{\partial\beta_n}{\partial x_V} & \frac{\partial\beta_n}{\partial y_V} \\
0 & 0 & 0 & 0 & \frac{\partial t_1}{\partial x_V} & \frac{\partial t_1}{\partial y_V} \\
0 & 0 & 0 & 0 & \frac{\partial t_2}{\partial x_V} & \frac{\partial t_2}{\partial y_V} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \frac{\partial t_n}{\partial x_V} & \frac{\partial t_n}{\partial y_V}
\end{bmatrix} \tag{40}
$$

Next, the $\bar{A}$ array is developed. Again the partial derivatives for α_1 and t_1 are presented to illustrate the expressions of which the matrix is composed.

$$
\begin{aligned}
\frac{\partial\alpha_1}{\partial x_1} &= -\frac{\rho'(y_U - y_1)}{(x_U - x_1)^2 + (y_U - y_1)^2} + \frac{\rho'(y_T - y_1)}{(x_T - x_1)^2 + (y_T - y_1)^2} \\
\frac{\partial\alpha_1}{\partial y_1} &= +\frac{\rho'(x_U - x_1)}{(x_U - x_1)^2 + (y_U - y_1)^2} - \frac{\rho'(x_T - x_1)}{(x_T - x_1)^2 + (y_T - y_1)^2} \\
\frac{\partial t_1}{\partial x_1} &= -\frac{\rho'(y_V - y_1)}{(x_V - x_1)^2 + (y_V - y_1)^2} \\
\frac{\partial t_1}{\partial y_1} &= +\frac{\rho'(x_V - x_1)}{(x_V - x_1)^2 + (y_V - y_1)^2}
\end{aligned} \tag{41}
$$

Because partials are taken with respect to the x and y coordinates of each of the n ship positions, the $\bar{A}$ matrix has $2n$ colums. Since there are three observations at each ship position, the row dimension is $3n$. The matrix array of the partial derivative elements is :

$$
{}_{3n}\bar{A}_{2n} = \begin{bmatrix}
\frac{\partial \alpha_1}{\partial x_1} & \frac{\partial \alpha_1}{\partial y_1} & 0 & 0 & \dots & 0 & 0 \\
\frac{\partial \beta_1}{\partial x_1} & \frac{\partial \beta_1}{\partial y_1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \frac{\partial \alpha_2}{\partial x_2} & \frac{\partial \alpha_2}{\partial y_2} & \dots & 0 & 0 \\
0 & 0 & \frac{\partial \beta_2}{\partial x_2} & \frac{\partial \beta_2}{\partial y_2} & \dots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \dots & \frac{\partial \alpha_n}{\partial x_n} & \frac{\partial \alpha_n}{\partial y_n} \\
0 & 0 & 0 & 0 & \dots & \frac{\partial \beta_n}{\partial x_n} & \frac{\partial \beta_n}{\partial y_n} \\
\frac{\partial t_1}{\partial x_1} & \frac{\partial t_1}{\partial y_1} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \frac{\partial t_2}{\partial x_2} & \frac{\partial t_2}{\partial y_2} & \dots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\
0 & 0 & 0 & 0 & \dots & \frac{\partial t_n}{\partial x_n} & \frac{\partial t_n}{\partial y_n}
\end{bmatrix} \qquad (42)
$$

Direction coefficients — Second structure

This array, of the general form $A_2 = \partial F_2/\partial X_a$, is obtained by taking the partial derivatives of the structure expressions with respect to each of the parameters. Since the shore point coordinate parameters are treated as observations with zero weight, the partial derivatives for these parameters are simply zero. For ship coordinates, the partial derivatives are :

$$
\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{\partial(-x_1)}{\partial x_1} = -1 \\
\frac{\partial f_2}{\partial y_1} &= \frac{\partial(-y_1)}{\partial y_1} = -1 \\
\frac{\partial f_3}{\partial x_2} &= \frac{\partial(-x_2)}{\partial x_2} = -1 \\
\frac{\partial f_4}{\partial y_2} &= \frac{\partial(-y_2)}{\partial y_2} = -1 \\
&\vdots \\
\frac{\partial f_{2n-1}}{\partial x_n} &= \frac{\partial(-x_n)}{\partial x_n} = -1 \\
\frac{\partial f_{2n}}{\partial y_n} &= \frac{\partial(-y_n)}{\partial y_n} = -1
\end{aligned}
\tag{43}
$$

Since the partial derivatives are taken with respect to the x and y coordinates of each of three shore points and n ship positions, the A_2 matrix has a column dimension of $2n+6$. Because there are two observations, an x and y coordinate, at each of the three shore points and n ship positions, the A_2 matrix also has a row dimension of $2n+6$. Further, it is a diagonal array with the only non-zero elements (— 1's) occurring in positions corresponding to ship coordinates.

Weight matrix — First structure

For sextant angles, it is assumed that the standard errors for all observations are equal. The same assumption is also made for compass directions. That is,

$$
\begin{aligned}
m_{\alpha_1} = m_{\beta_1} = m_{\alpha_2} = m_{\beta_2} \ldots = m_{\alpha} \\
m_{t_1} = m_{t_2} \ldots = m_t
\end{aligned}
\tag{44}
$$

Since all observations are considered to be independent, P_1 is a diagonal matrix of the reciprocals of the sextant angle and compass bearing standard errors squared. Since there are three observations at each ship position, the square dimension of P_1 is $3n$.

Weight matrix — Second structure

P_2 is the weight matrix for all observed parameters. Since the shore coordinates are treated as completely unknown parameters (observations with zero weight), P_2 takes the general matrix form :

$$ {}_{2n+6}P_{2\,2n+6} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Sigma_{\hat{x}}^{-1} \end{array}\right] \qquad (45)$$

$\Sigma_{\hat{x}}$ is the variance-covariance matrix associated with the ship coordinates. In its most generalized form, it would be entirely filled with variance and covariance estimates. If, however, it is assumed that systematic errors which would affect all x and y ship coordinates can be ascertained and removed or that the positioning method is being utilized as a relative "localized" system, the successive x coordinates and successive y coordinates can be considered as independently determined quantities. The array would then consist of "n" 2×2 blocks on the diagonal, each block representing a ship position. The four elements of each block are the estimated variances of the x and y coordinates on the principal diagonal and the covariance between x and y in the off diagonal positions. If for each ship position the x and y coordinate determinations are considered as independent quantities, then $\Sigma_{\hat{x}}$ becomes a diagonal matrix. Further, if all coordinate values are assumed to be of equal accuracy, $\Sigma_{\hat{x}}$ is a unit matrix. For the localized system 2×2 block form which is utilized in this problem, $\Sigma_{\hat{x}}$ is represented as follows :

$$ {}_{2n}\Sigma_{\hat{x}\,2n} = \begin{bmatrix} m_{x_1}^2 & m_{x_1y_1} & 0 & 0 & \cdots & 0 & 0 \\ m_{x_1y_1} & m_{y_1}^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m_{x_2}^2 & m_{x_2y_2} & \cdots & 0 & 0 \\ 0 & 0 & m_{x_2y_2} & m_{y_2}^2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & m_{x_n}^2 & m_{x_ny_n} \\ 0 & 0 & 0 & 0 & \cdots & m_{x_ny_n} & m_{y_n}^2 \end{bmatrix} \qquad (46)$$

Normals matrix — First structure

With the A_1 matrix partitioned into $\mathring{A}$ and $\bar{A}$, the normals matrix N_1 is formed as follows :

$$N_1 = A_1^T P_1 A_1 = \begin{bmatrix} \mathring{A}^T \\ \hline \bar{A}^T \end{bmatrix} [P_1]\ [\mathring{A} \mid \bar{A}] = \left[\begin{array}{c|c} \mathring{A}^T P_1 \mathring{A} & \mathring{A}^T P_1 \bar{A} \\ \hline \bar{A}^T P_1 \mathring{A} & \bar{A}^T P_1 \bar{A} \end{array}\right] \tag{47}$$

Now, setting $\mathring{A}^T P_1 \bar{A}$ equal to $\tilde{N}$, the expression for N_1 may be written as :

$$N_1 = \left[\begin{array}{c|c} \mathring{N} & \tilde{N} \\ \hline \tilde{N}^T & \bar{N} \end{array}\right] \tag{48}$$

Generalized normals matrix

The generalized normals matrix N_{gen} is formed by adding P_2 to N_1. The square dimension of N_{gen} is $2n+6$.

$$N_{gen} = \left[\begin{array}{c|c} \mathring{N} & \tilde{N} \\ \hline \tilde{N}^T & \bar{N} + \Sigma_{\hat{x}}^{-1} \end{array}\right] \tag{49}$$

Constant vector of normal equations

The vector U is formed as explained in the section above entitled "Adjustment Procedure" wherein the component matrices are as described in previous paragraphs of this section. For both the first and recycled adjustment, U consists of $2n+6$ elements representing the x and y coordinates of each shore point and ship position. For the first cycle, U is formed as follows :

$$U = A_1^T P_1 W_1 = \begin{bmatrix} \mathring{A}^T \\ \hline \bar{A}^T \end{bmatrix} [P_1]\ [W_1] = \begin{bmatrix} \mathring{A}^T P_1 W_1 \\ \hline \bar{A}^T P_1 W_1 \end{bmatrix} \tag{50}$$

When the adjustment is recycled, U becomes :

$$U = A_1^T P_1 W_1 - P_2 W_2 = \begin{bmatrix} \mathring{A}^T P_1 W_1 \\ \hline \bar{A}^T P_1 W_1 \end{bmatrix} - [P_2]\ [W_2] \tag{51}$$

Adjusted parameters

The corrections X obtained by solving (24) are combined with the approximate values of the shore points and observed ship positions as shown in (25) to obtain the finally adjusted values X_a of the terrestrial objects and ship position coordinates. Since there are n ship positions and three shore points, each with an x and y coordinate, the X_a vector is composed of $2n+6$ elements.

$$X_a = \begin{bmatrix} x_{T_a} \\ y_{T_a} \\ x_{U_a} \\ y_{U_a} \\ x_{V_a} \\ y_{V_a} \\ x_{1_a} \\ y_{1_a} \\ \cdot \\ \cdot \\ \cdot \\ x_{n_a} \\ y_{n_a} \end{bmatrix} = \begin{bmatrix} x_{T_o} + \Delta x_T \\ y_{T_o} + \Delta y_T \\ x_{U_o} + \Delta x_U \\ y_{U_o} + \Delta y_U \\ x_{V_o} + \Delta x_V \\ y_{V_o} + \Delta y_V \\ x_{1_o} + \Delta x_1 \\ y_{1_o} + \Delta y_1 \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ x_{n_o} + \Delta x_n \\ y_{n_o} + \Delta y_n \end{bmatrix} \tag{52}$$

Adjusted sextant angles and compass bearings

These values, L_{1_a}, are obtained by adding the residuals vector V_1 computed using (7) to the originally observed sextant angles and compass directions L_{1_b}. Since there are three such observations at each ship position, L_{1_a} consists of $3n$ elements.

$$L_{1_a} = \begin{bmatrix} l_{1_a} \\ l_{2_a} \\ l_{3_a} \\ \cdot \\ \cdot \\ \cdot \\ l_{3n-1_a} \\ l_{3n_a} \end{bmatrix} = \begin{bmatrix} l_{1_b} + v_1 \\ l_{2_b} + v_2 \\ l_{3_b} + v_3 \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ l_{3n-1_b} + v_{3n-1} \\ l_{3n_b} + v_{3n} \end{bmatrix} \tag{53}$$

ERROR ANALYSIS

In order to examine the accuracies of the adjusted values of the parameters and the observed quantities, the variance-covariance matrices for each set of values are obtained. The sequence of operations necessary to arrive at these arrays is explained below, continuing the matrix notations previously utilized.

Variance of unit weight

The variance of unit weight, denoted as m_0^2 is first determined. All weighting in the problem is based on the same variance of unit weight. This unit weight variance is expressed as the sum of squares of the weighted residuals divided by the degrees of freedom.

$$m_0^2 = \frac{V^T P V}{\text{degrees of freedom}} \tag{54}$$

The weighted residuals square sum for the entire problem is the sum of the square sums from each of the two mathematical structures.

$$V^T P V = V_1^T P_1 V_1 + V_2^T P_2 V_2 \tag{55}$$

In partitioned matrix form, (55) is expressed as :

$$V^T P V = [V_1^T \,\vdots\, V_2^T] \left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & P_2 \end{array}\right] \left[\begin{array}{c} V_1 \\ \hline V_2 \end{array}\right] \tag{56}$$

P_1 is explained in the two paragraphs above on "Weight matrix". The square dimensions of P_1 and P_2 are $3n$ and $2n+6$ respectively. V_1 is computed from (7) and is a column vector of $3n$ elements. V_2 is calculated using (8) and consists of $2n+6$ elements. The last $2n$ elements are simply the negative X vector of alterations to ship coordinates.

The number of degrees of freedom is determined as follows :

$$\text{degrees of freedom} = r + s - u \tag{57}$$

where

r = the number of sextant angle and compass observations;
s = the number of ship position x and y coordinates;
u = the number of shore point x and y coordinates.

For the general problem with three shore points and n ship positions,

$$r = 3n, \qquad s = 2n \qquad \text{and} \qquad u = 6 \tag{58}$$

Hence the number of degrees of freedom, substituting (58) into (57), is $5n-6$. Thus the variance of unit weight is finally expressed as :

$$m_o^2 = \frac{V_1^T P_1 V_1 + V_2^T P_2 V_2}{5n - 6} \tag{59}$$

Variance-covariance matrix of parameters

The weight coefficient matrix of the unknown parameters is simply the inverse of the generalized normals matrix (49)

$$Q_X = N_{gen}^{-1} \tag{60}$$

Thus the variance-covariance matrix Σ_X which indicates the accuracy of the adjusted values of the parameters is :

$$\Sigma_X = m_o^2 \, Q_X \tag{61}$$

It is a square array of dimension $2n + 6$.

Variance-covariance matrix of observations

The weight coefficient matrix for the adjusted values of the observed sextant angles and compass directions is :

$$Q_{L_{1_a}} = A_1 N_1^{-1} A_1^T \tag{62}$$

where A_1 and N_1 are as described under "Direction coefficients — First structures" and "Normals matrix — First structure" respectively. Its square dimension is $3n$ since two sextant angles and a compass direction are observed at each ship position. Multiplying (62) by the variance of unit weight from (59) :

$$\Sigma_{L_{1_a}} = m_o^2 \, A_1 N_1^{-1} A_1^T \tag{63}$$

NUMERICAL EXAMPLE

The unavailability of actual hydrographic survey data in a form which would lend itself to adaptation to this problem necessitated the compilation of fictitious information with which to illustrate the computation procedure. In order to simulate the actual observation situation as closely as possible, the following method was employed. A 28″ × 30″ plotting sheet prepared by the Naval Oceanographic Office was used as the basis for generating the required information. The plotting sheet contained a circular lattice net, longitude and latitude and a UTM X-Y grid coordinate system. Since standard computer program routines are available for converting coordinates from any of these three systems to the other two, the X-Y grid was utilized for the purposes of this example. In a practical situation, the electronic positions would be recorded and later converted with a computer program to the desired X and Y coordinate values.

The computational procedure for this problem was programmed in the Fortran IV language and run on the IBM 7094 computer at the Numerical Computation Laboratory of the Ohio State University.

Considerable effort was expended in designing the program for this problem in as general a form as possible to allow for variations in the types and amount of input information. Many combinations of ship headings and observation data were tested to insure that the program would produce proper results for all conceivable situations. The numerical example presented in this section is accomplished with three shore points and four ship positions. However, the number of each type of position can be increased as desired depending upon the amount of input data to be processed.

Problem input information

(a) Coordinates for each of four ship positions.

Source : arbitrarily placed on plotting sheet.

Data units : metres.

Ship position	X	Y
1	299865.0	1960800.0
2	299992.0	1961573.0
3	300081.0	1962378.0
4	300224.5	1963203.0

(b) Sextant angles and compass directions observed at each ship position.

Source : Measurements made on plotting sheet with three-arm protractor.

Ship position	α	β	t
1	31°33′	24°25′	325°.0
2	32°04′	29°00′	316°.5
3	29°37′	32°36′	305°.7
4	24°35′	32°20′	291°.2

(c) Side of survey ship from which angle and direction observation are made : Port side.

(d) Variance and covariance estimates for electronic positioning system coordinates of ship positions.

Source : Arbitrarily selected, based upon information obtained from various publications describing accuracies achievable with short-range electronic positioning systems.

Data units : metres squared.

Ship position	X variance	Y variance	Covariance
1	15.1	18.6	+ 2.1
2	20.8	19.4	+ 2.4
3	17.9	19.9	+ 1.8
4	16.1	20.8	+ 2.0

(e) Standard errors of sextant angle and gyro compass observations.

Sources: Sextant angle — from estimates given in JEFFERS [5] and FAGERHOLM and THUNBERG [6]. Gyro compass bearing — arbitrarily selected based on discussion presented in H.O. Pub. 9 of the U.S. Oceanographic Office [7].

Sextant angle observation standard error: 1'.0.

Gyro compass observation standard error: 5'.0.

Auxiliary information computed by program

(a) The azimuth from each ship position to each succeeding position.

ϕ_{12}	009°.330	ϕ_{14}	008°.509	ϕ_{24}	008°.118
ϕ_{13}	007°.794	ϕ_{23}	006°.309	ϕ_{34}	009°.867

(b) Approximate angles γ subtended at each shore position between each set of ship positions. A partial listing, for the angles between ship positions 1 and 4, is as follows:

At point T	At point U	At point V
34°.750	41°.717	33°.800

(c) Approximate coordinates of each shore point (in metres):

Point	X	Y
T	296810.8	1960748.4
U	296904.6	1962549.6
V	297421.0	1964290.4

Adjusted values and associated standard errors

(a) Ship positions (in metres):

Position	X	*m*	Y	*m*
1	299864.8	1.6	1960798.2	1.6
2	299990.8	1.2	1961574.4	1.4
3	300082.2	1.3	1962380.7	1.3
4	300224.8	1.5	1963201.7	1.6

(b) Shore points (in metres):

Point	X	*m*	Y	*m*
T	296810.2	7.0	1960747.8	2.3
U	296902.1	4.1	1962549.7	1.9
V	297416.6	4.1	1964292.4	3.2

(c) Sextant angles:

Position	α	m	β	m
1	31°33'.0	1'.0	24°25'.2	0'.7
2	32°03'.7	0'.9	29°00'.0	0'.8
3	29°37'.7	1'.0	32°35'.7	1'.0
4	24°34'.6	1'.1	32°20'.2	0'.9

(d) Compass directions:

Position	t	m
1	325°.02	5'.0
2	316°.44	5'.0
3	305°.75	5'.0
4	291°.17	5'.0

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