# SOME METHODS OF TIDAL ANALYSIS

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## 1. INTRODUCTION

A full knowledge of the harmonic constants for tidal constituents is important not only for theoretical studies and for improving astronomical tide predictions but more especially for making a better evaluation of variations in sea level due to meteorological causes. In fact, by eliminating the astronomic tide entirely from tide gauge records, as residue we have the effects due to seiches and to wind-produced accumulations of water.

With the classical tidal analysis methods of the Doodson type, based on a series of not very strictly selective filterings of the observed data, we obtain harmonic constants which include other effects besides those of the astronomic forces and are consequently in a certain measure variable. In such seas as the Adriatic, where tides are relatively small and perturbations (seiches, etc.) relatively strong, it was possible to compute harmonic constants for only six or seven constituents with sufficient accuracy, in spite of their variability.

These constituents are sufficient for prediction of the astronomical tide, but they don not permit a good isolation of variations in sea level due entirely to meteorological disturbances from tidal records.

In the Adriatic the periods of seiches are close to those of some of the tidal constituents, and moreover in the continuous or near-continuous spectral distribution of the seiches around their principal periods there exists energy whose frequency coincides (in several cases at least) with the tidal constituents. In consequence the harmonic analysis must be based on a series of very selective filterings so as to permit isolation of an oscillation having a maximum tide/noise ratio.

In the present article we shall be dealing with several methods — all are electronic computer oriented — enabling us to separate a certain number of tidal constituents by means of successive approximations, and thus to completely extract the astronomic tide from the tidal record.

All these methods were first worked out for use with the hourly values of tidal records obtained over a 3-year period (1966, 1967, 1968) at the Punta della Salute, Venice. We report here on the results obtained with these methods. The methods were later used for computing the tide for 8 Adriatic ports.

## 2. METHOD No. 1 (MOSETTI F., CARROZZO M. T., 1971)

We divided the frequency interval in which the tidal constituents occur into 14 wave groups — 7 diurnal and 7 semi-diurnal — the periods within each group being very close to each other, but sufficiently distinct from the periods of constituents in all other groups.

We first made a series of filterings (one for each group), so as to restore the effect of all the amplitudes of the constituents in a group and to exclude any contribution from the constituents in the remaining groups. Then by making a synthesis of the oscillations extracted at each filtering, we reconstructed the record for the astronomic tide without having to compute the harmonic constants.

In point of fact, the separation of tidal constituents into groups — and the combination of these groups for the purpose of computing a record of the astronomic tide — would be perfectly warranted if the frequency interval in which each of these filters is used contained no other effect. In other words, if the other oscillations of sea level had frequencies lying outside the tidal constituents' frequency band, and this could be the case in other seas.

For the Adriatic, with the results it is possible to achieve, we can as a first approximation evaluate the trends of seiches, their distribution in time and their development. Such results are not enough to enable us to obtain either a complete spectrum of seiche energy, or a perfect knowledge of the tides.

To have a better idea of the results obtainable merely by separating 14 groups of constituents, instead of plotting graphs of the oscillations obtained with each of the filters as a function of time, in figure 1, graph 1 we have shown the lines joining points of maximum and minimum amplitude for the oscillations in each group. The wave groups are designated by the conventional sign for one of the waves in the group, usually the one with the largest amplitude. Figure 1 deals with three groups only, $KJ_2$, $K_2$ (with $S_2$) and $L_2$.

Finally, the graphs themselves show the amplitude modulation for the wave obtained by each of the filters.

It can easily be seen that these modulations are all perturbed to some extent, and this is a sign that they are not merely due to interference phenomena from waves within the group (for in that case the modulation would be periodic and regular, much in the same way as for the $K_2$ group in which the tide/ratio is large), but rather to the occasional presence of waves lying in that part of the seiche spectrum situated in the interval of frequencies separated by each filter.

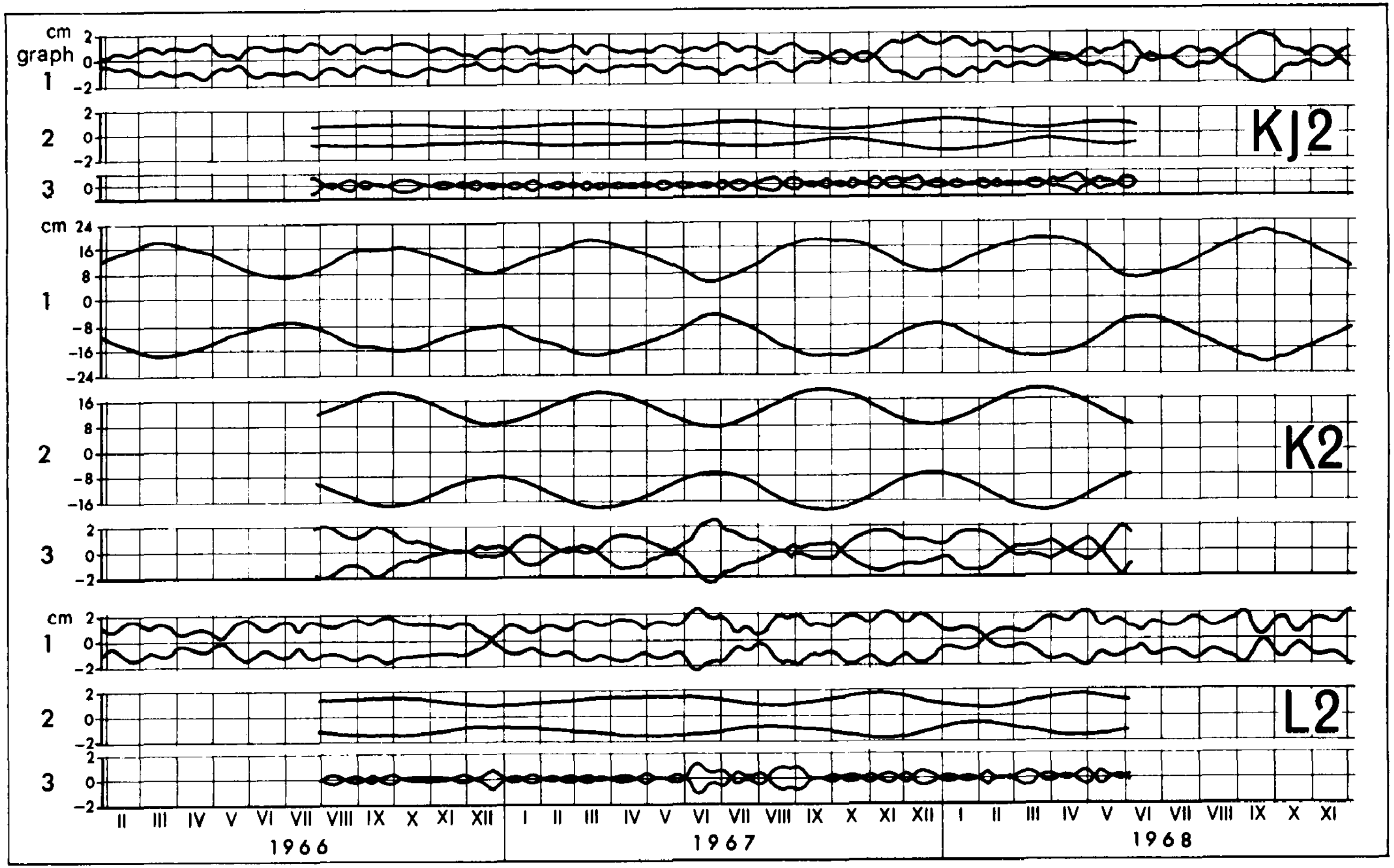


FIG. 1. — Lines joining the maximum and minimum oscillations, for Groups 12, 13, and 14. Each group is designated by the symbol of a representative constituent of the group. Original time scale : 1 mm per day.

Reading from top to bottom the graphs show :

1. The modulated amplitudes of oscillations after filtering each group of constituents (See MOSETTI, CARROZZO, 1971).
2. The filtered curve for the modulated amplitudes.
3. The residue (the difference between curves 1 and 2).

In some groups possessing tidal components of significant amplitude ($K_1$, $K_2$, $M_2$) perturbations are relatively small, whereas they are very strong where the constituents have small amplitudes and the constituents ($OO_1$, $SO_1$, $J_1$, $\theta_1$) have periods very close to the basin's own oscillation (21 to 23 hours).

Any attempt to use this method to separate the individual tidal constituents in order to construct a purer astronomical tide, one suitable both for a complete separation of the seiches and for determining their frequencies if these lie within the frequencies of the tidal constituents, is certain to be a failure. For filters it is in fact usually a case of filterings a fairly narrow band spectrum, but never a line spectrum unless long term tidal records (10 years of hourly values, for example) are available, for then it is possible to reduce the filter " window " to any desired extent. In order to avoid this shortcoming we have worked out two other methods which utilize the results obtained with Method 1, which we can say is a method of first approximation.

## 3. METHOD No. 2

With this method we compute the harmonic constants of the tidal constituents, by making a suitable analysis of the curves showing amplitude modulations for each of the 14 oscillations obtained with method 1 (see figure 1, graph 1).

As the frequencies of the tidal constituents present in each of the 14 groups are known we are able to determine the frequency with which the oscillation's amplitude varies under the effect of interference from the constituent waves.

In fact, if we consider the composition of two waves of different amplitude :

$$y_1(t) = a \cos \omega_1 t$$
$$y_2(t) = b \cos \omega_2 t \qquad (1)$$

by putting $\omega_2 = \omega_1 + \Delta\omega$, $\Delta\omega$ being very small as it obviously is in reality, we obtain :

$$y_2(t) = b \cos (\omega_1 + \Delta \omega) t \qquad (2)$$

and we note that the two constituents $y_1$ and $y_2$ are in phase when :

$$\Delta \omega t = k 2\pi, \quad k = 0, 1, 2 \ldots$$

and that their amplitudes are added together, giving a resultant wave of amplitude ($H = a + b$), whereas the two waves are in phase opposition when :

$$\Delta \omega t = (2k + 1)\pi, \quad k = 0, 1, 2 \ldots$$

and thus the amplitudes are subtracted from one another giving a resultant with an amplitude $h = a - b$ (see figure 2).

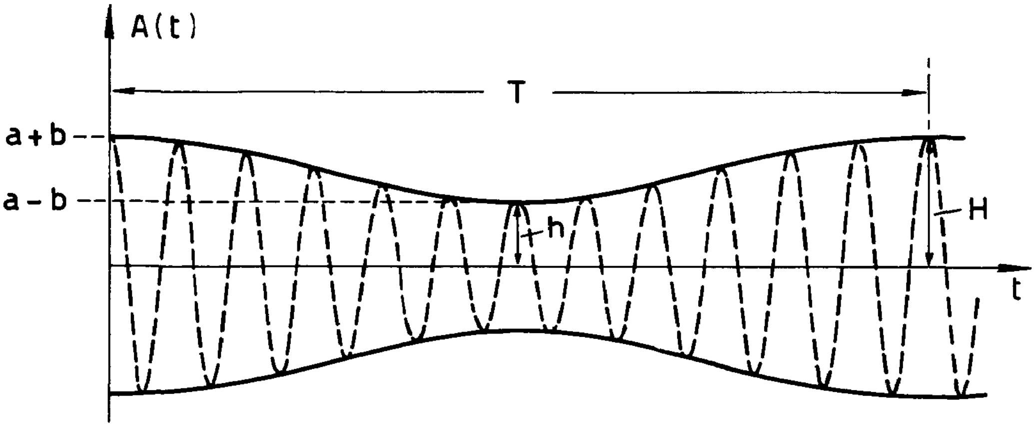


FIG. 2. — Graph of two waves whose frequencies are very close, but which have different amplitudes.

In brief, the amplitude of the resultant takes the same value H on every occasion that the time increases by :

$$T = \frac{2\pi}{\Delta\omega} \tag{3}$$

Computing these values of T separately for each of the 14 groups of constituents (*) we see that all these values fall within the span 4 000-5 000 hours (**). It is therefore possible to eliminate from the fluctuations in wave amplitude all the occasional perturbations which, as can be seen on graph 1 of figure 1, have very short periods. These perturbations are caused by the irregular occurrence of seiches and other phenomena, and are eliminated by applying the filtering operations not to individual hourly values but to the maximum and minimum values shown in graph 1 of figure 1. After we have obtained the filtered curves of the modulated amplitudes (figure 1, graph 2) we then scale the values for H and *h* on these curves and use these to compute the amplitudes of the two constituents.

Corrections to the modulated amplitudes through the use of filters could have equally well been by means of graphs, at least for the groups of constituents with the largest amplitudes where the tide/noise ratio is maximum. For the other groups, this graphical correction would not have been possible and consequently, in order to ensure uniformity of method, we deduced corrected curves by means of filtering all groups except the eleventh group where the only constituent with non-negligible amplitude is $M_2$ and thus has no amplitude modulation.

After the filtering, in the groups where the constituents have fairly large amplitudes the modulated amplitudes are plainly periodic, and the modulation period is given by expression (3) (see, for example, $K_2$ in

(*) With this method we limit ourselves to computing the harmonic constants for only two constituents in each group, i.e. those with the largest amplitudes, but the method could be extended to three or more constituents.

(**) For example, the periods of the groups in figure 1, composed of the constituents $\lambda_2$ and $L_2$, $S_2$ and $K_2$, $YY_2$ and $KJ_2$ are respectively 4941, 4383 and 4937 hours. If we include other constituents of closer frequency we obtain periods of over 8 000 hours.

figure 1). For the other cases it is seen that the modulation period is irregular when longer and stronger perturbations occur. In certain time intervals, however, the modulation period is regular and thus the advantage of this method is as follows. *If analyses extending over a fairly long period are available we are then able to evaluate the intervals on the record that are the least perturbed and where the amplitude varies with a regularity dictated by astronomic laws, and we can in consequence compute the harmonic constants for these very intervals.* In this way we eliminate the variability of harmonic constants. This variability prevents us applying any other method indiscriminately.

### 3.1. Computation of harmonic constants by Method No. 2

a) *Computation of amplitudes.*

Let us consider the most usual case — that of the composition of two sine waves whose frequencies are very close but whose amplitudes and phases are different :

$$\begin{aligned} y_1 &= a \cos (\omega_1 t + \varphi_1) \\ y_2 &= b \cos (\omega_2 t + \varphi_2) \end{aligned} \tag{4}$$

by putting :

$$\omega_1 = \omega + \delta$$

$$\omega_2 = \omega - \delta \qquad \text{with } \delta > 0$$

$$y(t) = y_1 + y_2$$

we obtain :

$$\begin{aligned} y(t) = (a+b) \cos \left( \delta t + \frac{\varphi_1 - \varphi_2}{2} \right) \cos \left( \omega t + \frac{\varphi_1 + \varphi_2}{2} \right) \\ - (a-b) \sin \left( \delta t + \frac{\varphi_1 - \varphi_2}{2} \right) \sin \left( \omega t + \frac{\varphi_1 + \varphi_2}{2} \right) \end{aligned} \tag{5}$$

The resultant from the two constituents is still an oscillation with a pulsation ω, and its amplitude A $(t)$ is not constant but varies slowly with time according to a specific law that depends on the value of δ and on the amplitudes and phases of the constituents, a law expressed by :

$$A(t) = \sqrt{a^2 + b^2 + 2\, ab \cos 2 \left( \delta\, t + \frac{\varphi_1 - \varphi_2}{2} \right)} \tag{6}$$

By putting :

$$\delta\, t + \frac{\varphi_1 - \varphi_2}{2} = \theta \tag{7}$$

we can see that amplitude A $(t)$ has a maximum H and a minimum $h$ respectively equal to :

$$\begin{aligned} H &= a + b \text{ for } \theta = k\pi \qquad \text{with } k = 0, 1, 2 \ldots \\ h &= a - b \text{ for } \theta = (2k + 1)\, \pi/2 \quad \text{with } k = 0, 1, 2 \ldots \end{aligned} \tag{8}$$

When the values for H and $h$ are known — and these can be obtained from the curves given in figure 1, graph 2 — for any group we are then immediately able to compute the amplitudes of both constituents (*) :

$$a = \frac{H + h}{2} \qquad b = \frac{H - h}{2} \tag{9}$$

b) *Phase computation.*

From equation (6) we see that the modulated amplitude of the resultant of the two constituents at a particular instant depends not only on $\delta$ but also on amplitudes $a$ and $b$ and on the phase difference $\varphi_1 - \varphi_2$ (**).

When the instant $t_m$ is known, this being reckoned from an initial instant $t_0$ for which the amplitude of the resultant A($t$) is maximum, with expression (7) we may compute the phase difference of the constituents at instant $t_0$ (if $t_m = 0$, i. e. that it coincides with the instant $t_0$, the two constitutent will have a phase difference equal to zero).

We have :

$$\delta\, t_m = \frac{\varphi_2 - \varphi_1}{2} \tag{10}$$

whence

$$\varphi_2 - \varphi_1 = 2\,\delta\, t_m = \Delta\,\omega\, t_m \tag{11}$$

where $\Delta\omega$ is known from the astronomic characteristics, and $t_m$ is obtained from the data (figure 1, graph 2).

Let us now consider time $\tau$ which elapses between instant $t_0$, the time origin, and the instant at which expression (5) becomes equal to zero. At this instant this expression then becomes :

$$a \cos(\omega_1 \tau + \varphi_1) + b \cos(\omega_2 \tau + \varphi_2) = 0 \tag{12}$$

Thus, knowing the phase difference (11), equation (12) can for instance be expressed as a function of phase $\varphi_1$.

By writing :

$$\omega_1 \tau = \alpha \qquad \omega_2 \tau + \Delta\,\omega\, t_m = \beta \tag{13}$$

we obtain :

$$\tan \varphi_1 = \frac{a \cos \alpha + b \cos \beta}{a \sin \alpha + b \sin \beta} \tag{14}$$

(*) This is true at a first approximation, since it is usual to consider the two constituents in each group with a larger amplitude than the others. In the case of groups 5 and 13, close to the constituents of larger amplitude ($K_1$ and $P_1$ for the 5th group and $K_2$ and $S_2$ for the 13th group) there are in addition other constituents whose amplitudes are non-negligible. However, since these constituents have differences of frequency amounting to only about 0.04° per hour, expression (3) gives another modulation period — i.e. 8 000 to 9 000 hours — which is eliminated by the filters which only separate periods of between 4 000 and 5 000 hours.

(**) $\varphi_1$ is the phase of the wave of higher pulsation $\omega_1 = \omega + \delta$, and $\varphi_2$ is the phase of the wave of lower pulsation, $\omega_2 = \omega - \delta$ ($\delta > o$).

where $a$ and $b$ are known from (9); the quantities (13) are known since $\omega_1$ and $\omega_2$ are known, and $\tau$ is computed with the aid of the data.

In a similar way we may also compute the second phase by expressing formula (12) as a function of $\varphi_2$. However, when we know the value of one phase, the second can be obtained more simply using expression (11).

The values of $\varphi$ obtained in this way could be a little doubtful since they could easily contain errors due to perturbations in the variation with time of expression (6), and the dephasings introduced change the value of the phase difference $(\varphi_2 - \varphi_1)$ of the constituents. This phase computation which is carried out with the value of expression (5) at instant $t_0$ can be repeated for other values of time taken at $\Delta t$ intervals each side of $t_0$, using the least squares method to obtain the phases. As the amplitudes are already known we obtain the most likely values for $\varphi$ — i. e. those that are the best suited to approximate the function under consideration.

The results have been obtained with $0^h$, 1 January 1968 as the time origin, $t_0$.

## 4. METHOD No. 3

In an Annex to the paper mentioned in the bibliography (MOSETTI F., CARROZZO M. T., 1971) we have set out criteria for computing a mathematic filter by a linear combination of $2n + 1$ symmetrical coefficients suitable for separating the groups of constituents from the tidal records.

A linear combination of coefficients computed for a particular $\omega_0$ leaves the frequency and phase of the constituent unchanged, whereas the amplitude is multiplied by an amplification factor :

$$M(\omega_0) = 2\,[a_0 + a_1 \cos(\omega_0 \Delta t) + a_2 \cos(2\,\omega_0 \Delta t) + \cdots + a_n \cos(n\,\omega_0 \Delta t)] \quad (15)$$

where $\omega_0$ is the pulsation to be chosen, $\Delta t$ the time interval of the sampled data, and $a_0, a_1 \ldots a_n$ the $n + 1$ coefficients of the linear combination.

By fixing this function beforehand by analytical means, for example $M(\omega) = 1$ for $\omega_1 \leqslant \omega \leqslant \omega_2$ (with $\omega_2 - \omega_1$ as small as necessary) and where $M(\omega) = 0$ is outside this interval, we can compute the coefficients $a_0$, $a_1$ ..., $a_n$, considering expression (15) as the Fourier expansion of merely the cosines (MOSETTI, 1959). This allows us to select a wave with pulsation $\omega_0$ lying within the interval $\Delta\omega = \omega_2 - \omega_1$.

Since this expansion does not go as far as infinity, but stops at the $n^{th}$ term, the function $M(\omega)$ takes the shape of a bell where the aperture $\Delta'\omega$ is larger than $\Delta\omega$ (see figure 3).

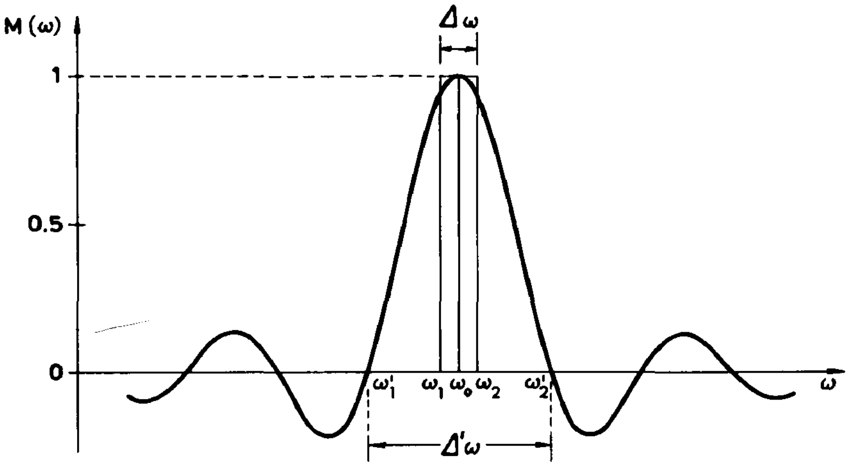


FIG. 3. — Curve showing the selectivity function M(ω).

We may see on figure 4 that $\Delta'\omega$ depends on $n$, the number of coefficients. We note that to increase the filter's selectivity it will be useless to increase the number of coefficients beyond a certain point.

Studying the pulsation values for the most important tidal constituents we see that in order to separate a constituent of pulsation $\omega_0$ the selectivity function must cancel out for the first constituent with a pulsation adjacent to $\omega_0$. With an hourly sampling interval we cannot separate the individual constituents since the differences between adjacent pulsations are between $\sim 4 \cdot 10^{-2}$ to $8 \cdot 10^{-2}$ degrees per hour, and for values of this magnitude we are already within the asymptotic area of the curve shown in figure 4.

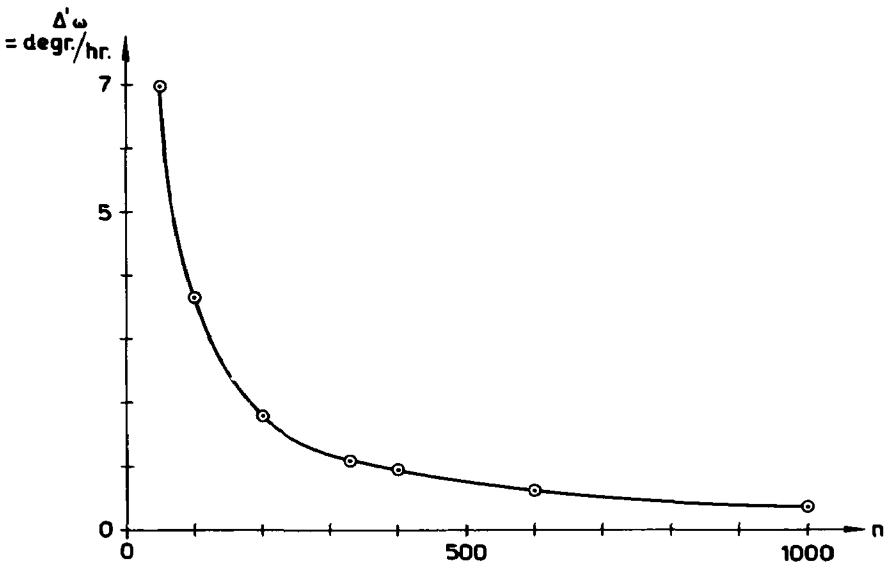


FIG. 4. — Graph of the width of the selectivity function $\Delta'\omega$ plotted against the number of coefficients. The values obtained from trials are indicated by the circles.

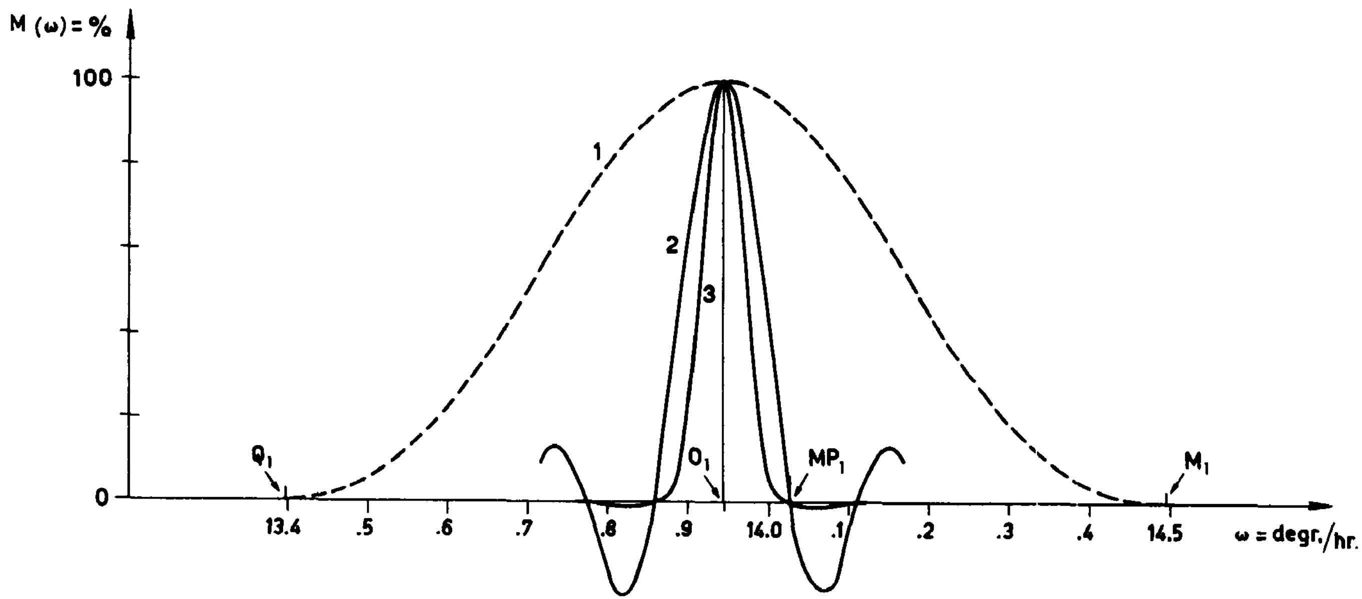


FIG. 5. — The selectivity function for the $O_1$ constituent in Group 3. The numbering on the curves indicates : 1) the filter used to separate all the constituents in Group 3 (see MOSETTI, CARROZZO, 1971) ; 2) the filter that only separates the $O_1$ constituent (first filtering) ; and 3) the filter separating the $O_1$ constituent only (third filtering).

However, the number of coefficients being equal, it is possible to make $\Delta t$ vary in a manner similar to the classic process of Doodson's method, and thus obtain more restricted selectivity functions enabling us to separate all the constituents in the group for so long as any remain.

As an example we have entered on figure 5 the selectivity function which restores the constituent $O_1$ to 100 % and cancels out at the same time as the first contiguous constituent ($MP_1$) of the third group. This function is obtained by a linear combination of 305 coefficients with $\Delta t = 7$ hours as sampling interval (i. e. instead of taking hourly values the interval is fixed at 7 hours). By choosing the most suitable combination of the $\Delta t$ and $n$ parameters according to the differences between the frequencies to be separated, we were able to compute similar filters for all the tidal constituents in the 14 groups.

As expression (15) is a periodic function defined within the intervals $0 - \pi$, $\pi - 2\pi$, etc., similar filters cannot be used directly on the tidal records. For instance, by sampling at an interval of $\Delta t = 8$ hours, the filter constructed for a particular part of the diurnal wave spectrum also separates a similar area of the spectrum falling within the semi-diurnal range, and vice versa. Hence the necessity for first of all separating the diurnal range from the semi-diurnal one, or even better, to separate groups of constituents in the two ranges (see Method 1) and then to proceed to the fine separation of the constituents in each group.

This is the method used for the Punta della Salute, Venice tidal records.

The results are largely pure sine curves with amplitudes that are constant over the whole of the remaining analysed interval which was in fact the central year of the 3-year analysis.

The small percentage of modulation still remaining in some constituents is due to the fact that the filters have a slight residual transparency for the other oscillations.

The constituents $2\,Q_1$, $\theta_1$, $XX_1$, $YY_2$, $XX_2$ (*) in which somewhat larger modulations still remain have not been taken into consideration for this study of the astronomical tide.

Applying the least squares method to the filtered data we computed the harmonic constants (i. e. the amplitudes and phases of the constituents), their phases being $0^h$ on 1 January 1968.

The results obtained are given in table 1.

(*) $XX_1$, $XX_2$ and $YY_2$ are constituents (without symbols) belonging respectively to the groups 4, 8 and 14.

TABLE 1

*Definitive harmonic constants for 33 constituents of the tide at the Punta della Salute, Venice tidal station.*
*The phases are referred to $0^h$ on 1 January 1968.*
*Constituents marked with an asterisk remain perturbed even after filtering (see section 3), and have not been taken into consideration for the computation of the astronomical tide.*

The 14 groups are those of Method No. 1 (see section 2).

| Group | Symbol | A(cm) | $\varphi^{\circ}$ |
|---|---|---|---|
| 1 | $2Q_1$ | 0.26 | 118.2* |
| | $\sigma_1$ | 0.32 | 182.1 |
| 2 | $Q_1$ | 1.16 | 173.4 |
| | $\rho_1$ | 0.43 | 332.4 |
| 3 | $O_1$ | 6.10 | 263.5 |
| | $MP_1$ | 1.00 | 194.5 |
| 4 | $XX_1$ | 0.30 | 324.3* |
| | $M_1$ | 0.91 | 227.3 |
| 5 | $\pi_1$ | 0.33 | 335.7 |
| | $P_1$ | 5.25 | 248.9 |
| | $S_1$ | 1.83 | 55.5 |
| | $K_1$ | 18.58 | 270.9 |
| | $\psi_1$ | 1.16 | 297.0 |
| | $\varphi_1$ | 0.66 | 159.6 |
| 6 | $\theta_1$ | 0.43 | 130.4* |
| | $J_1$ | 0.81 | 309.5 |
| 7 | $SO_1$ | 0.91 | 186.1 |
| | $OO_1$ | 1.82 | 80.3 |
| 8 | $XX_2$ | 0.06 | 184.3* |
| | $MNS_2$ | 0.27 | 198.6 |
| 9 | $2N_2$ | 0.18 | 192.3 |
| | $\mu_2$ | 0.71 | 303.7 |
| 10 | $N_2$ | 3.91 | 322.4 |
| | $\nu_2$ | 0.64 | 105.3 |
| 11 | $M_2$ | 22.07 | 26.5 |
| 12 | $\lambda_2$ | 0.46 | 157.4 |
| | $L_2$ | 0.94 | 251.7 |
| 13 | $T_2$ | 1.04 | 68.8 |
| | $S_2$ | 12.40 | 31.6 |
| | $R_2$ | 0.86 | 55.4 |
| | $K_2$ | 5.74 | 230.3 |
| 14 | $YY_2$ | 0.22 | 227.5* |
| | $KJ_2$ | 0.39 | 257.2 |

## 5. CONCLUSIONS

With this fairly high number of constituents the astronomic tide can be computed. As a result of eliminating the astronomic effect from the tidal record, all effects due to other causes are revealed. By way of example, in figure 6 we show the results from an analysis of a short period of tidal data from the Punta della Salute, Venice, station.

The astronomic tide has been computed by synthesizing a total of 28 tidal constituents. The residue contains simply the energy spectrum attributable to such non-astronomic causes as seiches.

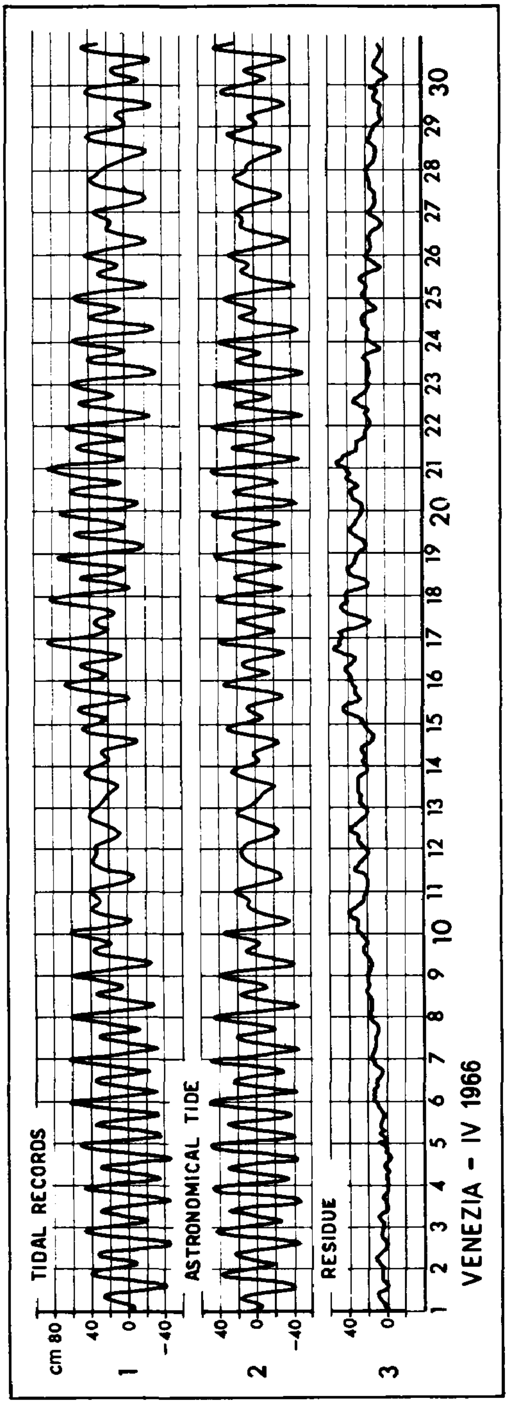


FIG. 6. — Analysis of the tide at Punta della Salute, Venice, April 1966. Reading from top to bottom : 1) the tidal record; 2) the astronomical tide computed from 28 tidal constituents (see table 1); and 3) the difference between the observed and computed tides, i.e. the residue showing the non-astronomic effects.

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[4] DOODSON A.T. (1921) : The harmonic development of the tide-generating potential. *Proc. Roy. Soc. A.*, Vol. **100**, pp. 305-329 (London).