*International Hydrographic Review*, Monaco, LIII (2), July 1976

# GRADIENTS ON THE MEAN AND INSTANTANEOUS SEA SURFACE LEVEL IN A SHALLOW WATER REGION

by D.T. PUGH and G.A. ALCOCK
Institute of Oceanographic Sciences, Bidston Observatory (UK)

## INTRODUCTION

The operations of both hydrographic and land surveyors would be enormously simplified if the sea surface were everywhere horizontal, corresponding to the geoid. However, this is only so for an ideal sea in the absence of currents, meteorological effects and horizontal density gradients. The real sea surface, even in the absence of waves, is affected by tidal and residual water movements, by the friction of the sea bed opposing the flow, by the surface stress due to wind on the sea surface, by variations of atmospheric pressure, and by temperature and salinity differences which affect the sea water density. These effects are likely to be more important in shallow water regions.

We have recently made quantitative estimates of these various factors in a shallow-water region, the Wash bay on the east coast of England. Using the hydrodynamic equations and only limited current observations, a datum level has been transferred to a gauge 12.9 km off shore in the centre of the Wash, across water of 7.5 m average depth, where the mean spring tidal range is 6.4 m. Mean values of the gradients over a lunar tidal cycle were computed. By calculating the datum level on an hourly basis as well, the apparent datum variations were used to investigate the limitations of the simple model applied. Level transfer is estimated to be accurate to within 0.04 m; the maximum uncertainty arises in estimating the residual gradients due to advective and bottom friction terms. The details of these calculations have been published elsewhere (ALCOCK and PUGH, 1975); the purpose of this additional discussion is to summarise the essential results for the hydrographic community, while placing additional emphasis on the physical significance of the various terms in the mathematical representation of the gradients.

The work was initiated as part of the feasibility study for a water storage scheme using bunded reservoirs, being undertaken for the Water Resources Board. For this study water level measurements were made over a period of a year at four sites using pneumatic tide gauges (figure 1). The coastal gauges at Tabs Head, West Stones and Hunstanton were levelled to Ordnance Datum Newlyn (ODN) using conventional techniques. We were asked to connect the gauge at Roaring Middle to the same datum for subsequent design studies, using the hydrodynamic equations. Alternative techniques were either insufficiently accurate or, in the case of hydrostatic levelling, too expensive for a single operation.

The gauge at Roaring Middle is several kilometres from the nearest coast, on a sand bank which dries only on extreme spring tides. The recorder was housed in a cabin at the top of a 14.0 m pile which had been jetted into the sand to a depth of 10.5 m. To the south and east, at the head of the bay, the area dries out extensively at low tide, apart from the river channels which disgorge into the bay, and there are also areas of extensive drying banks near Hunstanton. However, along the line from the Roaring Middle gauge to the Hunstanton gauge there is always a water connection even on extremely low spring tides. For hydrodynamic levelling the gauge at Hunstanton was chosen as a coastal reference, as water levels at Tabs Head and West Stones could be more affected by river discharge

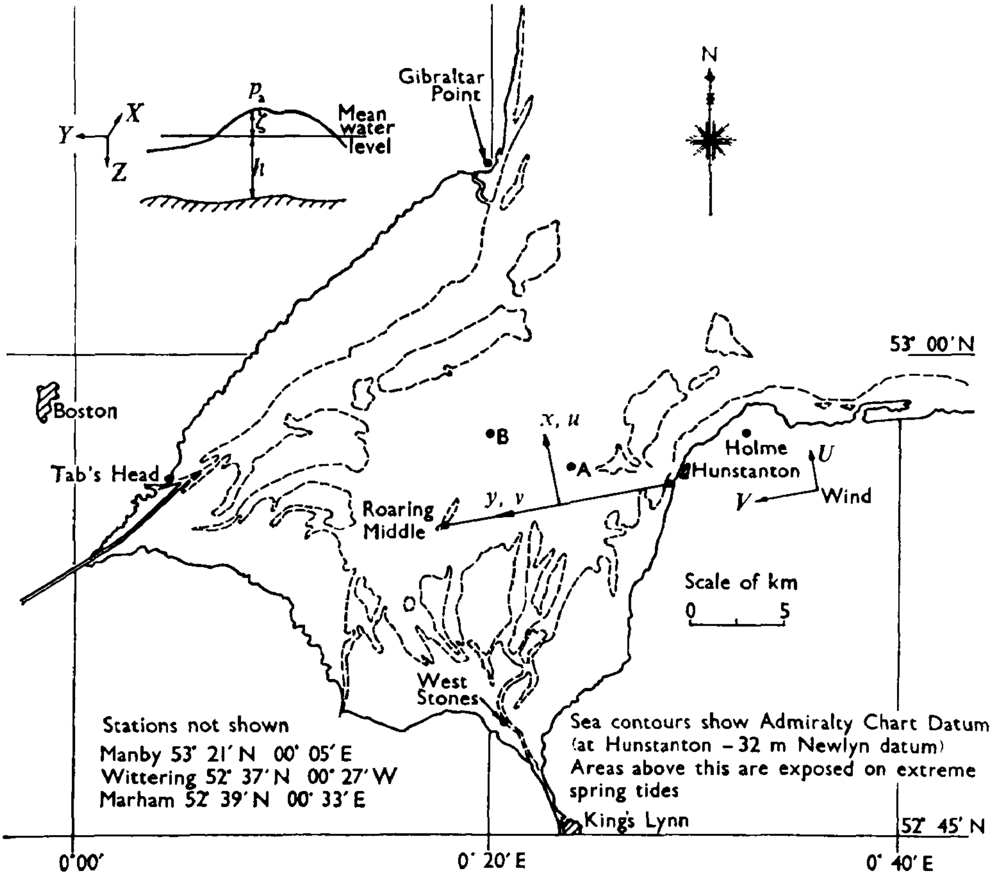


FIG. 1. — Location of pneumatic tide gauges and current meters in the Wash, and coordinate system.

and by the proximity of extensive drying areas, particularly at low tide. In addition, continuous current meter observations from close to this line (positions A and B in figure 1) were made available by courtesy of the Director of the Hydraulics Research Station, Wallingford.

## THE HYDRODYNAMIC EQUATIONS

The hydrodynamic levelling equations have been used by CARTWRIGHT and CREASE (1963) and by CHABERT d'HIERES and LE PROVOST (1970). We use a left-handed system of Cartesian coordinates as shown in figure 1, and consider the horizontal accelerations of a volume element of water due to the forces acting on it in the positive $Y$ direction from Hunstanton to Roaring Middle.

$$\underbrace{\underset{\textit{time}}{\frac{\partial v}{\partial t}} + \underset{\textit{advective}}{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}} + \underset{\textit{coriolis}}{fu}}_{\textit{accelerations}} = \underbrace{-\frac{1}{\rho}\left(\underset{\textit{pressure}}{\frac{\partial p}{\partial y}} + \underset{\textit{shear}}{\frac{\partial G}{\partial z}}\right)}_{\textit{forces}} \tag{1}$$

where $u$ and $v$ are the components of water velocity in the $X$ and $Y$ directions, assumed independent of depth,

$t$ is the time,
$f$ $= 2\,\Omega \sin \Phi$, the Coriolis parameter, due to the Earth's rotation at angular velocity $\Omega$, at a latitude $\Phi$,
$p$ is the external pressure force,
$\rho$ is the water density, and
$G$ is the force due to the horizontal shearing stresses.

We neglect vertical accelerations and direct gravitational forces — the latter are always less than 1.6 mm of level over the distance considered.

The external forces due to pressure gradients may be expressed:

$$p = p_a + \rho g\,(z + \zeta) \tag{2}$$

where :

$p_a$ is atmospheric pressure,
$g$ is gravitational acceleration,
$z$ is the depth of the element below mean water level, and
$\zeta$ is the excursion of the instantaneous water level from the mean level.

To calculate the surface gradient along the $Y$ axis we differentiate (2) with respect to $y$, substitute into (1) for $\partial p/\partial y$ and integrate from the surface to the bottom of the water column:

$$-\frac{\partial \zeta}{\partial y} = \frac{1}{g}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu\right] + \frac{1}{\rho g}\frac{\partial p_a}{\partial y} + \frac{h+\zeta}{2\rho}\frac{\partial \rho}{\partial y} + \frac{G_h - G_{-\zeta}}{g\rho\,(h+\zeta)} \tag{3}$$

where :

$G_h$ is the bottom stress,
$G_{-\zeta}$ is the surface wind stress.

The difference in elevation between Hunstanton and Roaring Middle is then given by:

$$\zeta_R - \zeta_H = \int_H^R \frac{\partial \zeta}{\partial y}\,\partial y$$

$$= -\frac{L}{g}\left[\overline{\frac{\partial v}{\partial t}} + \overline{u\frac{\partial v}{\partial x}} + f\overline{u}\right] \qquad \text{Due to accelerations of water along the line}$$

$$-\frac{{v_R}^2 - {v_H}^2}{2g} - \frac{p_{aR} - p_{aH}}{\rho g} - \frac{\overline{h + \zeta}}{2} \ln\left(\frac{\rho_R}{\rho_H}\right) \qquad \text{End effects} \qquad (4)$$

$$+ \frac{\overline{G_h} - \overline{G_{-\zeta}}}{\rho g} \int_H^R \frac{dy}{(h + \zeta)} \qquad \text{Surface and bottom stress}$$

The suffixes H and R refer to Hunstanton and Roaring Middle, and $L$ is their separation. Superior bars signify instantaneous mean values for the whole line. The terms fall naturally into three groups: those due to water velocity and acceleration, the terms which depend on conditions at the ends of the line, and the surface and bottom stress terms.

When averaged over an extended period several of the terms disappear or become negligible. The

$$\overline{\frac{\partial v}{\partial t}}$$

term must average to zero as sustained acceleration is impossible. It is worth noting that this term is a maximum at high water for a standing wave. In a bay such as the Wash, while currents are virtually zero at high water, there is a significant gradient on the sea surface due to this term, varying from 0.012 m per km on spring tides to 0.006 m per km on neap tides, along the north-east, south-west axis.

The

$$\overline{u\frac{\partial v}{\partial x}}$$

term is due to spatial accelerations (sometimes called advective accelerations). It represents the acceleration necessary, in the case of time-independent flow, to give a particle the velocity $v$ appropriate to its new position after a unit time interval, in which it will have travelled a distance depending on the velocity $u$. CARTWRIGHT and CREASE (1963) showed that it could also be expressed as:

$$u\frac{\partial v}{\partial x} = v\frac{\partial u}{\partial x} + (u^2 + v^2)\frac{\partial \theta}{\partial x}$$

by differentiating the relationship $v = u \tan \theta$. Here the right hand term is the product of the total current speed squared and the curvature of the flow -- and is easy to calculate from observations, as speed and direction are the parameters usually recorded. LACOMBE (1949) expressed the hydrodynamic equations in polar coordinates, showing clearly that it represents a centrifugal effect on sea levels due to the curvature of the stream lines. COURTIER (1933) calculated the effect of this centrifugal acceleration on the local mean sea level around the Cherbourg Peninsula to be as high as 0.8 m in 16 km.

The $fu$ term is the familiar Coriolis acceleration term due to the rotation of our coordinate system due to the Earth's rotation. The term involving $v^2/2g$ arises from the balance between the potential and kinetic energy of a water particle in accordance with Bernoulli's Theorem.

Assuming a constant gradient from Hunstanton to Roaring Middle the datum level at Roaring Middle is given by:

$$\left[D + \zeta_R\right] - \left[\zeta_H\right] = \left[L\left(\frac{\partial \zeta}{\partial y}\right)\right] + D \tag{5}$$

as shown by figure 2. The quantities in square brackets are the known parameters from which the value of $D$ may be determined. We have hourly values of these and so should be able to determine $D$ hourly; if our model were a complete representation the value of $D$ would be constant. In practice we made calculations for the average values of these parameters over a lunar month to eliminate tidal variations. We use the apparent variations of $D$ from a mean value to examine the limitations of the simple model and hence the accuracy of the results.

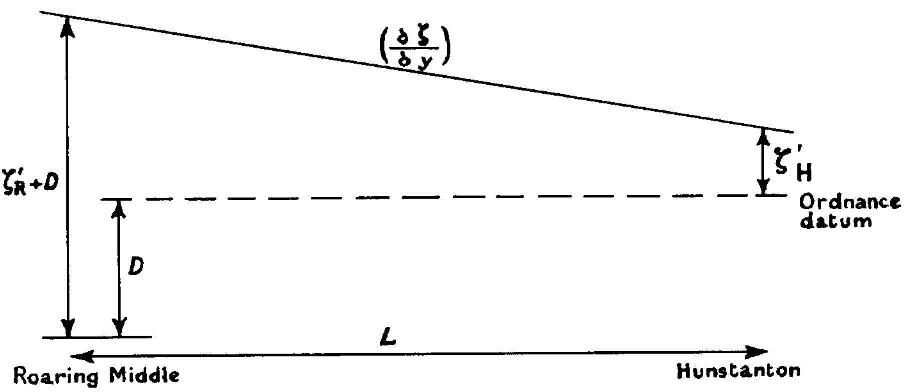


FIG. 2. — Scheme for calculating Roaring Middle gauge datum.

## PRACTICAL COMPUTATIONS OF GRADIENTS

Ideally, for practical computations, several pairs of current meters would be necessary, each pair monitoring the currents and current gradients along a small element of the line. In practice we had to make do with measurements at positions A and B and so were unable to consider detailed variations along the line.

The

$$\frac{\partial v}{\partial t}$$

term was computed using the $v$ component at A:
the

$$u \frac{\partial v}{\partial x}$$

term was computed using the $u$ component at A and the difference between the $v$ components at A and B; the $fu$ term was computed using the $u$ component at A and the $f$ value appropriate for 52°54′N. The end terms were very small and so did not require a detailed treatment. We computed the $v^2$ term from measurements of currents at the ends of the line over a tidal period and then related them to the currents measured at A. The density terms were similarly treated. Atmospheric pressure gradients were determined by fitting a pressure plane to readings at Manby, Wittering and Marham.

There are many different representations of the stress due to bottom friction, but because of the practical limitations of our model, the simplest commonly used representation, the quadratic law, was chosen:

$$G_h = K \rho v \sqrt{u^2 + v^2}$$

where $K$ is a dimensionless constant, taken here to be 0.0025.

The surface stress due to the wind was similarly represented:

$$G_{-\zeta} = K_a \rho_a V \sqrt{U^2 + V^2}$$

where $U$ and $V$ are the wind components at the centre of the line, calculated by linear interpolation from observations at Gibraltar Point, Holme and Marham. The maximum wind speed component along the section during the period of observations, from 22nd December 1972, to 23rd January 1973, was 10.7 $ms^{-1}$. The coefficient $K_a$ was also taken as 0.0025. $\rho_a$ is the air density.

The mean reciprocal depth was computed by dividing the line into twenty equal sections and summing the reciprocal depth over these sections at each state of the tide.

$$\int_H^R \frac{dy}{(h+\zeta)} \equiv \frac{L}{20} \sum_{n=1}^{20} \frac{1}{h_n + \zeta_H}$$

Figure 3 shows the relationship between the reciprocal mean depth and the mean depth. The difference is considerable at low water on spring tides. Preliminary calculations showed that the shallower water towards Hunstanton was giving overall corrections for stress effects at extreme low water levels which were unrealistically large when compared with the observed differences. Some adjustment was necessary to avoid over-emphasising the effects of sandbanks which, although almost dry, were separated from each other by channels of relatively deep water which joined the deeper water to the north and south of the section. This was done by removing a section from computations for reciprocal mean depth when the water depth in that section fell below 1.3 m.

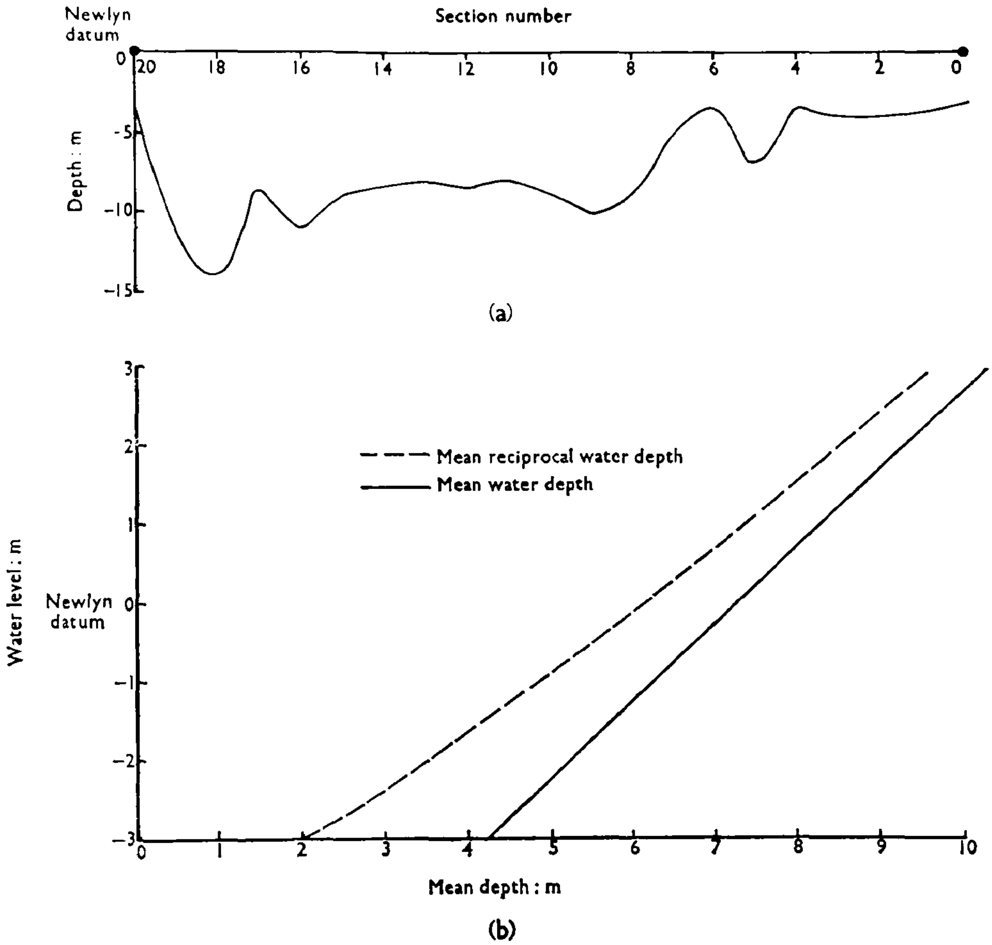


FIG. 3. — (a) Water depth profile from Hunstanton to Roaring Middle; (b) Relationship between mean and reciprocal mean water depths for the line.

## RESULTS

The results of performing the computations on an hourly basis are summarised in Table 1. Only two terms, the advective term and the bottom stress term, have a significant residual mean value, but the time derivative, Coriolis and wind stress terms all contribute to the variance of the hourly level differences. The maximum and minimum calculated differences in this case are interesting because they represent the theoretical errors which could have arisen in a hydrographic survey over Roaring Middle, when correcting for variable water levels by using the Hunstanton tide gauge records. The standard deviation of the calculated differences from the mean was 0.08 m, while, for the observed differences, the standard deviation was 0.14 m.

TABLE 1

*Comparison of contribution of different gradient terms; units are metres of difference $\zeta_R - \zeta_H$*

| Term | | | Non-tidal mean, $m$ | Maximum | Minimum | Variance about mean $m^2 \times 10^{-2}$ |
|---|---|---|---|---|---|---|
| Section water movements | 1 | $\frac{\partial v}{\partial t}$ | <0·001 | 0·093 | −0·061 | 0·135 |
| $\times \frac{L}{g}$ | 2 | $u \frac{\partial v}{\partial x}$ | 0·033 | 0·159 | −0·031 | 0·152 |
| | 3 | $fu$ | 0·003 | 0·141 | −0·131 | 0·496 |
| | 4 | $-\frac{v_R^2 - v_H^2}{2g}$ | 0·001 | 0·005 | 0·000 | 0·001 |
| End terms | 5 | $-\frac{(p_{aR} - p_{aH})}{\rho g}$ | 0·001 | 0·004 | −0·001 | <0·001 |
| | 6 | $\frac{h+\zeta}{2} \ln \left(\frac{\rho_R}{\rho_H}\right)$ | −0·003 | −0·002 | −0·004 | <0·001 |
| Friction terms $\times$ $\frac{1}{\rho g} \frac{L}{20} \sum_{n=1}^{20} \frac{1}{h_n + \zeta_H}$ | 7 | Bottom stress $G_h$ | −0·034 | 0·106 | −0·293 | 0·606 |
| | 8 | Wind stress $G_{-\zeta}$ | 0·001 | 0·100 | −0·137 | 0·047 |

Combining the mean values of surface gradients shows that, because of the apparently chance balance of the advective and bottom stress terms, the theoretical difference in mean water level between Roaring Middle and Hunstanton is only 0.001 m for the period of calculations. This gives the datum for the Roaring Middle gauge at −3.432 m to Newlyn Datum. When the data were treated in five separate blocks of approximately six days, this value varied from −3.407 m to −3.457 m, with a standard deviation of 0.016 m. However, systematic errors should also be considered.

Figure 4 shows the results of the computations for a spring tide with strong wind stress and a neap tide with little wind stress. The maximum discrepancy between the theoretical and observed differences occurs at low water on a spring tide. This is probably due to our model poorly representing the currents in the shallow water near Hunstanton by measurements in the deeper water at A. Note, however, that the radiational stress set-down due to waves, which is not included in our model, would also be greatest at low water. The advective (or centrifugal) term produces level variations at twice the semi-diurnal tidal frequency ($M_4$) and its value is almost always positive — hence its contribution to the mean value. Combining all sources

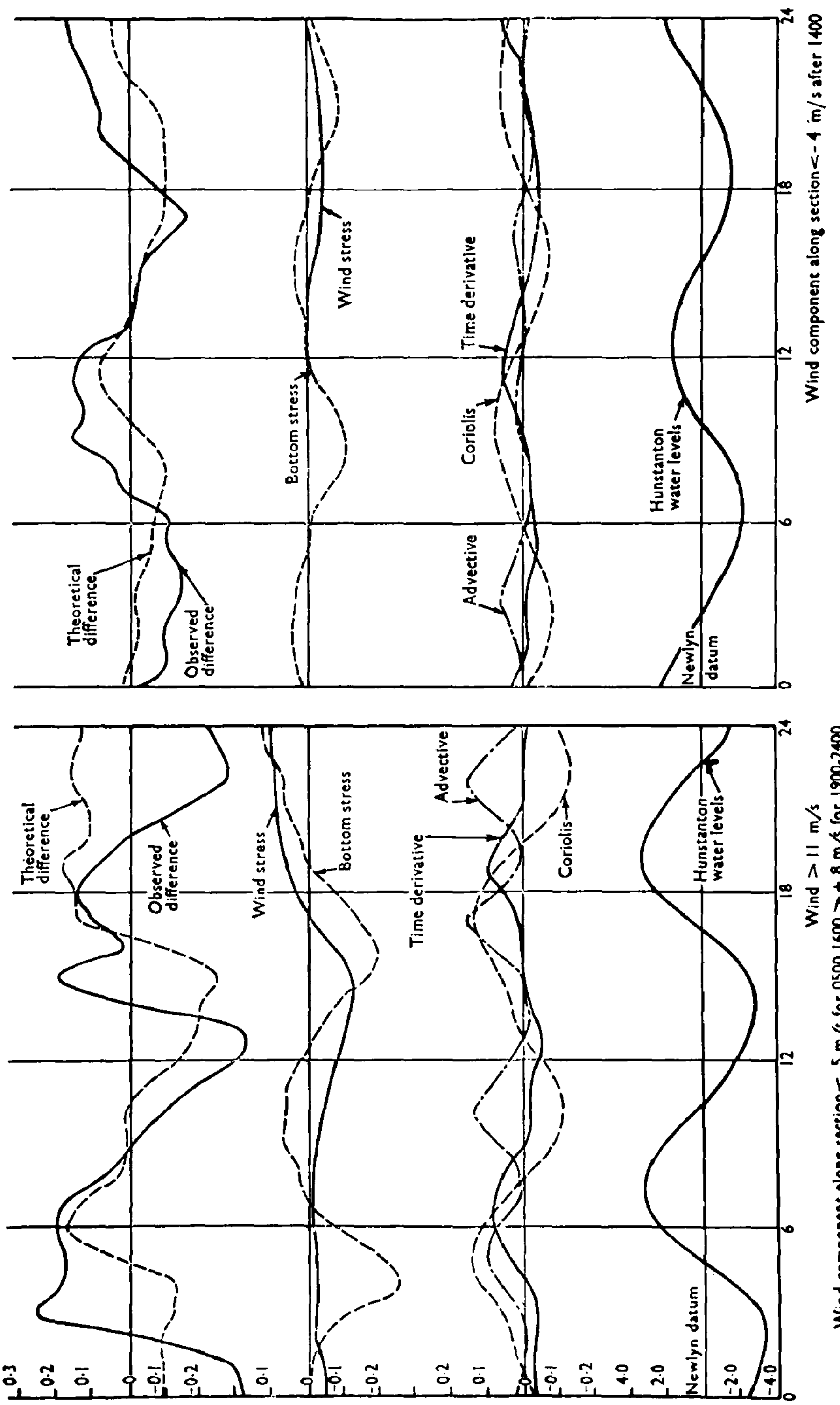


FIG. 4. — Hourly plot of observed and calculated level differences (Roaring Middle-Hunstanton), all heights in metres; a) spring tide, 20 January, 1973; (b) neap tide, 13 January, 1973.

of error suggests a probable error of ± 0.04 m on either side of the adopted value of − 3.43 m for the Roaring Middle gauge datum.

## CONCLUSIONS

Using a simple model and only limited data it has been possible to transfer a datum level to a gauge 12.9 km off shore. Over this distance the computed instantaneous differences in sea level were as large as 0.26 m, with a standard deviation of 0.08 m; observed instantaneous differences were greater than this, particularly on low water spring tides.

For the purpose of datum transfer the number of current meters required to describe properly the dynamic characteristics of the water in a region of channels and drying banks could be prohibitive, particularly if the depth mean currents were computed using meters at different depths. It is probably best to apply a relatively simple theoretical model to calculate average gradients, and to then estimate the accuracy of the final result by comparing the observed hourly differences with those predicted by the model.

Significant local variations of mean sea level occur over distances of a few kilometres. It is interesting to compare mean sea level at all four sites in the Wash (Table 2) based on a simultaneous year of data. The higher level at West Stones is almost certainly due to bottom friction, particularly at low water when, for levels below −2 m Newlyn datum, the discharge from the Great Ouse is confined to a relatively narrow channel for a further 10 km after passing West Stones.

TABLE 2

*Mean levels and tidal constants for Wash stations*

| Station | | Hunstanton | Roaring Middle | West Stones | Tab's Head |
|---|---|---|---|---|---|
| Mean level, *m* | | 0·046 | 0·074 | 0·339 | 0·117 |
| $M_2$ | *H* | 2·283 m | 2·385 m | 2·326 m | 2·386 m |
| | *g* | 179·4° | 175·9° | 183·3° | 180·6° |
| $S_2$ | *H* | 0·774 m | 0·821 m | 0·762 m | 0·795 m |
| | *g* | 228·1° | 225·5° | 234·5° | 231·6° |
| $M_4$ | *H* | 0·147 m | 0·133 m | 0·239 m | 0·211 m |
| | *g* | 304·7° | 305·3° | 311·8° | 311·3° |
| $MS_4$ | *H* | 0·130 m | 0·119 m | 0·209 m | 0·181 m |
| | *g* | 348·7° | 348·6° | 0·6° | 355·7° |

## REFERENCES

ALCOCK, G.A., PUGH, D.T. (1975) : Hydrodynamic levelling of an off-shore tide gauge. *Proc. Inst. Civ. Eng.*, Part 2, 59, pp. 123-137.

CARTWRIGHT, D.E., CREASE, J. (1963) : A comparison of the geodetic reference levels of England and France by means of the sea surface. *Proc. R. Soc.*, Series A, 273, pp. 558-580.

CHABERT d'HIERES, G., LE PROVOST, C. (1970) : Spatial variations of the mean sea level in the English Channel. Report of the Symposium on Coastal Geodesy, Munich, 20-24 July 1970. Ed. R. Sigl. Munich : Institute for Astronomical and Physical Geodesy. 644 pp.

COURTIER, A. (1933) : Note on the modification of mean sea level under the action of currents. *Int. Hydrog. Rev.*, X (2), pp. 124-129.

LACOMBE, H. (1949) : Détermination de la marée au large à partir des courants de marée. *Annales Hydrographiques*, Paris, 1361, pp. 1-39.

## DEEP SEA SOUNDINGS

"I venture to suggest to you three reasons why you are one of the happiest communities in the world. The first is that at an early age you rid yourselves of that form of ambition which ruins the happiness of so many men in a competitive service, the ambition which produces jealousy and prefers success to friendship. Your ambition lies elsewhere; it is to make the seas more safe for mankind — to assist your fellow men and brother officers.

A second reason is that, with few exceptions, men engaged in scientific pursuits are happy men. It is, I think, true, that if a man is a scientific labourer — a doctor, engineer or marine surveyor — he is impervious to the impulse to acquire wealth and is supremely content in his endeavours to add something to the knowledge of the human race.

A third reason is that marine surveyors, beyond any others who go down to the sea in ships, do see the wonders of the Lord in full measure ...

Robert Louis Stevenson said : "If a man love the labour of any trade apart from any question of success or fame, the gods have called him". I have been dining tonight with men who have been called by the gods".

Admiral Sir William JAMES, addressing a Reunion Dinner of Officers of the Royal Navy Surveying Service in London in 1937.
(Reproduced from his book *Hotch-Potch*, published 1968)