International Hydrographic Review, Monaco, LIV (1) January 1977.

SOME RECENT INVESTIGATIONS INTO THE HARMONIC SHALLOW WATER CORRECTIONS METHOD OF TIDAL PREDICTIONS

by M. AMIN
Institute of Oceanographic Sciences, Bidston Observatory,
Birkenhead, Merseyside, England

SUMMARY

To overcome the inadequacies of the harmonic method in the analysis and prediction of shallow water tidal regimes, DOODSON (1957) devised a Harmonic Shallow Water Corrections (H.S.W.C.) method to improve the quality of predicted times and heights of tidal turning points. This method proved to be very powerful where the constituent M_2 is relatively dominant in the tide. The theoretical background and technique of application as presented by DOODSON is devised for hand calculations and for use on mechanical harmonic analogue machines which were geared for conventional constituents, not H.S.W.C. constituents. In this paper the method is reformulated using a spectral analysis technique, thus providing a clear explanation of the fundamental ideas involved. In the spectrum of a finite time series record sampled at regular intervals, all the energy at frequencies above the Nyquist frequency is aliased with frequencies below the Nyquist frequency. The aliasing phenomenon when applied to high and low waters, occurring at intervals of approximately half a lunar day, has the inherent advantage that numerous constituents combine together, even eliminating the need for separate identification. Caution must be exercised, however, due to the fact that the time interval of half a lunar day is an approximation only. Any selected constituents can be resolved by use of the least squares method. This technique will be free from previous limitations of a fixed length data (355 days) requirement, and it will also handle effectively discontinuous data. An intensive comparison of Extended Harmonic Method (E.H.M.), Improved Response Method (I.R.M.) and H.S.W.C. method, shows that all these methods are approaching their theoretical limits. Examination of residuals indicates that they are similar in accuracy, but for some typical requirements one method can compute predictions marginally better than the others.

1. INTRODUCTION

The harmonic method of tidal analysis and prediction initiated by KELVIN and DARWIN, and further developed by DOODSON (1928), provided sufficiently accurate predictions for deep water ports. When this method was applied to the distorted tides of shallow water ports, it proved to be inadequate. The principal reason for this was that, in shallow water, various constituents interact strongly with each other, creating in turn the development of a large number of new constituents. It is difficult to resolve all these constituents without large computing facilities. The failure of the harmonic method was attributed to :

a) the slow convergence and consequently large number of higher-order constituents;

b) the fact that some of these are spectrally very close and difficult to resolve;

c) the fact that the implications of this shallow water interaction are as yet inadequately known, so that the specification of a comprehensive harmonic model is often difficult if not impossible.

To overcome the above mentioned problems, DOODSON (1957) devised a new method known as the Harmonic Shallow Water Corrections (H.S.W.C.) method to improve the resolution and prediction of times and heights of tidal turning points. The theoretical background is complex and the technique given by him is very sophisticated. More particularly, the method is orientated to meet the necessities and requirements of that time, i.e. biased to manual analysis and a mechanical predictor geared to different constituent speeds. The method was developed on the assumption that M_2 dominates the other constituents, and the procedure proved to be very powerful when such conditions prevail. However, some minor anomalies existed, such as non-zero mean values of residuals obtained from repredicting the analysed data. ROSSITER & LENNON (1968) have shown that the above problem exists, and in turn investigated the widely assumed reasons for failure of the harmonic method. If those reasons are valid, then with present computing facilities to help resolve large numbers of constituents, the harmonic method should be as accurate as the H.S.W.C. method. ROSSITER & LENNON (1968), after examining the whole tidal spectrum, extended the harmonic method to include 114 constituents, i.e. the Extended Harmonic Method (E.H.M.). ZETLER & CUMMINGS (1967) agree with this optimal number of constituents, but the performance of the H.S.W.C. method was found to be better than E.H.M. The efficiency of the H.S.W.C. method is associated with the fact that it concerns itself only with the turning points and not with the total time profile, which can be significantly distorted in shallow water. By giving additional weight to high and low waters, better predictions are achieved in those regions. CARTWRIGHT & ROSSITER (1972) in their comparative assessment of the H.S.W.C. method with E.H.M. and I.R.M. (an "Improved" version of the "Response Method" of tidal analysis, programmed by D.E. CARTWRIGHT about 1967 — see references contained in CARTWRIGHT & ROSSITER (1972)) have shown that for predictions of high and low waters, there is little to choose between H.S.W.C. and I.R.M., but E.H.M. gives large residuals. AMIN (1976) found that the comparatively poor performance of E.H.M. was due to :

a) the shallow water constituents which are very close to others and so cannot be adequately resolved from one year's data;
b) shallow water constituents which have the same speed as others, making it difficult to assume their relative contribution which may vary from place to place, leading to the use of incorrect nodal parameters.

This work was carried out to find :

a) the theoretical explanation of the H.S.W.C. method on a spectral basis;
b) a direct and simple computing technique;
c) the differences between E.H.M. and H.S.W.C. and under what circumstances they appear.

2. DEVELOPMENT OF SHALLOW WATER CONSTITUENTS

The fundamental differential equations relating to the motion of a homogeneous fluid in a one dimensional channel, when allowance is made for second order terms but neglecting bottom friction, are :

$$\frac{\partial \zeta}{\partial t} = - D \frac{\partial u}{\partial x} \quad \frac{\partial}{\partial x} (\zeta u) \tag{2.1}$$

$$\frac{\partial u}{\partial t} = - g \frac{\partial \zeta}{\partial x} - u \frac{\partial u}{\partial x} \tag{2.2}$$

where D is the mean depth of the fluid,
t is the time,
ζ is the elevation of the surface above the mean level,
u is the mean velocity of the fluid, and
g is the acceleration due to gravity.

In deep water ζ and u are small, therefore the product terms

$$\frac{\partial}{\partial x} (\zeta u) \text{ and } u \frac{\partial u}{\partial x}$$

can be neglected, and equations (2.1) and (2.2) to a first approximation become :

$$\frac{\partial \zeta}{\partial t} = - D \frac{\partial u}{\partial x} \tag{2.3}$$

$$\frac{\partial u}{\partial t} = - g \frac{\partial \zeta}{\partial x} \tag{2.4}$$

The Airy solution (LAMB, 1932, p. 281) of equations (2.1) and (2.2) for a tidal wave consisting of a single constituent, and GALLAGHER & MUNK's (1971) solution of these equations for the interactions of incident and reflected waves can be extended to a wave of two or more harmonic terms to examine the interactions of various constituents. Consider a simple tidal wave composed of two terms :

$$\zeta = Z_0 \cos \sigma_0 t + Z_1 \cos \sigma_1 t \tag{2.5}$$

entering the channel at $x = 0$.

The solutions of (2.3) and (2.4) which satisfy (2.5) are :

$$\zeta = Z_0 \cos \sigma_0 \left(t - \frac{x}{c}\right) + Z_1 \cos \sigma_1 \left(t - \frac{x}{c}\right) \qquad (2.6)$$

$$u = \frac{g}{c} \left[Z_0 \cos \sigma_0 \left(t - \frac{x}{c}\right) + Z_1 \cos \sigma_1 \left(t - \frac{x}{c}\right)\right] \qquad (2.7)$$

where : $c = (gD)^{1/2}$.

For the solutions of the non-linear equations (2.1) and (2.2) we replace the product terms in these equations by using (2.6) and (2.7), and get :

$$\frac{\partial \zeta}{\partial t} = - D \frac{\partial u}{\partial x} - \frac{g}{c^2} \psi (t , x) \qquad (2.8)$$

$$\frac{\partial u}{\partial t} = - g \frac{\partial \zeta}{\partial x} - \frac{g}{2c^3} \psi (t , x) \qquad (2.9)$$

where : $\psi (t , x) = Z_0^2 \sigma_0 \sin 2\sigma_0 \left(t - \frac{x}{c}\right) + Z_1^2 \sigma_1 \sin 2\sigma_1 \left(t - \frac{x}{c}\right)$

$$+ Z_0 Z_1 \left[(\sigma_0 - \sigma_1) \sin (\sigma_0 - \sigma_1) \left(t - \frac{x}{c}\right) + (\sigma_0 + \sigma_1) \sin (\sigma_0 + \sigma_1) \left(t - \frac{x}{c}\right)\right]$$

The solutions of which, consistent with (2.5), are :

$$\zeta = Z_0 \cos \sigma_0 \left(t - \frac{x}{c}\right) + Z_1 \cos \sigma_1 \left(t - \frac{x}{c}\right)$$

$$- \frac{3}{4} \frac{g}{c^3} x \left[Z_0^2 \sigma_0 \sin 2\sigma_0 \left(t - \frac{x}{c}\right) + Z_1^2 \sigma_1 \sin 2\sigma_1 \left(t - \frac{x}{c}\right)\right]$$

$$- \frac{3}{4} \frac{g}{c^3} Z_0 Z_1 x \left[(\sigma_0 - \sigma_1) \sin (\sigma_0 - \sigma_1) \left(t - \frac{x}{c}\right) + (\sigma_0 + \sigma_1) \sin (\sigma_0 + \sigma_1) \left(t - \frac{x}{c}\right)\right] \qquad (2.10)$$

$$u = \frac{g}{c} \left[Z_0 \cos \sigma_0 \left(t - \frac{x}{c}\right) + Z_1 \cos \sigma_1 \left(t - \frac{x}{c}\right)\right]$$

$$- \frac{1}{8} \frac{g^2}{c^3} \left[Z_0^2 \cos 2\sigma_0 \left(t - \frac{x}{c}\right) + Z_1^2 \cos 2\sigma_1 \left(t - \frac{x}{c}\right)\right]$$

$$- \frac{3}{4} \frac{g^2}{c^4} x \left[Z_0^2 \sigma_0 \sin 2\sigma_0 \left(t - \frac{x}{c}\right) + Z_1^2 \sigma_1 \sin 2\sigma_1 \left(t - \frac{x}{c}\right)\right]$$

$$- \frac{1}{4} \frac{g^2}{c^3} Z_0 Z_1 \left[\cos (\sigma_0 - \sigma_1) \left(t - \frac{x}{c}\right) + \cos (\sigma_0 + \sigma_1) \left(t - \frac{x}{c}\right)\right]$$

$$- \frac{3}{4} \frac{g^2}{c^4} Z_0 Z_1 x \left[(\sigma_0 - \sigma_1) \sin (\sigma_0 - \sigma_1) \left(t - \frac{x}{c}\right)\right.$$

$$\left. + (\sigma_0 + \sigma_1) \sin (\sigma_0 + \sigma_1) \left(t - \frac{x}{c}\right)\right] \qquad (2.11)$$

The equations (2.10) and (2.11) show that corresponding to two terms, there are four additional terms of speed : $2\sigma_0$, $2\sigma_1$, $\sigma_0 - \sigma_1$ and $\sigma_0 + \sigma_1$. If other terms of astronomical origin are included in (2.5), each will interact with the others and more terms will develop. A complete solution of differential equations (2.1) and (2.2) involves infinite series of which (2.10) and (2.11) contain the first few terms. The tidal profile will be more distorted as the amplitudes of the higher harmonics increase and further higher harmonics are generated through their interaction. Some of these may have the same speeds as in (2.5), thus an internal feedback system is set up. The significance of these terms, as we see from equations (2.10) and (2.11), depends upon the amplitudes of the interacting terms, the depth of water and the distance through which the tidal wave has progressed. The dependence of the amplitudes upon the distance x shows that the system may break down when the tide progresses through a sufficiently large distance, since the above solutions will not converge. In that case it will be difficult to establish any simple relation between two such points along the channel. In the differential equations we have not taken into consideration the effect of bottom friction which also helps to generate new constituents, but here the modes of interaction are different and it is difficult to establish any exact relationship between the constituents originating from the two effects. Thus the real tide, in which shallow water constituents are spread over a wide spectrum, is much more complex than the above solution, and it is very hard to relate the primary and shallow water constituents analytically. The whole system of predictions is based on the knowledge of these composition constituents, and due to lack of any comprehensive analytical technique their detection and identification depend upon resolving these constituents from the observed data. The difficulty of resolution from a short span of data or alternatively handling long data spans, causes us to look again at DOODSON's H.S.W.C. technique. Here one must bear in mind that the Doodson method was developed more than fifty years ago, that it was devised for hand calculations in so far as analysis is concerned, and for the limited capacity and inadequate gearing of the mechanical tide predictor.

3. DOODSON's H.S.W.C. METHOD IN SPECTRAL FORM

The Fourier expansion of the function $\zeta(t)$ which is continuous in the interval (0, T) is given by :

$$\zeta(t) = \alpha_0 + 2 \sum_{j=1}^{\infty} \left(\alpha_j \cos \frac{2\pi j t}{T} + \beta_j \sin \frac{2\pi j t}{T}\right) \tag{3.1}$$

where

$$\alpha_j = \frac{1}{T} \int_0^T \zeta(t) \cos \frac{2\pi j t}{T} \, dt \qquad j = 0, 1, 2, 3 \ldots$$

$$\beta_j = \frac{1}{T} \int_0^T \zeta(t) \sin \frac{2\pi j t}{T} \, dt \qquad j = 1, 2, 3 \ldots.$$

When function $\zeta(t)$ is sampled at points $t = r\Delta$, $r = 0, 1, \ldots 2L$, where Δ

is the sampling interval, $T = (2L + 1)\,\Delta$, then the corresponding set of values $\{\zeta_r = \zeta(r\Delta)\}$ is represented by the discrete Fourier series as follows:

$$\zeta_r = A_0 + 2\sum_{m=1}^{L}\left(A_m \cos\frac{2\pi m r}{2L+1} + B_m \sin\frac{2\pi m r}{2L+1}\right) \qquad (3.2)$$

where

$$A_m = \frac{1}{2L+1}\sum_{r=0}^{2L}\zeta_r \cos\frac{2\pi m r}{2L+1} \qquad m = 0, 1, 2, \dots L \qquad (3.3)$$

$$B_m = \frac{1}{2L+1}\sum_{r=0}^{2L}\zeta_r \cos\frac{2\pi m r}{2L+1} \qquad m = 1, 2, \dots\dots L \qquad (3.4)$$

Now multiplying both sides of (3.1) by $\cos 2\pi m t/T$ and substituting $r\Delta = t$, $\zeta(t)$ becomes ζ_r and summing over $r = 0, 1, \dots 2L$, it gives:

$$\sum_{r=0}^{2L}\zeta_r \cos\frac{2\pi m r}{2L+1} = \sum_{r=0}^{2L}\left[\alpha_0 + 2\sum_{j=1}^{\infty}\left(\alpha_j \cos\frac{2\pi j r}{2L+1} + \beta_j \sin\frac{2\pi j r}{2L+1}\right)\right]\cos\frac{2\pi m r}{2L+1}$$

$$= \alpha_0 \sum_{r=0}^{2L}\cos\frac{2\pi m r}{2L+1} + \sum_{j=1}^{\infty}\alpha_j\sum_{r=0}^{2L}\cos\frac{2\pi (j+m) r}{2L+1}$$

$$+ \sum_{j=1}^{\infty}\alpha_j\sum_{r=0}^{2L}\cos\frac{2\pi (j-m) r}{2L+1}$$

$$+ 2\sum_{j=1}^{\infty}\beta_j\sum_{r=0}^{2L}\sin\frac{2\pi j r}{2L+1}\cos\frac{2\pi m r}{2L+1} \qquad (3.5)$$

On using equation (3.3) and the orthogonality relationship:

$$\sum_{r=0}^{2L}\sin\frac{2\pi j r}{2L+1}\cos\frac{2\pi m r}{2L+1} = 0 \text{ for all } j, m$$

and

$$\sum_{r=0}^{2L}\cos\frac{2\pi (j \pm m) r}{2L+1} = \begin{cases} 2L+1 & \text{for } j+m = (2L+1)i \text{ where } i = 0, 1, 2, \dots \\ 0 & \text{otherwise}\end{cases}$$

equation (3.5) reduces to:

$$A_m = \alpha_m + \sum_{j=1}^{\infty}\left(\alpha_{(2L+1)j-m} + \alpha_{(2L+1)j+m}\right) \qquad (3.6)$$

for $m = 0$ it gives:

$$A_0 = \alpha_0 + 2\sum_{j=1}^{\infty}\alpha_{(2L+1)j} \qquad (3.7)$$

Similarly, multiplying by $\sin 2\pi m t/T$, (3.1) gives:

$$B_m = \beta_m + \sum_{j=1}^{\infty}\left(\beta_{(2L+1)j+m} - \beta_{(2L+1)j-m}\right) \qquad (3.8)$$

The results (3.6) — (3.8) show that in discrete Fourier series, the effects of higher frequencies than the Nyquist frequency f_c (= 1/2 Δ) fold back on the frequencies $0 - f_c$.

Now if the tidal height is represented as:

$$\zeta = Z_0 \cos(\sigma_0 t - \epsilon_0) + \Sigma Z \cos(\sigma t - \epsilon) \tag{3.9}$$

where zero subscripts denote M_2 and summation is over all the remaining significant lines;

substituting

$$\left.\begin{aligned} V_0 &= \sigma_0 t - \epsilon_0 \\ \theta &= (\sigma_0 - \sigma) t - (\epsilon_0 - \epsilon) \\ q_s &= \Sigma \frac{Z}{Z_0} \sin\theta \\ q_c &= \Sigma \frac{Z}{Z_0} \cos\theta \end{aligned}\right\} \tag{3.10}$$

the equation (3.9) becomes

$$\zeta = Z_0 R \cos(V_0 - \psi) \tag{3.11}$$

where

$$R = [(1 + q_c)^2 + q_s^2]^{1/2} \tag{3.12}$$

and

$$\tan\psi = q_s/(1 + q_c) \tag{3.13}$$

Following DOODSON's assumptions that if q_c and q_s are small, then since :

$$\psi = \tan\psi - \frac{1}{3}\tan^3\psi + \dots\dots\dots$$

the equations (3.12) and (3.13) give :

$$\psi = q_s - q_s q_c + q_s q_c^2 - \frac{1}{3} q_s^3 + \dots\dots\dots \tag{3.14}$$

$$R = 1 + q_c + \frac{1}{2} q_s^2 - \frac{1}{2} q_c q_s^2 + \dots\dots\dots \tag{3.15}$$

The equation (3.11) can be written in the form :

$$\zeta = (Z_0 + \delta) \cos \sigma_0 (\eta_0 - \eta) \tag{3.16}$$

$$\delta = Z_0 \left(q_c + \frac{1}{2} q_s^2 - \frac{1}{2} q_c q_s^2 + \dots\right) \tag{3.17}$$

where

$$\eta = \frac{\psi}{\sigma_0} = \frac{1}{\sigma_0}\left(q_s - q_s q_c + q_s q_c^2 - \frac{1}{3} q_s^3 + \dots\right) \tag{3.18}$$

$$\eta_0 = V_0/\sigma_0$$

As $d\delta/dt$ and $d\eta/dt$ are negligible in the vicinity of turning points, equation (3.16) is an adequate representation of heights and lags of the tide. Here η_0 gives the times of maxima and minima of the M_2 tide and η gives the shift in the maxima and minima of the M_2 resulting from the presence of other constituents. When q_c and q_s are small, δ and η will be small. Under such circumstances, modulation of the M_2 tide will be small and the observed tide will have a profile close to that of the M_2 tide, therefore consecutive high or low water will be almost equally spaced with time interval approximating to half a lunar day (HLD). The Nyquist frequency f_c of such a time series will be

$$f_c = \frac{1}{2}\Delta$$

$$= \frac{1}{2}\text{ cycle/HLD}$$

i.e. constituent M_1 will be at Nyquist frequency, and the terms on the higher frequency side of M_1 in the spectrum will be aliased with terms in the range $0 - \sigma(M_1)$ (speed of constituent M_1) according to (3.6) – (3.8), as shown in Table 1. HORN (1948) also considered this in relation to the

TABLE 1
Sets of aliased constituents and their speeds

No	Speed (deg/HLD)	Constituents
1	0.0	M_4, M_6, M_8, ...
2	0.5100967	Sa, MA_2, Ma_2, ...
3	1.0201934	Ssa, MKS_2, MSK_2, OP_2, ...
4	5.8565756	SN_4, MSN_6, $2MSN_8$, SNM_2, $M\nu_4$, ...
5	6.7614611	N_2, L_2, $2MN_2$, MN_4, ML_4, $2MN_6$, $3MN_8$, ...
6	12.1079635	T_2, ...
7	12.6180366	S_2, $2MS_2$, MS_4, $2MS_6$, $3MS_8$, ...
8	13.1281333	R_2, ...
9	13.6382313	$2MK_2$, MK_4, $2MK_6$, ...
10	18.4746122	$2SN_6$, $2SMN_8$, $M\nu S_2$, ...
11	19.3794977	MSN_2, MNS_2, $2MSN_4$, $2MNS_4$, ...
12	25.2360732	$2SM_2$, S_4, $2SM_6$, 2 $(MS)_8$, $3M2S_2$, ...
13	26.2562679	SK_4, MSK_6, $2MSK_8$, SKM_2, $3M(SK)_2$ . . . . .
14	27.2764626	$2KM_2$, K_4, ...
15	159.5426537	OO_1, KO_3, ...
16	160.5628484	SO_1, SK_3, ...
17	161.5830418	SP_3, ...
18	166.4194239	J_1, Q_1, MJ_3, MQ_3, ...
19	173.1808850	K_1, O_1, MK_3, MO_3, $2MK_3$, $2MO_3$, ...
20	173.6909817	S_1, MS_1, ...
21	174.2010784	P_1, MP_1, SO_3, ...
22	180.0	M_1, M_3, M_5, ...

similar problem of sampling tidal records in lunar time. Another account of HORN's principle is given in Appendix C of MUNK & CARTWRIGHT (1966).

DOODSON transformed the continuous function θ to a discrete function to conform to the fact that information about frequencies higher than the Nyquist frequency is lost due to aliasing, which in the time domain is equivalent to a loss of information in between the sampling points. Here the residual function will represent the envelope as shown in figure 1, whose common solution agrees with the tide at high or low waters only. Any attempt to retrieve informations about the tide from the envelope, at points other than the sampling points, obviously will be in error.

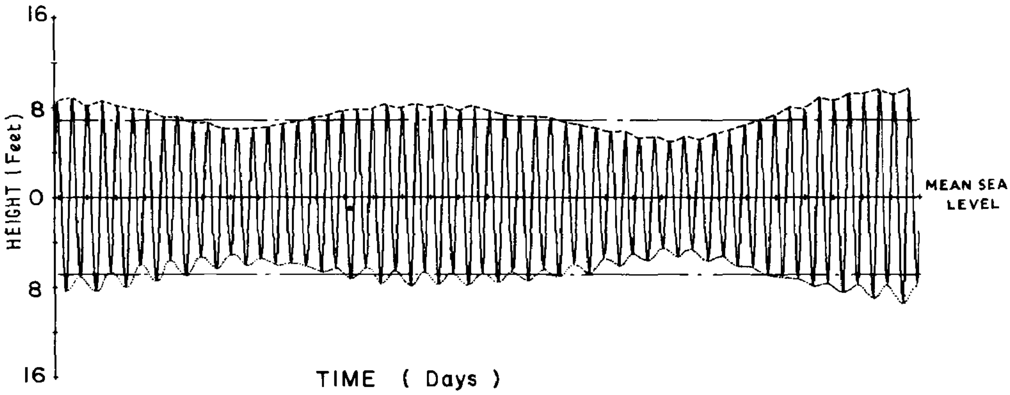

FIG. 1. — Tidal profiles : ——— total tide; —·—·— high and low waters of M_2 tide; - - - - - envelope of high waters; .... envelope of low waters.

The equations (3.17) and (3.18) show that since δ and η are functions of residual constituents, whose speeds are expressed relative to the speed of M_2, therefore they can be computed from differences of heights and times respectively of observed and basic tides. The details of the analysis technique are given in Appendix A.

Thus we conclude that:

(1) high or low water times and heights can be constructed in two steps : (a) computing a basic tide consisting of the M_2 constituent, or more constituents provided M_2 is dominant; (b) then improving the basic tide by using δ and η;

(2) similarly, the tide at one port can be computed from the basic tide of another port if the tidal characteristics of the two ports are comparable.

4. RESIDUAL ANALYSIS

To investigate the performance of the modified E.H.M. and the H.S.W.C. method based on the least squares technique, a comparative study was carried out. Three ports — Southend (*), Liverpool (*) (Princes Pier)

(*) The original data for Liverpool and Southend were in feet and the final results were converted into metres. Some differences under feet and metres, in both Table 2 and Table 3, are due to rounding off.

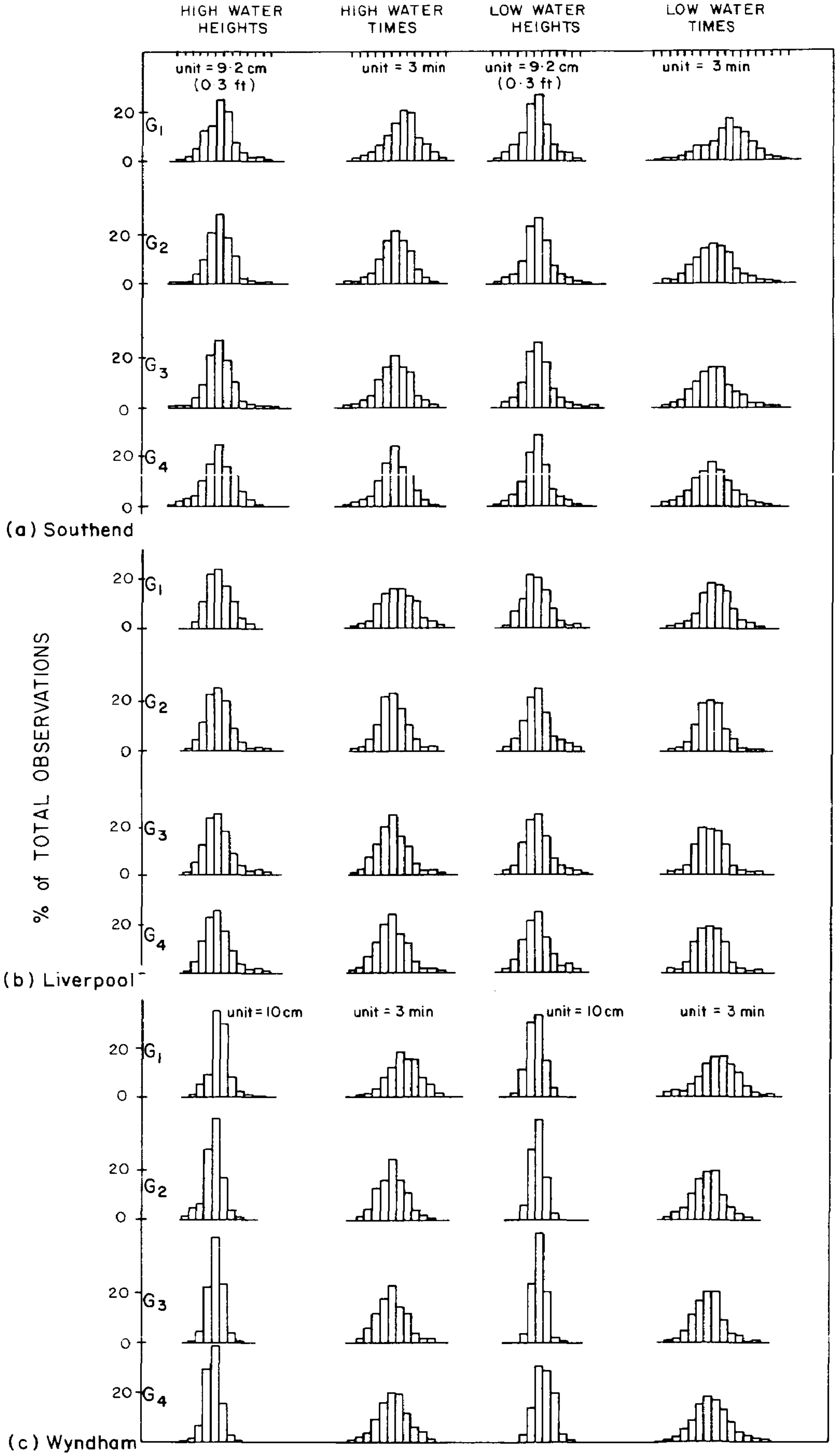
HIGH WATER HEIGHTS
HIGH WATER TIMES
LOW WATER HEIGHTS
LOW WATER TIMES
unit = 9·2 cm (0·3 ft)
unit = 3 min
unit = 9·2 cm (0·3 ft)
unit = 3 min
20
0
G1
G2
G3
G4
(a) Southend
(b) Liverpool
% of TOTAL OBSERVATIONS
unit = 10 cm
unit = 3 min
unit = 10 cm
unit = 3 min
(c) Wyndham

FIGURE 2

and Wyndham (Cambridge Gulf, Australia) — which have significant shallow water characteristics were selected. The tests were carried out on results obtained from analysing one year's data, this being the normal span of data available for tidal analysis. Analyses were performed by the H.S.W.C. method, using different numbers of basic constituents. The residuals (values of observed – predicted levels and times) were computed for each set of constituents. In figure 2 the distribution of errors in the high and low water times and heights are shown in histograms. Some differences between results from the E.H.M. and the H.S.W.C. method are expected when resynthesizing the dependent data because E.H.M. constituents are obtained from analysis of hourly heights whereas in the H.S.W.C. method, although hourly values are used for basic constituents, the emphasis is placed upon residuals at the turning points. The means and standard deviations of residuals are given in Table 2. ROSSITER & LENNON (1968) found that DOODSON'S H.S.W.C. technique gives non-zero means when predicting the same period as analysed, but in the revised method (Appendix A) near-zero means are obtained as expected since techniques of analysis and prediction are compatible.

Noting that 1971 data was analysed for Southend, some similar investigations were made on the data sets Jan. 1, 1962 to Jan. 7, 1963 and Sep. 1, 1967 to Apr. 23, 1968. These periods were used by CARTWRIGHT & ROSSITER (1972) so that, in addition, comparisons between the present methods and I.R.M. were possible. The I.R.M. constants were derived from analysis of the 3-year period 1959-61. The means and standard deviations of residuals are given in Table 3. The H.S.W.C. method gives consistent accuracy, as shown in Table 2 and figure 2, but its superiority over other methods is marginal and this requires explanation. This is considered due to overfitting. The size of the harmonic model in relation to the data is greater in the H.S.W.C. method than in conventional analysis. Consequently the noise content of the data sets is not sufficiently attenuated. E.H.M. gives slightly the best heights while I.R.M. gives marginally the best times. It has been observed, both at Liverpool and Southend, that in the case of very high tides, particularly equinoctial, H.S.W.C. predictions are better than E.H.M. predictions. Some of the observed and predicted equinoctial tides are listed in Table 4 to demonstrate this effect, which seems to be due to the fact that some distortion is associated with tides in the extreme range only, but not the total tide. When dealing with high and low profiles separately, it becomes possible to represent these variations harmonically. The residual spectra were calculated, as displayed in figures 3.1 - 3.4, showing that, on the average, residual power is almost identical. The variation in the peaks at the positions of L_2, S_2 and P_1 is due to the use of nodal parameters in the E.H.M. or basic tide. These parameters are computed on the basis of equilibrium relationships, but in the real tide, as shown by

---

FIG. 2. — Frequency distribution of errors in predictions of turning points using various sets of constituents.

G_1 : E.H.M. Cons. (110); G_2 : E.H.M. Cons. (110) + H.S.W.C.; G_3 : 60 basic cons. + H.S.W.C.; G_4 : 50 basic cons. + H.S.W.C. (for Wyndham, 60 basic cons. of Cape Domett + H.S.W.C.).

TABLE 2

The means and standard deviations of residuals obtained from re-predicting the analysed data.

(a) Southend 1971 ; (b) Liverpool (Princes Pier) from Nov. 27, 1972 to Nov. 27, 1973; (c) Wyndham (Cambridge Gulf, Australia) from July 12, 1973 to July 4, 1974

Prediction Constituents		HIGH WATER			LOW WATER		
		Times	Heights		Times	Heights	
		min.	metre	(ft)	min.	metre	(ft)
(a) *Southend*							
E.H.M. Cons. (110)	Mean	1.56	0.015	(–0.048)	4.24	–0.025	(0.082)
	Standard Deviation	0.98	0.173	(0.566)	10.15	0.184	(0.604)
E.H.M. Cons. (110) + H.S.W.C.	Mean	0.01	0.001	(0.002)	–0.01	0.00	(0.001)
	Standard Deviation	6.07	0.162	(0.539)	9.10	0.179	(0.588)
Basic (60) + H.S.W.C.	Mean	0.0	0.001	(–0.003)	–0.01	0.003	(0.001)
	Standard Deviation	6.28	0.166	(0.544)	9.01	0.175	(0.575)
Basic (50) + H.S.W.C.	Mean	0.09	0.0	(0.0)	0.07	0.0	(0.0)
	Standard Deviation	6.37	0.166	(0.545)	9.08	0.177	(0.581)
(b) *Liverpool*							
E.H.M. Cons. (110)	Mean	2.07	0.015	(0.049)	1.95	0.002	(0.007)
	Standard Deviation	7.42	0.168	(0.550)	7.39	0.200	(0.655)
E.H.M. Cons. (110) + H.S.W.C.	Mean	0.07	0.0	(0.0)	0.0	0.0	(0.001)
	Standard Deviation	6.31	0.162	(0.531)	6.81	0.193	(0.633)
Basic Cons. (60) + H.S.W.C.	Mean	0.05	0.0	(0.0)	0.0	0.0	(0.0)
	Standard Deviation	6.37	0.159	(0.523)	6.94	0.196	(0.642)
Basic Cons. (50) + H.S.W.C.	Mean	0.09	0.0	0.0	0.03	0.0	(0.001)
	Standard Deviation	6.32	0.161	(0.527)	6.99	0.196	(0.643)
(c) *Wyndham*							
E.H.M. Cons. (110)	Mean	3.93	0.026		2.59	0.028	
	Standard Deviation	7.09	0.116		8.80	0.140	
E.H.M. Cons. (110) + H.S.W.C.	Mean	0.00	0.016		0.01	0.013	
	Standard Deviation	6.09	0.100		7.39	0.126	
Basic Cons. (60) + H.S.W.C.	Mean	0.01	0.002		0.01	0.0	
	Standard Deviation	6.35	0.100		7.32	0.122	
Basic Cons. (60) of Cape Domett + H.S.W.C.	Mean	1.00	0.015		1.01	0.024	
	Standard Deviation	7.12	0.259		8.03	0.230	

TABLE 3

The means and standard deviations of Southend residuals from data sets:

(a) From Jan. 1, 1962 to Jan. 7, 1963; (b) From Sep. 1, 1967 to Apr. 23, 1968

Prediction Constituents		High Waters			Low Waters		
		Times	Heights		Times	Heights	
		min.	metre	(ft)	min.	metre	(ft)
(a) *1/I/62-7/I/63*							
E.H.M. Cons (110)	Mean	1.91	0.065	(−0.213)	3.12	−0.011	(−0.035)
	Standard Deviation	9.11	0.205	(0.672)	11.50	0.238	(0.784)
E.H.M. Cons. (110) + H.S.W.C.	Mean	0.43	−0.052	(−0.170)	−1.05	0.014	(0.047)
	Standard Deviation	9.05	0.206	(0.675)	11.32	0.243	(0.797)
Basic Cons. (60) + H.S.W.C.	Mean	0.22	−0.049	(−0.160)	−0.91	0.018	(0.060)
	Standard Deviation	9.28	0.208	(0.684)	11.14	0.244	(0.801)
Basic Cons. (50) + H.S.W.C.	Mean	0.21	−0.036	(−0.110)	−1.01	0.033	(0.109)
	Standard Deviation	9.53	0.209	(0.685)	11.25	0.244	(0.801)
E.H.M. Cons (110) (*)	Mean	1.56	−0.064	(−0.210)	2.46	−0.013	(−0.042)
	Standard Deviation	8.63	0.206	(0.676)	11.66	0.233	(0.765)
I.R.M. (**)	Mean	2.6		(−0.06)	2.7		(−0.17)
	Standard Deviation	8.2		(0.69)	11.1		(0.79)
(b) *1/9/67-23/4/68*							
E.H.M. Cons. (110)	Mean	1.03	0.050	(0.164)	1.73	−0,006	(−0.020)
	Standard Deviation	9.93	0.240	(0.786)	16.49	0.256	(0.841)
E.H.M. Cons. (110) + H.S.W.C.	Mean	0.09	0.092	(0.302)	−2.53	0.051	(0.166)
	Standard Deviation	10.05	0.246	(0.808)	15.65	0.264	(0.865)
Basic Cons. (60) + H.S.W.C.	Mean	0.06	0.077	(0.252)	−1.66	0.030	(0.099)
	Standard Deviation	10.34	0.251	(0.824)	14.64	0.264	(0.865)
Basic Cons. (50) + H.S.W.C.	Mean	−0.12	0.086	(0.282)	−1.67	0.048	(0.160)
	Standard Deviation	10.41	0.255	(0.835)	14.40	0.266	(0.872)
E.H.M. Cons (110)(*)	Mean	1.35	0.053	(0.173)	2.12	−0.010	(−0.032)
	Standard Deviation	10.12	0.237	(0.779)	14.98	0.259	(0.849)
I.R.M. (**)	Mean	2.2		(0.02)	3.6		(0.01)
	Standard Deviation	9.8		(0.82)	13.9		(0.89)

(*) Nodal terms of L_2 are not used.

(**) As in CARTWRIGHT & ROSSITER (1972).

AMIN (1976), these are perturbed by overlapping shallow water constituents. The evidence of figures 3.1-3.4 and Table 3 also suggests that the optimal number of constituents in H.S.W.C. is about sixty but it may slightly vary from port to port.

TABLE 4

Comparison of residuals of extreme high levels obtained from E.H.M. and H.S.W.C. method:

(a) Southend; (b) Liverpool

Date	Observation		Residuals			
			H.S.W.C.		E.H.M.	
	Time (G.M.T.)	Height (ft)	Time (Min.)	Height (ft)	Time (Min.)	Height (ft)
a) *Southend*						
6. 4.62	0058	20.6	– 10	0.3	– 12	0.1
	1333	20.6	– 12	0.0	– 7	– 0.1
14.10.62	0040	20.7	6	0.1	6	– 0.1
	1243	20.5	– 1	0.1	– 3	– 0.1
15.10.62	0143	20.5	6	– 0.1	6	– 0.4
	1331	20.2	3	– 0.3	1	– 0.7
6.10.67	0146	21.2	– 2	0.8	– 2	0.6
	1356	20.5	1	0.1	– 2	– 0.4
28. 3.71	0057	19.2	– 3	– 0.5	– 12	– 0.8
	1341	18.9	1	– 0.7	0	– 1.0
5.10.71	0035	19.6	5	0.0	11	– 0.4
	1241	18.7	5	– 0.9	7	– 1.4
(b) *Liverpool*						
8. 3.62	0012	31.0	5	– 0.3	– 5	– 0.2
	1237	32.6	7	0.0	1	0.2
16. 9.62	0014	32.4	0	0.0	1	0.3
	1236	31.1	8	0.0	– 5	0.2
28. 3.67	0020	32.4	– 7	0.7	5	1.0
	1232	32.9	7	0.4	– 10	0.7
4.10.67	1115	32.1	– 13	1.3	– 8	1.4
	2330	32.7	6	0.6	– 9	0.9
6. 4.73	0030	31.1	6	0.8	– 7	1.0
	1252	31.7	– 3	1.0	– 6	1.1
11.11.73	1122	29.7	10	– 0.2	5	– 0.1
	2341	30.0	8	– 0.5	3	– 0.4

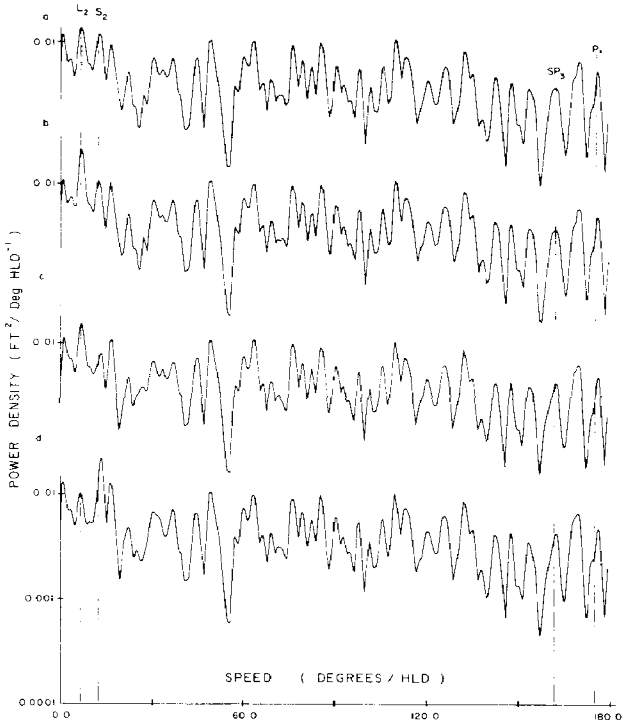

FIG. 3.1. — Aliased power spectra of high water heights residuals : (a) E.H.M. Cons. (110); (b) E.H.M. Cons. (110), without nodal terms of L_2; (c) 110 basic cons. + H.S.W.C.; (d) 60 basic cons. + H.S.W.C.

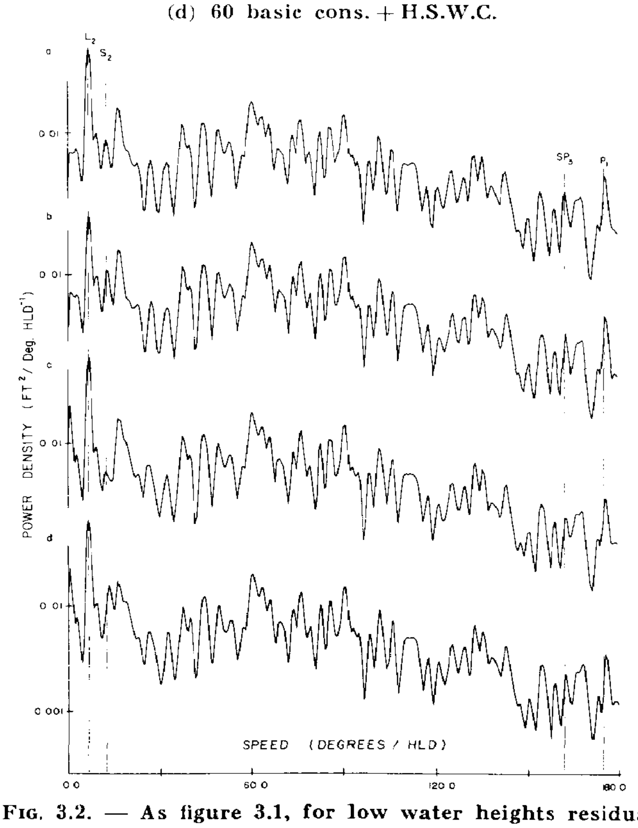

FIG. 3.2. — As figure 3.1, for low water heights residuals.

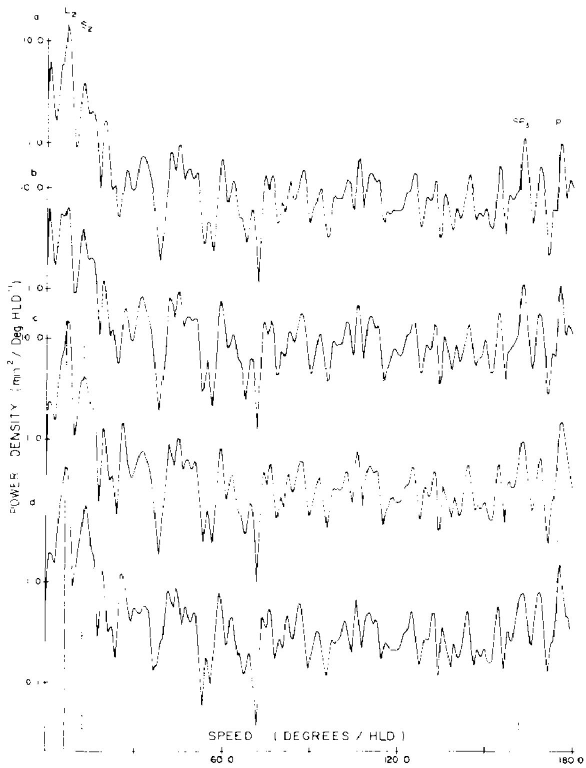

FIG. 3.3. — As figure 3.1, for high water times residuals.

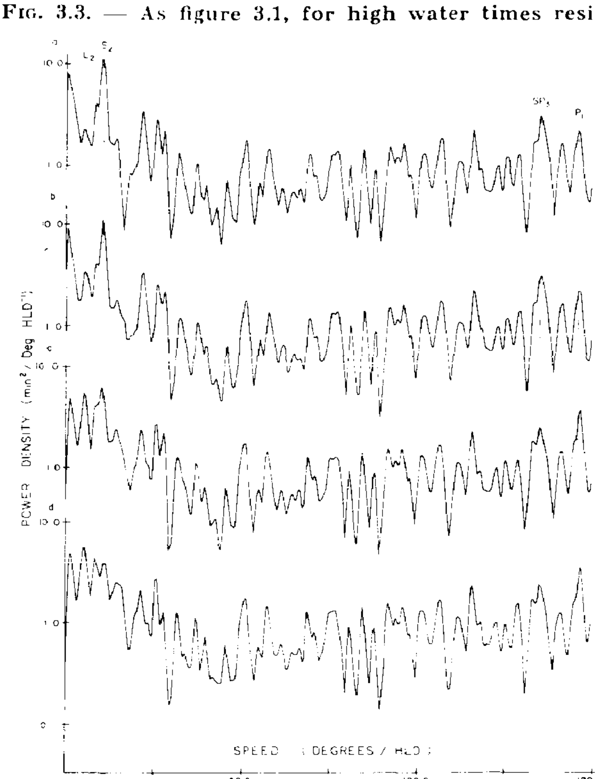

FIG. 3.4. — As figure 3.1, for low water times residuals.

Though the different methods give similar accuracy for a port, residual power varies significantly from port to port. This is due to non-tidal influences such as meteorological effects, fresh-water flow and flooding areas inundated at high tides. At Wyndham, the predicted heights are in good agreement with observations, times are generally quite accurate but errors increase at neaps. This characteristic error is very large in some cases, and appears to be related to local topography. The possibility of obtaining improved predictions for Wyndham using the basic predictions for Cape Domett, which is a deep port and exposed to open sea, was also investigated. However, the best results are achieved by the H.S.W.C. method based on its own basic sixty constituents. The Wyndham problem represents a special case, and it may be possible to explain and solve the problem of large time residuals at neaps on the basis of a mathematical model which can account for inflow and outflow of water from surrounding areas.

Liverpool and Southend predictions give greater residuals than Wyndham. The tidal regimes are highly distorted. Meteorological conditions induce large perturbations; after the perturbing force disappears, a damped oscillatory residual persists for some time.

5. CONCLUSIONS

The results from Cartwright & Rossiter (1972) and the present work suggest that all methods, from both the theoretical and applied points of view, are stretched to their limits. The main problems affecting these analyses and prediction methods are the non-tidal effects in the observed data arising from meteorology. Though the analyses of residuals show that accuracy is similar, it has been observed that for specific purposes one may be given preference over others. I.R.M. at Southend, for example, predicts better times, while E.H.M. gives better heights. The differences are marginal and the choice of one rather than the other is difficult, as computations of times and heights are invariably linked in user requirements. Though the statistical evidence indicates that E.H.M. and H.S.W.C. are similar in accuracy, nevertheless in cases studied, experience suggests that using H.S.W.C. extreme high waters are predicted rather better than those of E.H.M. Using E.H.M., Southend extreme high levels are predicted anomalously high, while Liverpool extreme tides are predicted lower than observations. Since E.H.M. deals with the tidal profile as a whole, and moreover since extreme high and low waters seem to be associated with the maximum distortion of the profile, it is perhaps not surprising that large residuals occur here. These extreme levels are sensitive events for the user concerned with flood warning and matters of coastal defence, so that the superior performance of H.S.W.C. is significant. Although the statistical analysis of residuals does not argue strongly for H.S.W.C. by comparison with other methods, nevertheless it does seem to remove the systematic error associated with extreme high levels, and to the typical user this is significant enough to favour the H.S.W.C. method. H.S.W.C. is a "two-step method" similar to predictor-corrector methods in the numerical integration of differential equations. The basic constituents give an estimate of high and low water times and heights, and the H.S.W.C. constituents are used to improve these estimates. The solution has limitations in that ideally it calls for two contradictory requirements:

a) in the basic tide, the divergence of high or low waters from a regular time interval of half a lunar day should be minimal, since the theoretical speeds of all H.S.W.C. constituents make this assumption. This can be achieved by using M_2 only as the basic tide;

b) the fit of the basic tide to the actual should be good, otherwise the harmonic variation of the actual about the basic will again significantly distort the theoretical time interval of half a lunar day. An average time offset which remains reasonably constant is however acceptable and can be modelled by the H.S.W.C. constituents. This condition can only be achieved by incorporating in the basic tide a comprehensive set of constituents in addition to M_2.

Yet failure to meet either condition introduces an unwelcome noise in the results. In practice it is found that an optimum solution is achieved by the use of about sixty basic constituents, but this assumes a predominant M_2. In diurnally-dominated tides, where K_1 approaches the magnitude of M_2, a satisfactory solution cannot be achieved and the H.S.W.C. method becomes inappropriate. A similar problem arises in certain areas of high shallow-water interaction, which manifests itself in the form of double high or double low waters. There the problem can be overcome by omitting from the basic constituent set certain of the high-frequency tides until the basic profile is simplified into an unambiguous semidiurnal form. In such cases the first and second events are treated as separate time series, each with their own H.S.W.C. analysis and prediction procedure. It is recognised that the presence of shallow-water interaction increases dramatically the problem of predictions and often implies a high susceptibility to additional non-tidal perturbations, which cannot in any case be predicted significantly in advance of real time. Despite the fact that one can expect discrepancies between predicted and observed tide, it has been shown that the H.S.W.C. method can be used effectively to remove systematic bias in the residuals. This in itself is a significant achievement of great value to the user.

APPENDIX A

Analysis Technique

The equations (3.17) and (3.18) show that δ and η, the differences between basic predictions and observed tides both in heights and times, are functions of residual constituents. If the δ's and η's are separated such that:

$\delta^{(H)}$ are the differences of high water heights,
$\delta^{(L)}$ are the differences of low water heights,
$\eta^{(H)}$ are the differences of high water times, and
$\eta^{(L)}$ are the differences of low water times;

then the residual constituents are grouped together to form a small number of constituents called the H.S.W.C. constituents. This feature arises due to aliasing according to the equations (3.6) - (3.8) and as shown in Table 1. Any of these series can be expressed in the form:

$$\delta_i^{(H)} = \sum_k (H_k \cos (W_k + w_k t_i - \chi_k) + n(t_i) \qquad (A.1)$$

where summation is over all the significant lines in the spectrum and

$n(t)$ is the noise in the series,
H is the amplitude,
W is the initial phase, as later defined,
w is the speed (deg/HLD), and
χ is an arbitrary phase lag
of the H.S.W.C. constituent.

The initial phase of the H.S.W.C. constituent depends upon the initial phases of the individual constituents contributing to it, which are given by

$$V = r\tau + as + bh + cp + dN' + ep_1 + \phi \qquad (A.2)$$

Here r, a, b, c, d and e are integers which represent the argument number of the constituent,

ϕ is a phase constant; and
τ = local mean lunar time reduced to angle;
s = the mean longitude of the Moon;
h = the mean longitude of the Sun;
p = the mean longitude of the Moon's perigee;
N' = the negative of the mean longitude of the ascending node of the Moon;
p_1 = the mean longitude of the Sun's perigee.

Taking the high water of the astronomical tide occurring at moon's lower transit, $\tau = 0$, the corresponding phase equation (A.2) becomes:

$$V = W + \phi \qquad (A.3)$$

where

$$W = as + bh + cp + dN' + ep_1 \qquad (A.4)$$

Although it is seen that a single H.S.W.C. constituent comprises the contributions from a number of conventional harmonic constituents, it is notable that the part of the argument number of the latter which determines W in equation (A.4) is in each case identical in magnitude, though possibly opposite in sign. Therefore the initial phase of all the component constituents which are grouped within a single H.S.W.C. constituent can be considered equal. This is possible due to the symmetry of the harmonic cosine function about zero. In this way, constituents of speed w are associated with an initial phase W, and those of speed $-w$ are associated with an initial phase $-W$, although in practice this distinction is irrelevant. In equation (A.1) these are represented by a single constituent such as :

$$H_k \cos (W_k + w_k t - \chi_k) = Z_1 \cos (W_k + w_k t + \phi_1 - \epsilon_1) + Z_2 \cos (-W_k - w_k t + \phi_2 - \epsilon_2) + \cdots\cdots\cdots \quad (A.5)$$

where

$$H_k = \{[Z_1 \cos (\phi_1 - \epsilon_1) + Z_2 \cos (\phi_2 - \epsilon_2) + \ldots]^2 + [-Z_1 \sin (\phi_1 - \epsilon_1) + Z_2 \sin(\phi_2 - \epsilon_2) + \ldots]^2\}^{1/2}$$

$$\chi_k = \tan^{-1} \left[\frac{-Z_1 \sin (\phi_1 - \epsilon_1) + Z_2 \sin (\phi_2 - \epsilon_2) + \ldots}{Z_1 \cos (\phi_1 - \epsilon_1) + Z_2 \cos (\phi_2 - \epsilon_2) + \ldots} \right]$$

Thus the initial phase W of the constituent can be calculated by using the argument number of any constituent of the group.

To calculate s, h, p, N′ and p_1 at the moon's lower transit, these parameters are first calculated for 0 hour, preferably on the 3rd day of the appropriate data set using standard formulae, then the time correction t_l for the moon's lower transit is given by:

$$t_l = \frac{s - h}{14.492} \quad (A.6)$$

A negative value here will indicate the time of the previous lower transit and must not be adjusted; s, h, p, N′ and p_1 can then be incremented by the time difference t_l to give the initial phases of H.S.W.C. constituents when substituted in equation (A.4) for high water series $\delta^{(H)}$ and $\eta^{(H)}$. The origin is fixed at the astronomical high water occurring at time t_l and for low water series, $\delta^{(L)}$ and $\eta^{(L)}$, at the low water which follows t_l. But observed and astronomical tides are not in phase, therefore the above initial phases are applied to the nearest observed high water time t_o such that:

$$|t_o - t_c| = \text{is a minimum}, \quad (A.7)$$

where

$$t_c = t_l + \frac{\epsilon_0}{\sigma_0}$$

Here ε_0 is the phase lag on the astronomical tide and σ_0 is the speed of M_2 used in basic predictions.

Equation (A.6) shows that when h is greater than s, t_l will be negative, therefore if s, h, p, N′ and p_1 are for the 0 hour of the first day of the span of data then t_l will be out of the time scale of data. To avoid this, initially, s, h, p, N′ and p_1 must be calculated for the 0 hour of the third day from the start of data.

Now the equation (A.1) can be written as :

$$\delta_i^{(H)} = \sum_k (A_k \cos w_k t_i + B_k \sin w_k t_i) + n(t_i) \quad (A.9)$$

where

$$\left.\begin{aligned} A_k &= H_k \cos (W_k - \chi_k) \\ B_k &= -H_k \sin (W_k - \chi_k) \end{aligned}\right\} \quad (A.10)$$

The H.S.W.C. constituents can be obtained by solving the redundant system (A.9) for A's and B's by the least squares technique so that the noise effect is minimal. The whole technique of analysis is summed up in the following algorithm:

Algorithm for the H.S.W.C. Analysis

(1) Tabulate observed times and heights of turning points.
(2) Analyse the hourly heights for the basic constituents.
(3) Compute the basic predictions for turning points from the constituents (obtained in step 2).
(4) Form the difference series of times and heights by subtracting the basic predictions (step 3) from the observations (step 1).
(5) Separate the difference series into four series of (a) high water heights, (b) low water heights, (c) high water times and (d) low water times.
(6) Compute s, h, p, N' and p_1 for 0 hour of day 3 of the observations.
(7) Calculate the time correction t_l of moon's lower transit which is given by:

$$t_l = \frac{s \quad h}{14.492}$$

(8) Correct s, h, p, N' and p_1, (step 6) for time t_l, and use these values in equation (A.4) to get initial phases.
(9) Compute the time of the nearest high water as:

$$t_c = t_l + \frac{\epsilon_0}{\sigma_0}$$

and fix the origin at observed high water time t_0 such that $|t_0 - t_c|$ is a minimum.
Initial phases obtained in step 8 correspond to this origin.
(10) Compute A's and B's unknown parameters, from equation (A.9) using the least squares technique, and obtain H's and χ's from (A.10).

ACKNOWLEDGEMENT

I am grateful to Dr. D.E. CARTWRIGHT and Mr. G.W. LENNON for their helpful criticism and suggestions on the work presented in this paper.

REFERENCES

AMIN, M. (1976) : The fine resolution of tidal harmonics. *Geophys. J.R. Astr. Soc.*, **44**, pp. 293-310.

CARTWRIGHT, D.E. & J.R. ROSSITER (1972) : A comparison of modern tidal predictions for Southend Pier, National Institute of Oceanography, Internal Report, **A. 55.**

DOODSON, A.T. (1928) : The analysis of tidal observations. *Phil. Trans. R. Soc. Lond.*, **A 227**, pp. 223-279.

DOODSON, A.T. (1957) : The analysis and prediction of tides in shallow water, *Int. Hydrogr. Rev.*, XXXIV (1), 85-126.

GALLAGHER, B.S. & W.H. MUNK (1971) : Tides in shallow water : spectroscopy, *Tellus*, **23**, pp. 346-363.

HORN, W. (1948) : Über die Darstellung der Gezeiten als Funktion der Zeit. *Deutsch. Hydr. Zeitschr.*, **1**, pp. 124-140.

LAMB, H. (1932) : *Hydrodynamics*, Cambridge University Press.

MUNK, W.A. & D.E. CARTWRIGHT (1966) : Tidal spectroscopy and prediction. *Phil. Trans. R. Soc. Lond.*, **259**, pp. 533-581.

ROSSITER, J.R. & G.W. LENNON (1968) : An intensive analysis of shallow water tides. *Geophys. J.R. Astr. Soc.*, **16**, pp. 275-293.

ZETLER, B. & R.A. CUMMINGS (1967) : A harmonic method for predicting shallow water tides. *J. Mar. Res.*, **25**, pp. 103-114.