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OCEANIC TIDES

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ABSTRACT

Tidal research has had a long history, but the outstanding problems still defeat current research techniques, including large-scale computation. The definition of the tide-generating potential, basic to all research, is reviewed in modern terms. Modern usage in analysis introduces the concept of tidal 'admittance' functions, though limited to rather narrow frequency bands. A 'radiational potential' has also been found useful in defining the parts of tidal signals which are due directly or indirectly to solar radiation.

Laplace's tidal equations (LTE) omit several terms from the full dynamical equations, including the vertical acceleration. Controversies about the justification for LTE have been fairly well settled by MILES' (1974) demonstration that, when regarded as the lowest order internal wave mode in a stratified fluid, solutions of the full equations do converge to those of LTE. Solutions in basins of simple geometry are reviewed and distinguished from attempts, mainly by PROUDMAN, to solve for the real oceans by division into elementary strips, and from localized syntheses as used by MUNK for the tides off California.

The modern computer seemed to provide a 'breakthrough' in solving LTE for the world's oceans, but the results of independent workers differ, mostly because of the inadequacy in their treatment of friction and for the elastic yielding of the Earth. The difference between estimates of the rate of working of the Moon on the tides, about 3.5×10^{12} W, and of the power loss in shallow seas, about 1.7×10^{12} W, marks a serious unidentified sink of energy. Conversion to internal tides has been suggested as a sink, but calculations are at present inconclusive.

Computed models of the global tides do, however, agree well with recent estimates of the Moon's deceleration in longitude. Calculations based on

such astronomical measures suggest an even higher figure for the rate of working, 4.3×10^{12} W. This is considerably greater than the figure of 2.7×10^{12} W accepted until about 1970, thus accentuating the problem of the missing energy sink.

Future progress will require more extensive programmes of pelagic tidal measurement. The technology for this is well developed but slow. Altimetry of the sea surface from satellites could provide the much needed global coverage if properly backed up by high-precision tracking of the instrumented satellite.

1. HISTORICAL RETROSPECT

Tidal science as a branch of dynamical oceanography began with the work of LAPLACE (1775), which he later summarized with subsequent work in his *Mécanique Celeste* (1824). Of course, the foundations of the subject in gravitational theory go back still further to NEWTON's *Principia* (1687), which inspired several extensive series of tidal measurements and their correlation with lunar motions (DEACON, 1970, CARTWRIGHT, 1972). It is therefore a science of venerable age, as was recognized by the IAPSO (*) when their 'Committee on Tides', under the late Professor PROUDMAN of Liverpool, compiled a complete bibliography of tidal publications from 1665 to 1939, a work which has since been updated to 1969 (IAPO 1955, 1957, 1971). This compilation reveals the following counts of publications over 50 year intervals since 1670 :

1670-1719	1720-1769	1770-1819	1820-1869	1870-1919	1920-1969
27	29	37	262	1 121	2 100

Thus, for a fairly uniform subject matter, the quantity of research has been vigorously, if not exponentially, rising for the last 150 years. To be fair, there are signs that the output for the next half-century will not exceed the last figure — the 1960 decade produced about the same number of papers (410) as the 1920 decade — but evidently research in this old subject is still flourishing.

The above facts may surprise the many educated people who think of the tides as a phenomenon of limited interest whose mysteries were largely resolved in the 19th century. Such people are quick to point out that tide-tables, which forecast the times and heights of High Water and Low Water for every port and seaside pier, have been produced satisfactorily for over 50 years. Further, discrepancies between such tables and the observed phenomena are more often the result of the less predictable effects of weather on the sea surface than of errors in the formulation of the astronomically induced tides. Why then should oceanographers continue to investigate the motions induced by the Moon and the Sun ?

The reason in brief is that interest has shifted from the definition of tides in time to their definition in space. Tides are unique among natural

(*) International Association of the Physical Sciences of the Ocean, then known as IAPO.

physical processes in that one can predict their motions well into the future with acceptable accuracy without learning anything about their physical mechanism. When DARWIN (1883-86) first formulated the rules of harmonic tide prediction he virtually produced a cook-book recipe for producing tide-tables for a given place without requiring any reference to Laplace's tidal equations or any kind of hydrodynamical law. All that is needed is to obtain a record of the tide at the place in question for a few lunar cycles — typically a year — and extract from this record the amplitudes and phases of a number of harmonic terms at known lunar and solar frequencies. These can be extrapolated with minor adjustments for many decades into the future. There have, of course, been several advances in technique since Darwin's day (summarized by GODIN, 1971) and new approaches based on modern concepts of time series analysis (MUNK & CARTWRIGHT, 1966), but tidal prediction remains an essentially empirical art. The search for improved accuracy in the prediction of sea level and currents has been more profitably diverted into the study of storm surges, a subject outside the scope of this review.

But since the time of Laplace, scientists have realized that the tides as a mechanical system will never be fundamentally understood until we can define spatially the great complex of tidal waves which oscillate over the world's oceans, where they are generated. The great majority of tidal records in the past have been taken at coastal ports and river estuaries, mostly in shallow seas where the tides are dissipated. These give the least possible information about the behaviour of the tides in the ocean. To this day no satisfactory 'cotidal map' of the oceans has yet been produced. Such a map requires a combination of exacting measurements from the open sea and the solution of a set of partial differential equations with irregular boundaries to satisfy the wave mechanics. The problems involved have taxed mathematicians for over a century and are only now beginning to look possible with the aid of modern oceanographic technology and large computers.

It is relevant to note that the very simplest oceanic tidal problem posed, and only partially solved, by LAPLACE (1824), namely 'what waves are possible on a thin sheet of fluid of uniform depth on a rotating sphere ?' has only recently been completely solved by LONGUET-HIGGINS (1968). Even the foundations of Laplace's dynamical equations for the oceanic tides, which make certain convenient but questionable approximations, were the subject of renewed discussion in the 1940s (PROUDMAN, 1942, 1948, JEFFREYS 1943), and again very recently (MILES, 1974, PEKERIS, 1975). Evidently, the subject is still under close scrutiny. New developments and the failure of old methods have caused scientists to re-examine the fundamentals.

By 1955, despite many ingenious mathematical solutions for tides in idealized basins, PROUDMAN chose as the title of his presidential address to the IAPO : 'The unknown tides of the ocean'. Classical mathematical analysis seemed to have been carried as far as it would go in this field : a new approach was needed. This new approach came shortly afterwards from the numerical techniques pioneered by HANSEN (1949) and PEKERIS & DISHON (1961) and from the deep-ocean pressure sensors pioneered by EYRIÈS *et al.* (1964) and SNODGRASS (1968). A new spirit of optimism was awakened in a field which had previously lost its appeal to the younger researchers. The international Scientific Committee for Oceanic Research (SCOR) set up a new working group devoted to oceanic tides under the chairmanship of W.H. MUNK (later succeeded by the author). In 1966 this group aimed to produce definitive global cotidal maps by about 1972.

But the good intentions proved over-optimistic. After the initial excitement of new results it was noticed that they were not really good enough. Results from independent researchers disagreed with each other and with nature. New corrections, previously ignored, were found to be important and these greatly complicated the solution of cotidal maps. Oceanographic tidal measurements have proved slow and expensive, and their rate has seriously dwindled under the present economic stress. By 1977, the subject has returned to a state of uncertainty, awaiting new ideas. The situation is ideal for a review.

Although most of the developments reviewed here will be recent ones, it is inevitable with a subject of this age to include some discussion of older, well-tried material. This is necessary as a foundation for the later work, and as an introduction to the majority of readers who are not actively concerned with the subject.

2. THE TIDE-GENERATING POTENTIAL

The potential of the tide-raising forces on the Earth's surface due to the Sun or the Moon is fundamental to all tidal studies. It may be defined as :

$$V = \gamma M/r - \gamma M/d \qquad (2.1)$$

where γ is the gravitational constant, M is the mass of the celestial body B, r is the distance of its centre of gravity from a given point P on the Earth's surface, and d is the corresponding distance BC from the Earth's centre (figure 1).

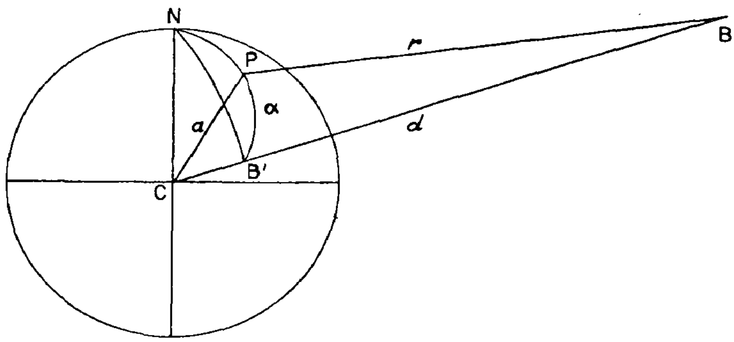

FIG. 1. — Earth, centre C, North pole N. Moon or Sun at B, distance d from C, at zenith angle α relative to point P on Earth's surface.

The first term in (2.1) is simply the (negative) potential of the Newtonian attraction at distance r and the second term is a convenient arbitrary constant in space. The vector tide-raising force per unit mass acting at P is given by the spatial gradient of V, but we do not need to expound its well-known properties here since all useful results may be expressed more simply in terms of the scalar V. It suffices to say that the component of the tide-raising force which is effective in accelerating the water mass is that tangential to the Earth's surface : the vertical component is negligible compared with the Earth's own vertical gravitational forces.

Equation (2.1) is the negative of the strictly physical potential, and is taken so conventionally because a positive increment in V is balanced by a positive rise in the water level at P to retain an equipotential surface. The potential due to the centrifugal force of the Earth's rotation :

$$V_E = \frac{1}{2} \Omega^2 r_n{}^2$$

where r_n is the normal from P to the Earth's axis NC and Ω is the rate of rotation, is merely additive to V. Being independent of the position of B it does not cause a tide but a constant equatorial bulge in the shape of the Earth. (A minor exception is the 'pole tide', caused by the variation in V_E due to the Chandler motion of the position of N, of 1.2 yr periodicity).

Treating the Earth as a sphere of radius a (corrections for a spheroid can be made, but they are not important), (2.1) can be expanded in terms of α, the zenith angle of the body B :

$$V = ga(M/E)\,[\xi^2 \cos\alpha + \xi^3 P_2(\cos\alpha) + \xi^4 P_3(\cos\alpha) + O(\xi^5)] \qquad (2.2)$$

Here g is the gravitational acceleration at P, E is the Earth's mass, $\xi = a/r$ is the body's equatorial parallax, and $P_n(\cos\alpha)$ are the Legendre polynomials :

$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1) \qquad P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)$$

The first term in (2.2) corresponds to a constant force parallel to CB : it is balanced by the orbital accelerations and has no tidal interest. The second term is the principal part of the tide-generating potential, geometrically a symmetrical oval with major axis along CB. Since ξ is about 1/60 for the Moon, the term in P_3 is considerably smaller than the principal term, but its presence can be detected in the oceanic tides. The principal tidal term for the Sun is, on average, 0.45 times that of the Moon, but having ξ of the order of 10^{-4}, its P_3 term is quite negligible.

From the spherical triangle NPB′, where B′ is the sub-lunar point, we may more conveniently express α in terms of NP $= \theta =$ the co-latitude of the place P, NB′ $= D =$ the co-declination of the body, and the angle :

$$\text{PNB}' = \lambda - R + h + 2\pi\left(t - \frac{1}{2}\right)$$

where λ is the East longitude of P, R is the Right Ascension of the body, h is the mean longitude of the Sun, and t is Greenwich Mean Time in days. We shall restrict our attention to the principal tidal term involving $P_2(\cos\alpha)$ and express the "*equilibrium tide*" $\overline{\zeta} = \overline{\zeta}_2 + \overline{\zeta}_3 + ... = V/g$ in terms of the un-normalized spherical harmonics $P_n{}^m(\theta)\exp(im\lambda)$, where :

$$P_2{}^0(\theta) = \frac{1}{2}(3\cos^2\theta - 1) \qquad P_2{}^1(\theta) = 3\sin\theta\cos\theta \qquad P_2{}^2(\theta) = 3\sin^2\theta$$

We obtain for the principal term of the equilibrium tide :

$$\xi_2 = A_2^0(t)\,P_2^0(\theta) + \sum_{m=1}^{2} [A_2^m(t)\,P_2^m(\theta)\cos m\lambda + B_2^m(t)\,P_2^m(\theta)\sin m\lambda] \quad (2.3)$$

where

$$A_2^0(t) = a(M/E)\,k_2^0\,\xi^3 P_2^0(D)$$

$$A_2^m(t) + iB_2^m(t) = a(M/E)k_2^m(-1)^m \xi^3 P_2^m(D)\exp[im(R - h - 2\pi t)] \quad (2.4)$$

and k_2^m is a simple normalizing parameter (note that MUNK & CARTWRIGHT (1966) and works stemming from that paper use a more sophisticated normalization than that implied by (2.3)).

Equation (2.3) separates the time variables from the geographical coordinates. Equation (2.4) defines the three 'species' of time variable $(A, B)_2^m$. Species $m = 0$ varies with the monthly or yearly frequencies of ξ and D multiples thereof, causing the 'long-period' tides. Species $m = 1, 2$ vary primarily at a 'carrier' frequency near m cycles per day with low-frequency modulation from ξ and D, causing the 'diurnal' and 'semi-diurnal' tides respectively.

The functions A, B are used in two different ways. In the first way, they are computed directly from (2.4) as arbitrary functions of time t for direct correlation with tidal measurements. The position of the celestial body is computed with respect to the ecliptic from standard orbital formulae and these are simply transformed to ξ, D and R. Thus in the case of the Sun, if we ignore planetary and other very minor perturbations, the ecliptic longitude ρ and parallax ξ are defined by :

$$\begin{aligned} \rho &= h + 2e\sin(h - p') + \frac{5}{4}e^2\sin 2(h - p') + \dots \\ \xi &= \bar{\xi}[1 + e\cos(h - p') + e^2\cos 2(h - p') + \dots] \end{aligned} \quad (2.5)$$

where e is the eccentricity of the Earth's orbit, p' is the longitude of perihelion, and $\bar{\xi} = 8''.794$ is the Sun's mean parallax. h and p' are defined from astronomical ephemerides as nearly linear functions of time with very small accelerative terms. e and ε (the obliquity of the ecliptic) are nearly constant numbers with slow secular trends. The Sun's declination D and right ascension R are then derived from (2.5) by the equations :

$$\sin D = \sin\rho\sin\epsilon \qquad \cos D\cos R = \cos\rho \qquad \cos D\sin R = \sin\rho\cos\epsilon \quad (2.6)$$

and the results used in (2.4).

The case of the Moon is more complicated because it has a non-trivial latitude variation about the ecliptic, which involves the longitudes of the node n, and also a number of orbital terms depending on the position of the Sun. Besides n, we have two additional secular variables, the mean longitude of the Moon, s, and the mean longitude of perigee, p. These are best grouped in terms of BROWN's variables $Q_1 = s - p$, $Q_2 = h - p'$, $Q_3 = s - n$, $Q_4 = s - h$. The Moon's parallax, longitude and latitude are all expressed in the form :

$$\sum_i R_i \begin{matrix}\cos\\ \sin\end{matrix} \left(\sum_{j=1}^{4} M_j^{(i)} Q_j\right) \quad (2.7)$$

where $M_j^{(i)}$ are sets of small integers including 0, and R_i are amplitudes derived from the tables of BROWN (1919) or as re-computed by ECKERT *et al* (1954). Parallax (cos) and longitude (sin) use the same secular arguments : latitude (sin) use other sets $M_j^{(i)}$. D and R are finally computed from formulae similar to (2.6) but with terms involving the Moon's latitude.

Modern tidal workers differ in the number of terms used in (2.7) and hence in the accuracy of their ephemerides. One certainly does not need the several thousand terms which are used for astronomical work. LONGMAN (1959) used only 4 harmonic arguments. MUNK & CARTWRIGHT (1966) used 7, derived from an older expansion than BROWN'S. The author, as described in CARTWRIGHT & TAYLER (1971), currently uses about 110 arguments for the Moon, giving an accuracy of about 10^{-5} in units of radians or mean parallax. Such accuracy is better than necessary for most purposes. For ordinary analysis of tidal data, about ten terms in parallax and longitude and six in latitude give good results.

The second and older method of referring to the equilibrium tide (2.3) is in the form of a harmonic expansion for $A_n^m(t)$. (The harmonic expansion for $B_n^m(t)$ is identical except for a $\frac{1}{2}\pi$ phase change). The first thorough harmonic series was produced by DOODSON (1921), who used algebraic expansions based on BROWN'S (1919) series (2.7). CARTWRIGHT & TAYLER (1971) and CARTWRIGHT & EDDEN (1973) re-computed them with modern astronomical constants and greater accuracy, using spectral analysis of time series of $A_n^m(t)$. The results are of the form :

$$A_n^m(t) = \sum_i H_i \begin{matrix}\cos\\ \sin\end{matrix} \left(\sum_{j=1}^{6} N_j^{(i)} S_j \right) \tag{2.8}$$

where H_i are tabulated amplitudes, $N_j^{(i)}$ are sets of small integers, and S_j are the six secular arguments $(2\pi t - \pi + h - s)$, s, h, p, $-n$, p' respectively. These six arguments increase almost linearly in time and complete one cycle in a lunar day, a sidereal month, a tropical year, 8.85 yr, 18.61 yr and 21×10^3 yr respectively.

The frequencies of the summed arguments in (2.8) are conveniently arranged in a hierarchy, depending on the fineness of splitting. All terms with the same value of N_1 constitute a 'species' : the species are separated from each other by about one cycle per lunar day. Terms with the same (N_1, N_2) constitute a 'group' : the groups are separated by one cycle per month. Similarly, $(N_1\, N_2, N_3)$ form a 'constituent' at one cycle per year separation. Because of the importance of BROWN'S argument Q_1, all terms of a given constituent conventionally have the same value of N_4 also. Non-zero values of N_5 supply 'nodal modulation' to a constituent. Non-zero N_6 effectively implies a nearly constant phase shift.

As with the harmonic expansion of the lunar orbit, some hundred terms in (2.8) are necessary to define the potential to, say, 1 % accuracy, but a few terms have dominantly large amplitudes. These have been allocated symbols (originally by Darwin) which are frequently used in tidal literature. The following table lists particulars of the four largest terms (constituents) in species 1 (diurnal) and 2 (semi-diurnal). Species 0 (long-period) constituents are not included because they are relatively unimportant. For the given terms $N_5 = N_6 = 0$: ω_r represents 2π times the listed frequency.

Darwin symbol	N_1 N_2 N_3 N_4	Frequency cycles per solar day	Origin	Amplitude (m)	Geodetic factor
Q_1	1 −2 0 1	0.8932441	Moon	0.01293	
O_1	1 −1 0 0	0.9295357	Moon	0.06752	$P_2^1(\theta)$
P_1	1 1 −2 0	0.9972621	Sun	0.03142	$\sin(\omega_1 t + \lambda)$
K_1	1 1 0 0	1.0027379	Both	0.09497	
N_2	2 −1 0 1	1.8959820	Moon	0.01558	
M_2	2 0 0 0	1.9322736	Moon	0.08136	$P_2^2(\theta)$
S_2	2 2 −2 0	2.0000000	Sun	0.03785	$\cos(\omega_2 t + 2\lambda)$
K_2	2 2 0 0	2.0054758	Both	0.01030	

As pointed out by CARTWRIGHT & TAYLER (1971), the amplitudes of the tidal harmonic expansion are not constant on a long time scale but have secular trends of the order of 10^{-4} per century, due principally to the trend in obliquity ε. The 'frequencies' of course also vary slowly with the secular accelerations of the Moon's elements, owing to planetary perturbations and tidal friction.

3. OCEANIC ADMITTANCES

We need to know the response of the ocean to the tide-generating potential. Since the latter is conveniently expanded in a series of low-order spherical harmonics (2.3) it suffices to investigate the response to these spatial harmonics individually. Such a response is a function of time and position. In this section I shall outline the nature of the response in time at a given position on the globe. This sort of response is naturally determined by time series analysis of tidal records, and in fact has rarely been treated by analytical mathematics, which are usually for convenience restricted to a single periodic term such as the M_2 harmonic constituent.

The oceanic tidal response is strongly linear. This is evident from hydrodynamic considerations (§ 4) and can be easily demonstrated by the fact that the spectra of tidal records from places near the deep ocean contain strong lines at the major harmonic frequencies present in the potential (figure 2) and almost negligible energy at multiples of those frequencies. Non-linear effects do become evident in shallow seas and these are important in computing tide predictions for coastal and estuarial ports. However, they have little influence on the global mechanics of the oceanic tides and will be neglected in this review, except in considering the nature of the frictional forces (§ 5).

We expect a relationship of the form :

$$\zeta(t) = \mathrm{Re} \int_0^{\infty} [A(t-\tau) + iB(t-\tau)]\, R(\tau)\, d\tau \qquad (3.1)$$

where ζ is the elevation of the sea surface above its mean level at a given place, A, B are the time-dependent parts of the potential (or equilibrium tide (2.4)) for a particular spherical harmonic, and R is a response function characteristic of the ocean dynamics. I have omitted the suffices m, n for convenience. Or equivalently, if $H(f)$ and $G(f)$ are the complex 'spectra' of ζ and A respectively in the domain of frequency f, we expect :

$$H(f) = Z(f)G(f) \qquad (3.2)$$

where $Z(f)$ is the complex 'admittance' of the ocean to the equilibrium tide, being the Fourier transform of $R(t)$.

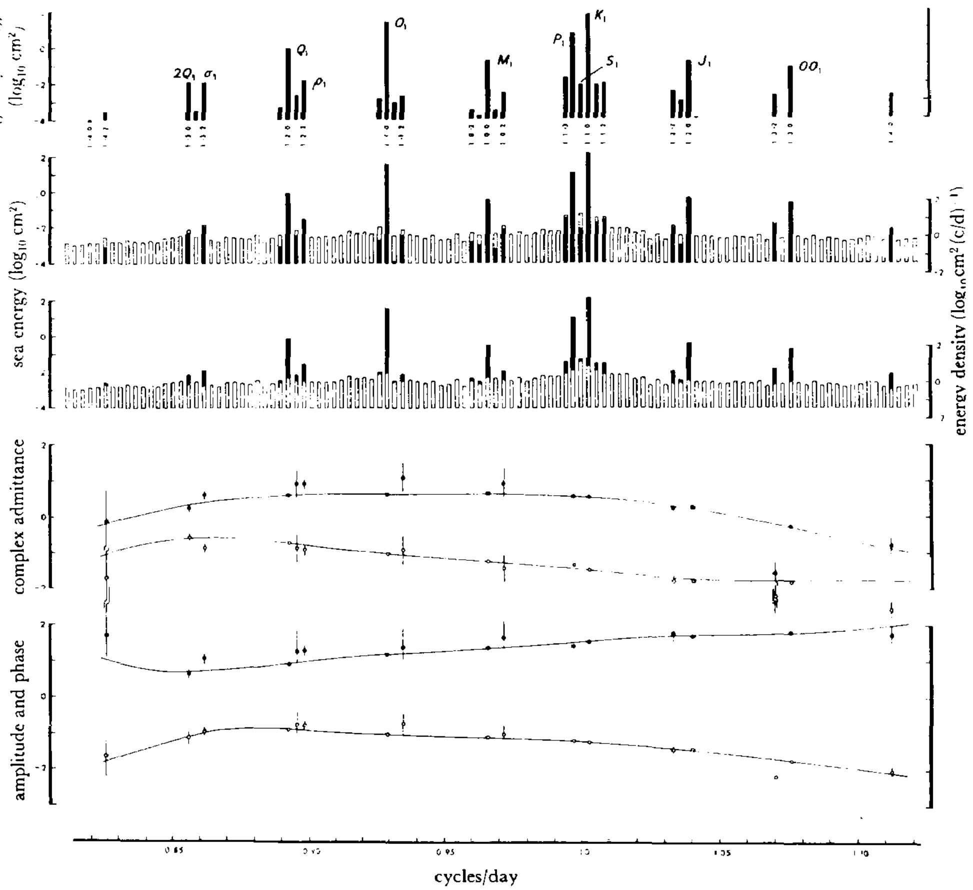

FIG. 2. — Spectroscopy of principal part of gravitational potential (top) and of 18 years' sea level at Honolulu. Black portions of panels 2 and 3 denote part of spectrum coherent with potential, white portions incoherent part. The lowest panels represent the corresponding admittance, spot estimates and smoothed estimates (from MUNK & CARTWRIGHT 1966).

A complete definition of the response function $R(t)$ would tell us a lot about oceanic behaviour, but it is impossible to determine this function from analysis of the tides alone. It would be equivalent to determining the admittance Z at all frequencies : this is clearly impossible since any one tidal species occupies only a narrow band of frequencies, a few cycles per month wide. It is tempting to evaluate $Z(f)$ for a number of tidal species, say diurnal ($f \sim 1$) and semi-diurnal ($f \sim 2$), and interpolate between them to obtain wider coverage of $Z(f)$. This has occasionally been done for rough illustrative purposes, using the geodetic factors as a sort of normalization (e.g. WUNSCH, 1972), but the procedure is invalid because the spherical harmonics associated with the two species are different, with different associated admittance functions. Nevertheless, it is useful to know what these admittances are like, even within the narrow band of each tidal species. Are they nearly constant, smoothly varying or sharply peaked ?

Figure 2 shows a direct spectral comparison between the diurnal tide at Honolulu and the corresponding potential function $A_2^1(t)$, evaluated by MUNK and CARTWRIGHT (1966) — one of many similar analyses which have since been carried out for other places. The top panel shows a spectrum $G(f)$ at 1 cycle per year resolution, computed as a Fourier transform of $A_2^1(t)$ for a span of time simultaneous with the sea-level data. The 'groups' of tidal constituents described in § 2 are obvious. The second panel shows the spectrum $H(f)$ of $\zeta(t)$ — top level of each small column — and also that part of $\zeta(t)$ which is 'coherent' with $A_2^1(t)$ in the statistical sense — lower, black level. The third panel shows the same data as the second but with the 'incoherent' (white) portion of $H(f)$ underneath. The differences between the white and black levels (logarithmic scale) show the important fact, still ignored by many tidal analysts, of a noise continuum in the sea-level spectrum, filling the gaps between the tidal groups and setting a limit to the resolvability of the weaker constituents. It is the continuum which prevents us getting any more information outside the frequency range between groups (1 ± 4) shown in the figure, although the diurnal potential contains weak spectral energy outside this range. The continuum level rises at lower frequencies, as in most geophysical processes. The long-period tides are therefore almost completely swamped by it, unless records of very long duration are analysed or special techniques are employed to reduce the noise level (WUNSCH, 1967 ; CARTWRIGHT, 1968).

The lowest two panels of figure 2 show the estimates of $Z(f)$ derived from the spectral analysis, firstly as real and imaginary parts, and secondly as amplitude ratio $|Z|$ and phase lead arg (Z). Evidently, the function is not constant, but neither does it show any detailed structure within the given frequency band. It is best described as 'smoothly variable'. Most tidal admittances which have been evaluated could be similarly described, although some show a single peak or trough in $|Z|$, suggesting a broad resonance or anti-resonance of a particular ocean basin. This is interesting, because calculations of normal wave modes on a rotating sphere (LONGUET-HIGGINS, 1968) or realistic ocean (PLATZMAN, 1975) show a dense range of such modes at frequencies occupied by the tidal bands. These should produce a corresponding number of peaks in admittance functions, with rapid changes in phase, provided the 'Q factor' is high enough. The fact that such peaks are not in general observed suggests that the oceanic Q must be rather low and dissipation relatively high (GARRETT & MUNK, 1971). On the other hand, WEBB (1973 a) has identified a very narrow peak with associated phase change at Cairns, NE Australia, indicating a rather lightly damped resonance of the Coral Sea (see also MCMURTREE & WEBB, 1975). The most famous tidal resonance, in the Bay of Fundy, Nova Scotia, which produces the largest recorded tidal amplitudes in the world, has a much broader admittance peak, affecting the whole semi-diurnal species. This appears to be a combination of a general magnification of the semi-diurnal tides in the North Atlantic Ocean and more local resonances involving the Gulf of Maine and the Bay itself (GARRETT, 1973 ; HEAPS & GREENBERG, 1974). CARTWRIGHT (1971) has identified an anti-resonance of the diurnal tides in the southwest Atlantic, bringing the admittance amplitude to zero near the frequency of O_1 at Simons Bay.

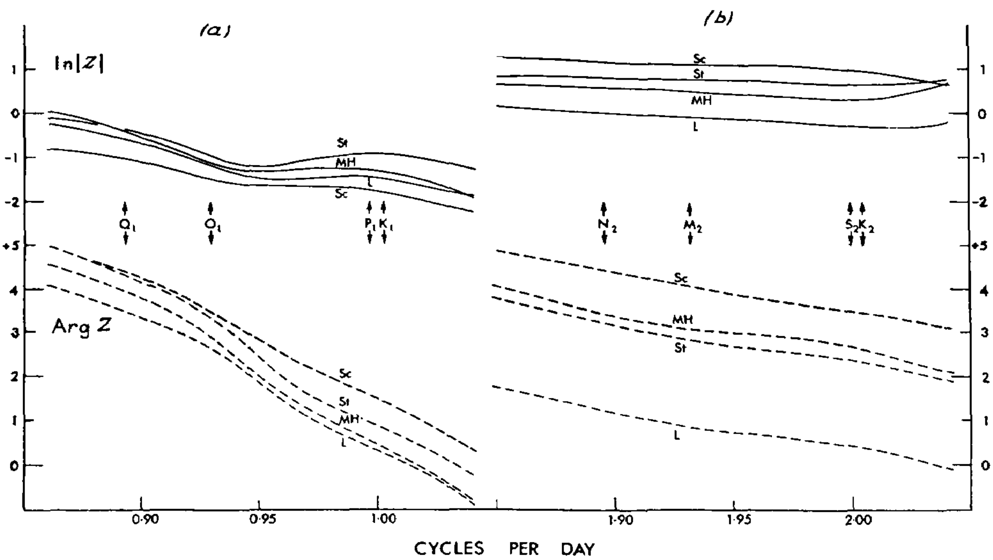

FIG. 3. — Log amplitudes and phases of admittances ((a) diurnal tides, (b) semi-diurnal tides) at four widely spaced locations in the British Isles; Lerwick (Shetland), Stornoway (Lewis), Malin Head (Eire) and Scilly Isless (Cornwall). Arrows denote the frequencies of the principal harmonic constituents.

Figure 3 gives an impression of how little the admittance function varies in character over a distance of the order of 1 000 km. The natural logarithm of Z is drawn for diurnal and semi-diurnal tides for four places on the oceanic coast of the UK, from St Mary's, Scilly Isles to Lerwick, Shetland. $\ln Z = \ln |Z| + i \arg(Z)$ has the characteristic that a general magnification or change of phase lead over the whole tidal species merely shifts the curves representing the real and imaginary parts up or down, without altering their general shape. In fact, one sees that the shapes are remarkably similar at the four places. This applies to the tides of the whole of the northwest European seaboard, when allowances are made for non-linear effects induced by shallow water.

One unusual characteristic shown in figure 3 is the pronounced negative slope of diurnal phase lead, nearly 2π across the species bandwidth. This phenomenon is observed over the northeastern Atlantic from the Azores to the Faeroes, but vanishes on the North American coast. It has never been physically explained. A less pronounced phase slope is common in the semi-diurnal tides throughout the world with some exceptions, and is responsible for the long-observed phenomenon of tidal 'age' whereby the spring tides lag the times of Full and New Moon by a day or so, due to the difference in phase of M_2 and S_2. Tidal age is now firmly attributed to the effect of tidal friction (PROUDMAN, 1941 ; GARRETT & MUNK, 1971 ; WEBB, 1973 b) but the mechanism whereby it is produced is too complicated for a simple heuristic explanation.

On a more technical level, there is some interest in methods of transforming a band-limited tidal admittance function as discussed above into a time relation between $\zeta(t)$ and $A(t)$, $B(t)$. Although we cannot fully define a response function $R(t)$ as it appears in (3.1), there are various ways of defining a finite series of terms like :

$$\tilde{\zeta}(t) = \sum_{k=0}^{K} w_k L_k [A(t) + iB(t)] \quad (3.3)$$

where L_k is a series of simple linear operators and w_k is a set of complex 'weights', whose Fourier transform fits the admittance (3.2) optimally within its valid band of frequencies. MUNK & CARTWRIGHT (1966) used a series of time-lagged terms :

$$L_k [C(t)] = C(t - k\Delta\tau)$$

with $\Delta\tau = 2$ days, equivalent to fitting a Fourier series to $Z(f)$ of periodicity 0.5 cycles per day in frequency. ZETLER & MUNK (1975) have improved on this slightly by writing the right-hand side as $C(t - T - k\Delta\tau)$, where T is the tidal 'age' appropriate to the species. WEBB (1974) proposed :

$$L_k[C(t)] = \exp(i\omega_m t)(-i\partial/\partial t)^k [\exp(-i\omega_m t)\, C^*(t)]$$

equivalent to a simple power series development of $Z(f)$ about a central frequency $f = \omega_m/2\pi$. Yet another scheme has been proposed by GROVES & REYNOLDS (1975), in which L_k is a sequence of specially constructed orthogonal functions of time, called by the authors 'orthotides'. Their advantage is that their associated 'orthoweights' w_k are, unlike in the other schemes mentioned, numerically stable, with a tendency to diminish in magnitude as k is increased.

3.1 Radiational tides

When admittances to the gravitational tide-generating potential are evaluated from a set of tidal data, it is generally found that a small but definite anomaly appears at the frequencies of the solar tides, near 1 and 2 cycles per solar day and at 1 cycle per year. These anomalies have been known for many years to tidal analysts and generally ascribed to 'meteorological effects' without inspiring much curiosity. Now the tides in the atmosphere are almost entirely at the solar frequencies, due to the thermal effect of radiation (CHAPMAN & LINDZEN, 1970). The lunar atmospheric tides are extremely small in comparison with the solar, showing that the gravitational tidal forces are ineffective : part of the lunar tide is even derived from coupling with the ocean (HOLLINGSWORTH, 1971). MUNK & CARTWRIGHT (1966) postulated that the solar anomalies in the ocean were also the result of solar radiation, without specifying the precise physical mechanism. There are various possible mechanisms : direct radiation pressure (small), direct surface heating and cooling, and onshore winds caused by heating and cooling of coastal land, coupling with the atmospheric tide through surface pressure. In order to deal with any of these in a general way, MUNK defined a 'radiational potential', whose gradient equals the amount of radiation received per unit area at a given point of the Earth's surface, neglecting transient atmospheric losses. Thus :

$$U(\theta, \lambda, t) = \begin{cases} S(\xi/\bar{\xi}) \cos \alpha & (0 \leqslant \alpha \leqslant \frac{1}{2}\pi, \text{ day}) \\ 0 & (\frac{1}{2}\pi < \alpha \leqslant \pi, \text{ night}) \end{cases} \tag{3.4}$$

where S is the solar constant, taken as unity for convenience, ξ is the Sun's parallax and $\bar{\xi}$ its mean value, and α is the zenith angle of the Sun at the point P as in figure 1.

U may be expanded in spherical harmonics as was the gravitational potential in § 2. The principal features are the appearance of harmonics P_1^0, P_1^1 owing to the asymmetry of the function (3.4), and the slow convergence of the harmonics of degree 2, 4, 6, ... owing to the night-time cutoff. The zonal harmonic P_1^0 has a strong annual term, otherwise weakly represented in the gravitational tide. This provides what is more usually called the 'seasonal variation in mean sea level', whose global distribution was intensively studied by PATULLO *et al.* (1955). Its physical cause is certainly a combination of the last three of the four mechanisms mentioned earlier, in comparable proportions.

The P_1^1 term, proportional to $e^{i\lambda} \cos \theta$ (θ is the latitude, λ is the longitude) provides a term of solar daily period which is again very weakly represented in the gravitational tide. The terms P_2^0, P_2^1, P_2^2 are similar to the corresponding gravitational terms but may be distinguished from them in the oceanic tide by the fact that they lack contributions from the Moon at their respective frequencies, especially in the harmonic constituents K_1 and K_2. The full harmonic development of all the leading terms in the radiational potential has been tabulated by CARTWRIGHT & TAYLER (1971).

Analyses of tidal records employing both gravitational and radiational tidal potentials ((2.2) and (3.4)) by CARTWRIGHT (1968), ZETLER (1971) and CARTWRIGHT & EDDEN (1977) strongly suggest that the major component of the radiational anomaly is forcing from the atmospheric tide. In the first place, the diurnal anomaly (S_1) is always much weaker than the semi-diurnal (S_2). This may sound strange for a response to the diurnal radiation pattern, but it is a well-known feature of the tide in the surface atmospheric pressure. As explained by CHAPMAN & LINDZEN (1970), the tide in the upper atmosphere is indeed principally diurnal with higher harmonics, but the vertical wave form is such that at ground level the diurnal component is suppressed and the first harmonic (S_2) predominates. In the ocean the amplitude of S_1 is typically 10 mm or so (and in some cases could be partially attributed to a diurnal defect in the recording mechanism of the conventional tide gauge), while that of S_2 is about 17 % of the gravitational S_2 component, typically 50-100 mm.

Another feature of the S_2 radiational tide supporting its derivation from the atmosphere is the fact that its phase tends to lead that of the gravitational S_2 component by an angle near 240°. This tallies with the well-known property of the semi-diurnal atmospheric tide, that its minimum (corresponding to a positive static rise in sea level) occurs at about 4 o'clock, local time, everywhere on the globe. However, since the amplitude of the atmospheric tide is only of the order of 1 mbar, corresponding statistically to about a 10 mm rise in sea level, the dynamics of the transfer mechanism cannot be said to be properly understood.

4. LAPLACE'S TIDAL EQUATIONS AND SOLUTIONS FOR IDEALIZED GEOMETRY

We now have to consider the dynamical equations on which so much analytical tidal work has been based over the past 150 years. Laplace's equations for continuity and momentum are physically plausible and have been verified in several limited seas. However, they do not perfectly represent the mechanical system. The neglected terms are small, but it is not at all obvious at what stage of solution of the complete equations they should be set equal to zero, when oceans of global dimensions are considered. Scepticism was first raised by BJERKNES *et al.* (1933), who pointed out that in neglecting vertical acceleration, Laplace's equations did not permit vertical cellular motions which could be important features of tidal motion at certain critical latitudes, especially for the diurnal tides in the atmosphere. There followed a period of controversy in which the principal defender of the conventional approximations was PROUDMAN (1942). He showed that in certain spherical basins for which simple solutions to both the 'complete' and Laplace's equations were possible, Laplace's equations gave a very good approximation everywhere except near the poles for the K_2 constituent and near the equator for long-period tides. Later, following some work of HYLLERAAS (1939), PROUDMAN (1948) showed that when density stratification is allowed for, even these restrictions are removed. Most recently, the limiting processes in ignoring vertical acceleration and other terms in Laplace's equations have been rigorously analysed by MILES (1974), who arrived at a justification which, when expressed in simple terms, is fairly similar to that of PROUDMAN (1948).

In this review it would be impossible to give an adequate account of MILES' analysis, but it is appropriate to point out the ways in which Laplace's equations differ from a complete mechanical description and under what circumstances the relevant terms may be justifiably neglected. We take fixed local coordinates (x, y, z) in the south, east and vertically upwards directions respectively, and corresponding velocity components (u, v, w). Let also $\boldsymbol{v}$ denote the two-dimensional horizontal vector with components (u, v) and ζ the elevation of the surface above its mean level $z = 0$. We assume the bottom, $z = -h(x, y)$, to be fixed at this stage ; the elastic yielding of the Earth will be considered in § 6.1.

In the first place, we ignore products of the variables, such as occur in the distinction between Eulerian acceleration $\partial \boldsymbol{v}/\partial t$ and the particle acceleration $D\boldsymbol{v}/Dt = \partial \boldsymbol{v}/\partial t + \boldsymbol{v}\,\text{grad}\,.\,\boldsymbol{v}$; we also ignore horizontal viscosity. Both may be shown to be very small in comparison with the leading terms in the oceanic tides, and moreover they do not radically alter the character of the motion. Vertical viscosity $\nu \partial^2 w/\partial z^2$ may enter the equations when integrated over the depth, giving a resultant horizontal stress from bottom friction, but even this is negligible in the main body of the ocean with which we are concerned at present. The equation of volume continuity is :

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = -\rho^{-1}\,\partial \rho/\partial t \qquad (4.1)$$

where ρ is the density. If the fluid is taken to be homogeneous, the only contribution to $\rho^{-1}\partial\rho/\partial t$ is through compressibility. This term may be shown to be of the order of (gh/c^2) times typical terms on the left, where c is the speed of sound in water (1400 km s^{-1}). Since h is typically 4 km, the factor quoted is of the order of 1/50 and may be neglected because again it does not alter the character of the motion. Putting the right-hand side of (4.1) equal to zero and integrating vertically, we obtain the usual expression of continuity :

$$\partial(h\bar{u})/\partial x + \partial(h\bar{v})/\partial y + \partial\zeta/\partial t = 0 \tag{4.2}$$

where $(\bar{u}, \bar{v})$ are the depth-averaged horizontal components of velocity.

With no more than the above-mentioned restrictions the equations of horizontal and vertical acceleration are respectively :

$$\partial \mathbf{v}/\partial t + 2\boldsymbol{\Omega}'\wedge\mathbf{v} + 2\boldsymbol{\Omega}''\wedge\mathbf{w} = -\operatorname{grad}(p/\rho - g\bar{\zeta}) \tag{4.3}$$

$$\partial \mathbf{w}/\partial t - 2\boldsymbol{\Omega}''\wedge\mathbf{v} = (-\rho^{-1}\partial p/\partial z - g)\mathbf{k} \tag{4.4}$$

where $\boldsymbol{\Omega}'$, $\boldsymbol{\Omega}''$ are the vectors representing the local vertical and horizontal components of the Earth's rotation, of magnitudes $\Omega\cos\theta$, $\Omega\sin\theta$ (θ is the co-latitude) respectively, $\mathbf{w}$, $\mathbf{k}$ are the vertical vectors of magnitude w, 1, and p is the pressure. $g\bar{\zeta}$ is the generating potential as discussed in § 2. The terms involving w and the second term of (4.4) are in general small compared with the leading terms and are often omitted, as they are in Laplace's equations. However, it is just these omissions which have raised scepticism.

To illustrate the difficulties, in the simplified case of periodic motions in constant depth h and uniform density, Laplace's equations (4.2) and (4.9) reduce to a freewave equation in ζ :

$$\nabla^2\zeta + (gh)^{-1}(\omega^2 - 4\Omega'^2)\zeta = 0 \tag{4.5}$$

where $2\pi/\omega$ is the period and $\Omega' = |\boldsymbol{\Omega}'| = \Omega\cos\theta$ ($2\Omega'$ is often denoted by the symbol f). Equation (4.5) is elliptic and has solutions which change character from trigonometric to exponential at the 'critical co-latitudes' given by $\cos\theta = \omega/2\Omega$ (near 30° latitude for diurnal periods, near the poles for semi-diurnal). The full equations (4.1), (4.3) and (4.4), however, reduce to an equation which is hyperbolic if $\omega^2 < 4\Omega^2$, and their solutions near the critical latitudes do not reduce to those of (4.5) as h approaches zero. As PROUDMAN (1942) showed, the solutions of the full and the approximate equations approach each other uniformly only in certain zones remote from the critical latitudes.

By allowing for the realistic complication of density stratification the dilemma is resolved (PROUDMAN, 1948, MILES, 1974). Equations (4.1), (4.3) and (4.4) still stand with $\rho = \rho(z, t)$, and in addition we have the equation for mass continuity :

$$\partial\rho/\partial t - wN^2\rho/g = 0 \tag{4.6}$$

where :

$$N^2 = -g\rho^{-1}\partial\rho/\partial z \tag{4.7}$$

(N is the 'buoyancy frequency' which is the natural frequency of internal oscillations of the fluid, typically about 10 times the semi-diurnal tidal frequency). The non-static terms in the equation for vertical motion (4.4) are now balanced by buoyancy forces through the term ρ^{-1}. The solution of (4.1), (4.3), (4.4) and (4.6) allows an infinite set of wave modes with different vertical structure according to their mode number. The lowest order mode, having motion nearly uniform with respect to the vertical, is found to agree closely with the solution to the simplified barotropic (i.e. Laplace's) equations at all latitudes. Near the critical latitudes the previous discrepancy is replaced by a weak coupling between the barotropic motion and the internal motions represented by the highest order modes.

Confidence in the traditional approximation for obtaining global tidal solutions is thus restored, since the ocean is always well stratified. Laplace's equations may be derived by setting $\rho = \bar{\rho}$, the mean density of the sea, assumed constant, and ignoring the term in w in (4.3) and the left-hand side of (4.4). Vertical integration of (4.4) then gives the ' hydrostatic ' approximation :

$$p = \bar{\rho} g(\zeta - z) + p_0 \qquad (4.8)$$

where p_0 is the atmospheric pressure, assumed uniform. Substituting (4.8) in (4.3) and again averaging in the vertical gives :

$$\partial \bar{\mathbf{v}}/\partial t + 2\mathbf{\Omega}' \wedge \bar{\mathbf{v}} = -g \text{ grad } (\zeta - \bar{\zeta}) \qquad (4.9)$$

Equation (4.9) combined with the continuity equation (4.2) and expressed if necessary in terms of co-latitude and longitude constitute Laplace's tidal equations for the independent variables $\bar{u}$, $\bar{v}$, ζ (PEKERIS (1975) has recently given a more formal derivation).

Much mathematical effort was devoted from the time of Laplace to about 1950 in deriving solutions for the tidal equations in an ocean of uniform depth covering a complete sphere or simple portions of a sphere. Reviews of the older work may be found in LAMB (1932) and DOODSON (1958). Here I outline only the principal results as far as they have a bearing on modern work. HOUGH (1897) improved on Laplace's method of solution for a complete sphere, and showed that the family of eigensolutions divides itself into two classes : I, those which merge into ordinary gravity waves as Ω tends to zero : and II, those which tend to steady currents with little asociated surface elevation. Waves of class II exist only at low frequencies and correspond with what we now know as 'Rossby' or 'planetary waves', in which the main restoring force is the variation of the Coriolis force $2\mathbf{\Omega}' \wedge \mathbf{v}$ with latitude (LONGUET-HIGGINS, 1964). Other class II waves have since been identified as being associated with abrupt changes in topography (varying h) such as the ' continental shelf waves ' which have been shown to exist at tidal frequencies (CARTWRIGHT, 1969, HUTHNANCE, 1974).

The complete range of possible eigenfunctions for the spherical ocean was computed by LONGUET-HIGGINS (1968). Figure 4 shows a plot of the eigenfrequencies as a function of $(gh/a^2)^{1/2}$, both normalized with respect to 2Ω (a is the radius) for spherical harmonic waves denoted by parameters n and s, with in this case $s = 2$. They are divided firstly into a group of modes travelling eastward (a), in which all waves are of class I, and those travelling westward (b) which are further subdivided into a highfrequencies group, again of class I, and a lower frequency group which are of class II. A numerical value of the abscissa for typical oceanic depths is 0.2 and the semi-diurnal tidal frequency gives about ± 1 in the ordinate. We see that these values fall well within the range of normal modes, so it is not surprising that oceanic resonances to the tide-generating forces occur. LONGUET-HIGGINS also identified eigensolutions corresponding to negative values of h. These are not possible physically as free waves, of course, but they would become relevant in matching a given forcing field with a complete set of normal modes.

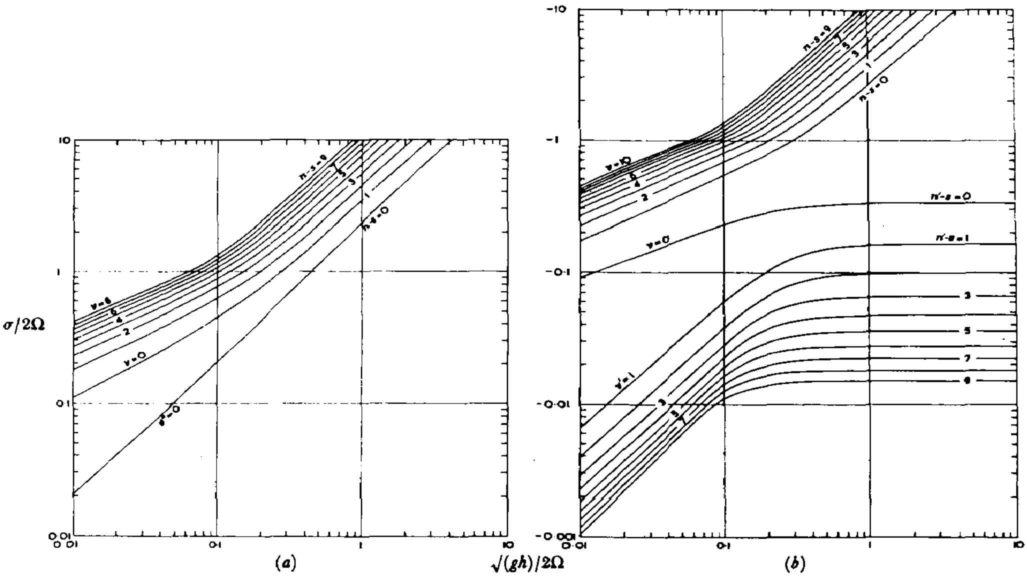

FIG. 4. — Eigenfrequencies of free modes of oscillation on a sphere proportional to exp $(2i\lambda)$. (a) Modes travelling eastwards, (b) modes travelling westwards. In the abscissa, h is the uniform depth of the fluid in units of the Earth's radius. The ordinate gives frequency in units of the inertial frequency at the pole (from LONGUET-HIGGINS 1968).

Solutions for a bounded portion of a sphere come a little nearer to representing the effects of the continental boundaries to the real oceans. The necessity to set the normal velocity to zero along the boundaries greatly complicates the mathematics. In an important series of papers, PROUDMAN (1917, 1931) set out general theorems by which the normal modes and forced solutions may be calculated for oceans bounded by arbitrary coasts. GOLDSBROUGH (1927) solved for the forced K_2 tidal constituent in a sea bounded by two meridians 60° apart, somewhat like the Atlantic Ocean. He found that resonance occurred at depths rather less than that of the actual Atlantic Ocean. PROUDMAN & DOODSON (1936, 1938) solved for both the K_1 and K_2 tidal constituents in an ocean bounded by meridians 180° apart, somewhat like the Pacific. As the assumed mean depth was reduced, successive resonances were passed, at each of which a new node of zero elevation appeared in the solution, more nodes for K_2 than for K_1. It is now known that at least 2 nodes exist in the Pacific Ocean for K_1 and 6 for K_2. The complete set of normal modes for such a hemispherical ocean has been computed by LONGUET-HIGGINS & POND (1970), in a manner similar to LONGUET-HIGGINS' treatment of the spherical ocean. All these calculations confirm that elevations tend to be highest near the bounding coast, somewhat after the manner of the well-known 'Kelvin wave' in simpler geometry.

5. TIDAL CALCULATIONS IN OCEAN OF MORE GENERAL SHAPE

Exact solutions for simple portions of a sphere aid our understanding of how tides behave in the real ocean, but do no more than that. Irregular boundaries and depth topography greatly complicate the situation and require other techniques for solution. We should distinguish here between oceanic solutions and solutions for shallow semi-enclosed seas. In the latter case, the gravitational forcing function represented by $g\bar{\zeta}$ in equation (4.9) can be ignored and the tides can be treated as free waves driven by the wave form at the sea's openings to the ocean, which in many cases can be determined from measurement. The elevations round the coast are, in any case, determinable from measurement. If the tidal currents are also known these also provide boundary gradients of ζ through equation (4.9).

Even simpler is the case of an elongated gulf or channel like the Adriatic or Red Seas. There we may take the x axis along the gulf, and assume that transverse currents v are much smaller than u (a procedure justified with some rigour by PROUDMAN (1925 a)). The equations of continuity and acceleration ((4.2) and (4.9)) reduce to :

$$\partial(h\bar{u})/\partial x = -\,\partial\zeta/\partial t \qquad \text{and} \qquad \partial\bar{u}/\partial t = -\,g\partial\zeta/\partial x - \tau(\bar{u})/\rho h$$

respectively, where $\tau(u)$ is the frictional stress on the bottom, and $h = h(x)$ is the mean depth of each transverse section of the sea. With a linear form $\tau = \rho c\bar{u}$ for the frictional stress, which is usually adequate at this degree of approximation, and a tidal constituent of period $2\pi/\omega$ for which $\bar{u} = U \exp(-\mathrm{i}\omega t)$, $\zeta = Z \exp(-\mathrm{i}\omega t)$, the above equations yield :

$$\mathrm{i}\omega Z = \partial(hU)/\partial x \qquad (\omega^2 - \mathrm{i}\omega c)\,U + g(\partial/\partial x)[\partial(hU)/\partial x] = 0$$

These equations may easily be solved numerically for a given sequence of depths $h(x)$, with the boundary values $Z = Z_0$ at the open sea end(s) of the gulf and $U = 0$ at a closed end. Finally, the gradient of elevation across the width of the gulf may be obtained from the transverse equation involving simply the 'Coriolis' or 'geostrophic' stress :

$$\partial\zeta/\partial y = -\,2\Omega' U(x)/g$$

DEFANT (1961) shows several examples of cotidal maps for elongated seas derived essentially by this method with hand computation.

But such methods, even when extended to two dimensions, cannot deal directly with seas of oceanic scale because of the complications of spherical geometry, the tide-generating forces, and the relative lack of boundary data. While the full problem can only be tackled by finite-difference methods with large modern computers, it is worth briefly considering some of the semi-analytical methods which have been applied to limited portions of open oceans. Several approaches were developed by PROUDMAN and his associates in pre-computer days. His 'tidal theorem' (PROUDMAN, 1925 b) relates surface integrals of the forced tides over the area of any sea on a sphere to line integrals round its boundaries, after the manner of Green's potential theorem. In its more practical form, if a certain family of free waves can be defined in the sea area APA′ (figure 5 (*a*)) but unrestricted by the boundaries, then the elevations and currents of forced tides along the parallel of latitude AA′ can be determined by a series of explicit integrals involving the family of free waves, the tide-generating potential, and the (coastal) tidal elevations round the boundary APA′. The process can then be repeated for the zone BAA′B′, and so continued to cover possibly an extensive sea area. Unfortunately, however, the free-wave solutions can only be solved tractably for a sea of constant depth, so at least each latitudinal strip must be approximated by a uniform depth, which is sometimes unrealistic. The method appears to have been applied seriously only in one case, to part of the Indian Ocean (FAIRBAIRN, 1954).

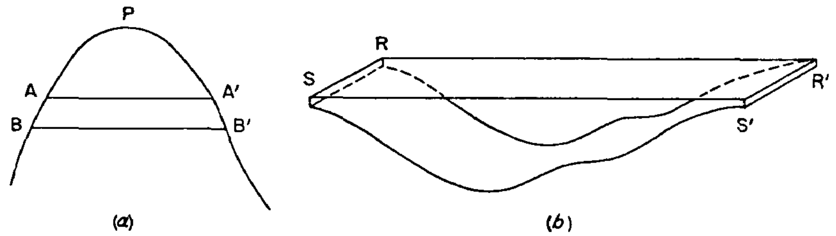

FIG. 5. — (*a*) Illustrating PROUDMAN's tidal theorem. (*b*) Geometry of PROUDMAN's (1946) general expansion for tides in a two-dimensional latitudinal section.

A method which involves dividing an oceanic area into a small number of rectangles, each of constant depth, was applied by GRACE (1932, 1935) to the Gulf of Mexico and the Bay of Biscay. It has the disadvantage that the tidal currents must be assumed constant along each of the rectangle's sides, which tend to be rather long for such a property to be realistic.

PROUDMAN (1946) developed a more powerful approach. A narrow strip of ocean is bounded by two parallels of latitude, RR′ and SS′ (figure 5 (*b*)). Within the strip, the depth topography may be any function of longitude, but must be constant along each meridian. An important touch of realism is added by allowing the tides within the strip to radiate outwards into adjacent shallow water at R and R′ at an arbitrary rate determined by the depths at those ends (possibly zero) ; this allows for frictional loss of energy. Given any distribution of tidal elevation and current along RR′, PROUDMAN (1946) derived explicit formulae for the tides in the interior, including the other boundary SS′. He went on to apply this formalism to the central Atlantic Ocean between 35° S and 45° N, by dividing the whole area between the two continents into 5° strips, each of appropriate depth topography. Starting with an arbitrary functional form for the tides along the southern boundary, he calculated its effect propagating through each successive strip to the northern boundary. He did this for a range of starting functions, and then found a linear combination which best fitted the known tidal elevations along the continental edges. As related in his

George Darwin Lecture (PROUDMAN, 1944) (*) the result was only partly successful, but could have been improved by taking a longer series of boundary functions which were essentially a series of free Kelvin and Poincaré wave forms, with an added wave forced by the generating potential. However, it would appear that the promise of the computer-based solutions produced soon after by HANSEN (1949) discouraged PROUDMAN from pursuing his more analytical method any further.

PROUDMAN would have been more successful if he had had some direct measurements along his initial boundary. More recently MUNK *et al* (1970) have used a method related to PROUDMAN's (1944) to rationalize the tidal system in the Pacific Ocean off California using deep-ocean tidal measurements (see § 9) as well as those from ordinary coastal tide gauges. They treated the coastline as straight (the x axis) and the ocean as semi-infinite with a uniform underwater profile independent of x, stretching in the y direction from a narrow shelf structure to a uniformly deep region some distance from the coast. The effect of varying latitude on the vertical component of the Earth's rotation Ω' was ignored. Over this simplified topography they considered all possible wave forms :

$$f(y) \exp [i(\beta x - \omega t)]$$

which satisfy Laplace's tidal equations. The frequency ω was taken as positive, so positive values of β represent waves travelling along the coast with shallow water on their left, and vice versa. The principal solutions divide into two groups according to whether $f(y)$ behaves at a distance like $\exp(-\mu y)$ or like $\exp(i\mu y)$. The first case represents 'trapped' waves, of which the 'Kelvin' wave, for which $\mu = 2\Omega'(gh)^{-1/2}$, is a classic example (LAMB, 1932) and these, like the Kelvin wave, are associated with a unique wavenumber β for a given frequency ω. If the frequency is less than the inertial frequency (e.g. the diurnal tides polewards of 30° latitude), then β must be negative to maintain geostrophic balance in the northern hemisphere (positive in the southern). At higher frequencies, positive but numerically smaller wavenumbers β are also possible.

The other principal family of waves is oscillatory in the offshore direction and represents the 'leaky' modes, of which the classic example is the 'Poincaré' wave (LAMB, 1932). These waves exist only at frequencies ω higher than the inertial frequency $2\Omega'$ and their wavenumbers, positive or negative, are generally lower than the trapped waves at the same frequency. Also, since μ is arbitrary in this case, the wavenumbers β may take *any* value between certain limits for a given frequency. There is thus a 'continuum' region of permissible leaky wave modes in the (β, ω) plane.

The complete range of solutions for the shelf geometry considered by MUNK *et al* is represented in figure 6, in which β is scaled to L^{-1} where L is the shelf width, and the vertical ω axis is scaled to the inertial frequency $2\Omega'$. The trapped waves as described above are represented by the loci named 'discrete edge waves, $n = 0$'. Similar loci for $n = 1$ and 2 are higher mode trapped waves possessing oscillations in $f(y)$ between the shelf edge

(*) This paper logically follows PROUDMAN (1946), which was actually presented in 1942.

and the coast. The leaky mode 'continuum' is shown bounded by a hyperbola with apex at $\omega/2\Omega' = 1$. Within this continuum, the authors identified certain discrete waves for which the amplitudes near the coast are magnified (full circles) or reduced (open circles). The former are considered to be more likely to represent the observed tidal conditions. From these free-wave solutions, to which they added a waveform directly forced by the generating potential, the authors managed to select a combination which convincingly fitted a number of onshore and offshore tidal measurements in amplitude and phase. Both diurnal and semi-diurnal tides were treated. The semi-diurnal tidal waveform indicated an amphidrome (node) about 2 000 km offshore, which was later confirmed by supplementary measurements.

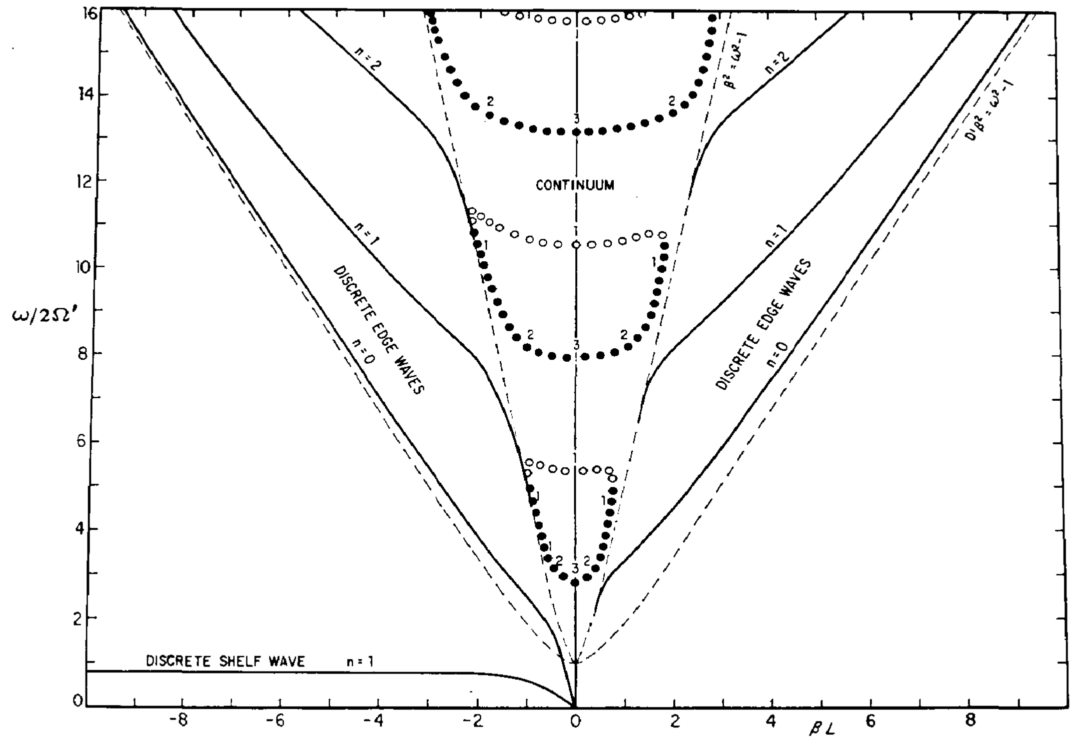

FIG. 6. — Dispersion diagram for long waves off a model oceanic shelf. The abscissa represents long-shore wavenumber with positive values progressing with shallow water to the left (northern hemisphere). The ordinate represents frequency in units of the local inertial frequency $2\Omega'$. Full curves denote permissible (trapped) modes whose amplitude decreases exponentially towards the deep ocean. Waves of Poincaré-type (periodic in all directions) are possible anywhere inside the inner hyperbola but are likely to be large near the coast at the full circles, small at the open circles (from MUNK *et al.*, 1970).

At the bottom left of figure 6, another species of freewave is described as a 'discrete shelf wave'. This is otherwise named a 'continental shelf wave' or 'topographical Rossby wave', and was referred to in § 4 as a newly discovered (ROBINSON, 1964) example of HOUGH's 'class II' waves, consisting principally of currents. As figure 6 suggests, shelf waves exist only at frequencies below the inertial, and travel only with shallow water on their right (northern hemisphere). Although MUNK *et al* measured tidal currents at their pelagic sites, they did not find any evidence for the existence of such waves at tidal frequencies off California. CARTWRIGHT (1969, 1974), however, found that their properties matched those of some unusually strong diurnal tidal currents off the west of Scotland, a phenomenon which had been virtually forgotten since first publicized in an early volume of the Royal Society's transactions by Sir Robert Moray. HUTHNANCE (1974) has also examined some enhanced diurnal currents on Rockall Bank in terms of shelf wave theory. In both areas mentioned the semi-diurnal tidal frequency is higher than $2\Omega'$, so cannot include continental shelf waves or leaky modes of Poincaré-type. They appear to have the properties of a trapped Kelvin-like wave, although the complicated form of the coast prevents any simple interpretation.

6. MODERN GLOBAL SOLUTIONS AND ATTEMPTS TO MODEL DISSIPATION

The development of automatic computers in the 1950s at last opened up the possibility of finding realistic solutions of Laplace's tidal equations over all oceans and seas without any idealization of geometry. In principle, the procedure is straightforward. One divides the ocean by a network of elementary areas, small enough to define the tidal structure. In each area specified by (i, j) (figure 7) the depth h_{ij} is specified from a good bathymetric chart, depth-mean current $\bar{u}_{ij}$, $\bar{v}_{ij}$ are assumed at the central point (cross), and elevation $\zeta_{i+1/2,\,j}$ and $\zeta_{i,\,j+1/2}$ at the edges. In the continuity equation (4.2), $\partial(h\bar{u})/\partial x$ is approximated by :

$$\Delta_{ij}^{-1}(h_{ij}\bar{u}_{ij} - h_{i-1,j}u_{i-1,j})$$

where Δ_{ij} is the mesh length and is associated with $\partial(\zeta_{i-1/2,\,j})/\partial t$. In the acceleration equation (4.9), $\partial\zeta/\partial x$ is approximated by :

$$\Delta_{ij}^{-1}(\zeta_{i+1/2,j} - \zeta_{i-1/2,j})$$

and associated with $\partial(\bar{u}_{ij})/\partial t$ and $2\Omega'\bar{v}_{ij}$, the frequency $2\Omega'$ being defined of course by the latitude, as is the generating potential $g\bar{\zeta}$. At land boundaries

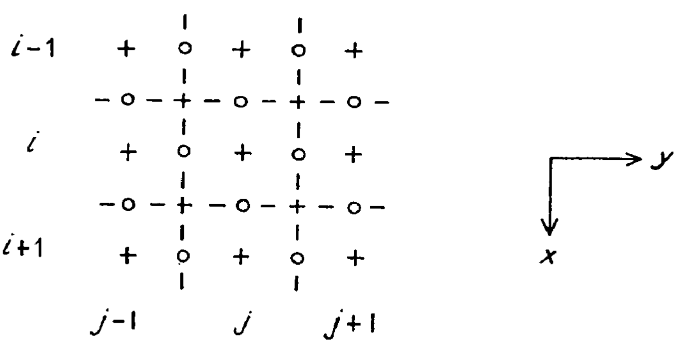

FIG. 7. — Basic network commonly used for finite-difference tidal models, with elevations calculated at the circles and current components at crosses.

the component of velocity normal to the coast (not necessarily parallel to a mesh side) is equated to zero. At open sea boundaries, either ζ or some linear combination of $\bar{v}$ and ζ must be specified by empirical data.

If the linearized form of the equations is used, as in (4.2) and (4.9), a common time factor $\exp(-i\omega t)$ may be removed, and the problem is expressed as an array of ordinary linear equations determined by the boundary conditions (a 'boundary-value problem'). Such a method was followed by PEKERIS & ACCAD (1969). More usually, the time factor is retained and the solution approached by time-stepping from an assumed condition of rest everywhere until a steady state of oscillation is reached. This enables non-linear terms, such as a quadratic friction law, to be retained (ZAHEL, 1970). There are also questions of numerical stability, choice of time increment, etc, which are outside the scope of this review.

If we exclude various attempted solutions for individual oceans, which invariably suffer from lack of reliable data along the open boundaries, at least five independent global solutions have been proposed for the M_2 tide alone during the last decade. To these we should add the very creditable empirically drawn cotidal maps by DIETRICH (1944), later modified by VILLAIN (1952), which are still invoked in comparisons with pelagic measurements (ZETLER *et al*, 1975). HENDERSHOTT (1973) is an excellent comparative review : see also ZAHEL (1977). Unfortunately, the solutions differ considerably from each other and from the measured tides in various parts of the ocean. It is therefore a matter of considerable importance to try to understand where each method has gone wrong. In the first place, the methods and types of solution can be divided into two classes, according to whether the solution is derived solely from the tide-generating potential and oceanic topography, or whether it is constrained to agree with tidal measurements along the coastal boundaries. The maps of PEKERIS & ACCAD (1969) and ZAHEL (1970, 1977) are of the former class : those of BOGDANOV & MAGARIK (1967), TIRON *et al* (1967) and HENDERSHOTT (1972) are of the latter class. A typical example from each class is shown in figures 8 and 9.

A 'cotidal map' as exhibited here defines the oceanic loci of equal phase lag (co-phase lines), here typically in steps of 30°, and the loci of equal amplitude (co-amplitude lines) of a harmonic constituent of the tide, in this case M_2. The nodal points, traditionally known as 'amphidromes', stand out as centres of zero amplitude round which the co-phase lines rotate, usually but not invariably clockwise in the southern hemisphere and anticlockwise in the northern hemisphere. Less obvious, but perhaps more important, are the 'anti-amphidromes', zones of locally maximum amplitude with little change of phase, such as shown in both figures 8 and 9 south of India, though with different amplitude. The two maps agree roughly in some other gross features ; the amphidromes of the North Atlantic and Central Pacific, and anti-amphidromes somewhere in the equatorial Pacific, and the rotatory system ('virtual amphidrome') centred on New Zealand. But the resemblances are clearly outnumbered by the many differences, some quite radical. The two amphidromes west of the North American continent in figure 8 rotate in opposite direction from those in figure 9, although both contrive to show a northward progression of phases along the Californian coast. (The map of PEKERIS & ACCAD

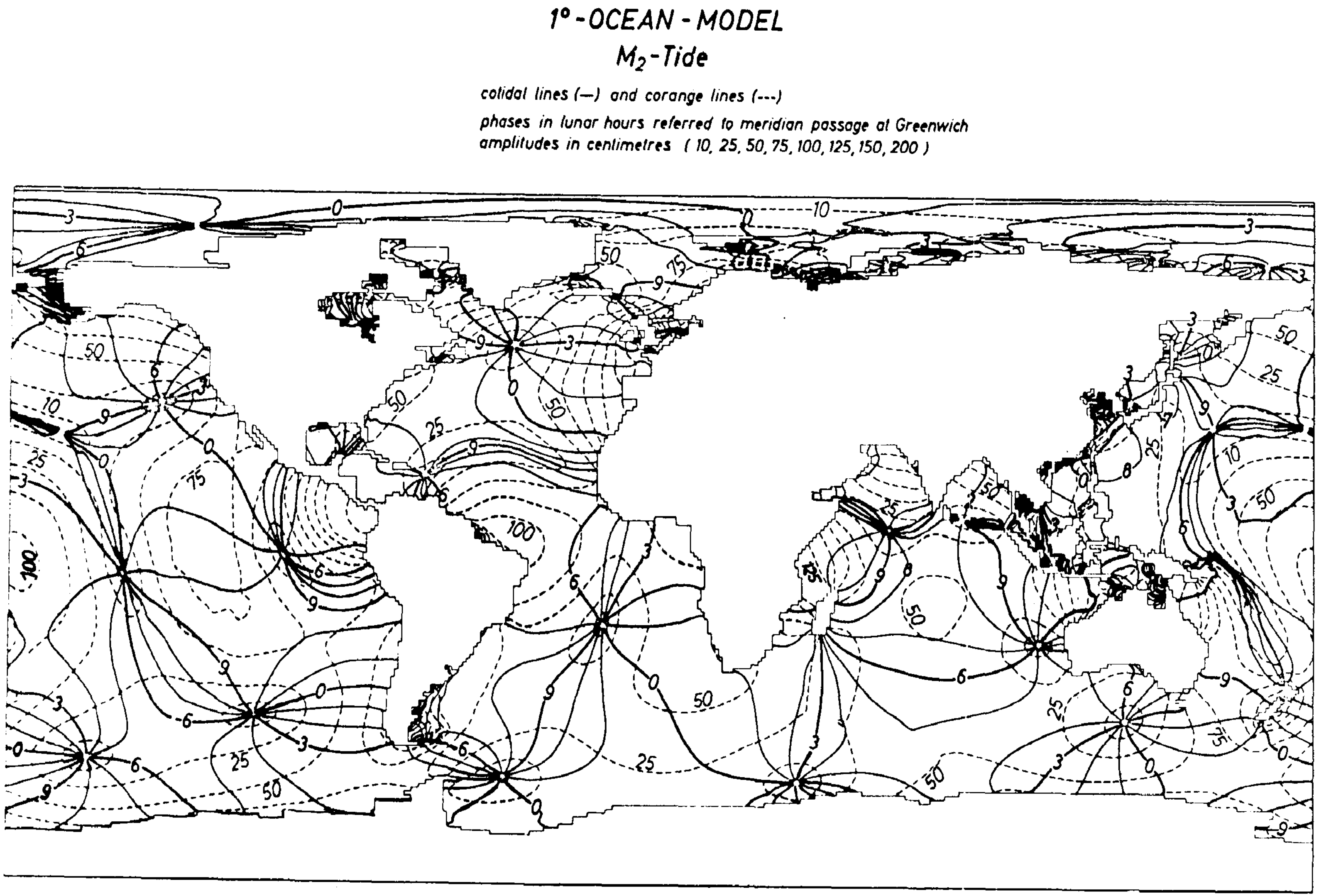

FIG. 8. — The M_2 tide in the world oceans from the computations of ZAHEL using a 1° mesh size, and no empirical tidal data. Full curves are loci of equal phase (lunar hours) ; broken curves are loci of equal amplitude (centimetres) (from ZAHEL 1977).

(1969) gives a southward propagation there, contrary to observation). Figure 8 agrees with PEKERIS & ACCAD in placing a prominent anticlockwise amphidrome in the central south Atlantic, which is absent from figure 9 and from all maps constrained to agree with coastal observations. PEKERIS & ACCAD (1969) showed that this interpretation was consistent with data from the four islands in this ocean area — a point further investigated by CARTWRIGHT (1971) — but it is difficult to reconcile it with the observed northward progression of phase along the Brazilian coast.

Comparing free and coastally constrained tidal solutions naturally tends to emphasize differences, although it is unfortunate that such differences even exist. But the constrained solutions also differ from each other, indicating errors in physical representation. In fact, all the authors mentioned differ in their treatment of friction, which is a factor of fundamental importance as shown in § 7. Basically, the force per unit mass due to friction may be added to equation (4.9) in the form :

$$-c\mathbf{v}|\mathbf{v}|/h \tag{6.1}$$

where c is a non-dimensional constant known from direct measurement in shallow water to be about 0.0025 (BOWDEN & FAIRBAIRN, 1956). $\mathbf{v}$ should properly represent a velocity about 1 m above the bottom though the depth-mean velocity $\bar{\mathbf{v}}$ is often assumed. The direct effects of (6.1) are therefore practically confined to the shallow seas, where $\mathbf{v}$ is large and h is small. However, global tidal computations are mostly confined to deep-water zones for practical reasons (e.g. shallow-water tides require a much finer mesh owing to their smaller scale of horizontal variation), and so dissipation can only be properly represented by radiation of energy out of the model into bounding seas.

The importance of boundary radiation was first pointed out seriously by PROUDMAN (1941) when he showed that the analytical solution for a frictionless hemispherical ocean was radically altered when the usual boundary condition of zero normal velocity was replaced by a radiation condition. In essence, if a tidal wave of elevation ζ and mean normal velocity v_n propagates from an ocean of depth h to a shelf of depth h' where its energy is completely absorbed by friction, the following relations hold at the shelf edge :

$$\zeta = \zeta' \qquad hv_n = h'v_n' = \zeta'(gh')^{1/2} \tag{6.2}$$

that is :

$$hv_n/\zeta = (gh')^{1/2} \tag{6.3}$$

where the prime denotes the wave parameters on the shelf. Thus (6.3) is the appropriate boundary relation instead of $v_n = 0$. It implies normal velocity in phase with the tidal elevation. In general, not all the energy propagated across a given stretch of boundary is absorbed, so a relation like :

$$hv_n = \lambda\zeta \tag{6.4}$$

where λ is a complex parameter, is appropriate. In the real ocean, λ can usually only be assigned empirically.

Figure 9 was computed by HENDERSHOTT (1972) with boundary relations like (6.4) where possible, the body of the ocean being assumed friction-

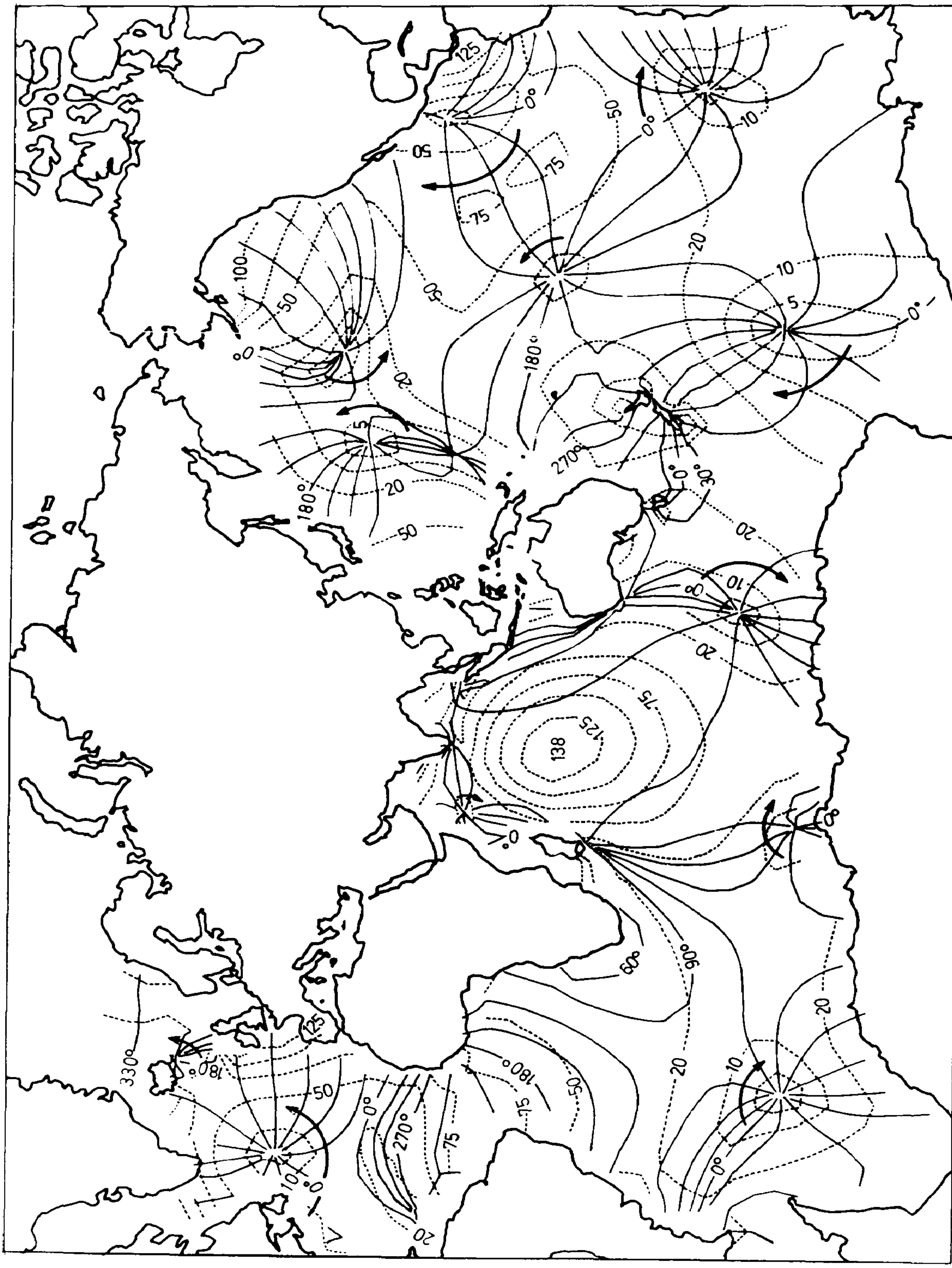

FIG. 9. — As figure 8 but from the computations of HENDERSHOTT for a rigid Earth,

less. $\lambda = 0$ was, of course, assumed round continents like Africa whose shelf zone is very narrow. Figure 8 was computed by ZAHEL (1977) using direct friction (6.1) in the field equations which, unlike HENDERSHOTT'S, extended to depths as small as 50 m. Most dissipation occurs in seas much shallower than 50 m, but following TREPKA (1967), ZAHEL added an arbitrary term representing horizontal shear stress, approximately :

$$A \partial^2 v / \partial x^2 \tag{6.5}$$

The physical significance of such a term as (6.5) is hardly known from oceanographic measurements, but it serves as a stabilizing parameter which also accounts for a good deal of dissipation not directly represented in the numerical model. In fact, the arbitrary factor A was adjusted so that the cotidal map agreed as nearly as possible with coastal data.

PEKERIS & ACCAD (1969) restricted their area of computation to depths of a kilometre or more and, like ZAHEL, used reflecting boundaries. Some dissipation had to be introduced artificially, and since they had an interest in keeping the equations linear, they adopted a form $-\alpha\omega v(h_0/h)^2$ where $h_0 = 1$ km, and α took a sequence of trial values in the range 0.1-0.5. Physically, this frictional term was much greater than the true frictional stress at any given point of the ocean, but was intended to compensate for the lack of genuine shallow-water friction. The resulting cotidal maps were only qualitatively successful, but from PEKERIS & ACCAD'S account, it appears that they regarded this work as a pioneering exercise, to be improved later.

The earlier published global maps by BOGDANOV & MAGARIK (1967) and by TIRON *et al.* (1967) apparently were derived from frictionless equations, on the assumption that the effects of friction are incorporated in the empirical boundary data to which their solutions were constrained. This assumption is partly true, but with later knowledge one sees that the use of reflective boundary conditions everywhere makes for solutions which fall short of realism. Their results agree with neither figure 8 nor figure 9. However, both groups of Russian authors also give cotidal maps for S_2, K_1 and O_1. Otherwise only ZAHEL (1973) has computed for a constituent (K_1) other than M_2.

Apart from this troublesome question of friction, the diversity of results has led some scientists to investigate other neglected physical factors which might prove surprisingly important on a global scale. One such factor whose importance has recently been demonstrated is the elastic yielding of the Earth's crust to tidal loading. This introduces the Earth tide into our discussion and requires a separate sectional heading.

6.1 The Earth tide

It has long been known from precise measurements of gravity (still longer in theory) that the Earth's crust itself yields elastically to the tidal forces of the Moon and Sun. The normal modes of oscillation of the Earth, unlike those of the ocean, have frequencies at least an order of magnitude higher than the tidal frequencies, so the response is virtually static. If ζ_e

is the tidal elevation of a portion of the Earth's crust above its mean position, due to direct yielding :

$$\zeta_e(\theta, \lambda, t) = h_2 \bar{\zeta}(\theta, \lambda, t) \qquad (6.6)$$

where $\bar{\zeta}$ is the 'equilibrium tide' defined in § 2 and h_2 is the elastic constant ('Love number') now known to have the value 0.61. Further, as a result of this yielding the tidal potential on the Earth's surface is increased by $k_2 g \bar{\zeta}$, where $k_2 = 0.30$ is another Love number.

ζ_e as defined by (6.6) is known as the 'body tide', the rise and fall of the Earth's crust which would occur in the absence of the oceans. Since the equilibrium tide due to the Moon has an average amplitude near the equator of about 0.24 m, the corresponding amplitude of the body tide is about 0.15 m. The actual rise and fall is, however, considerably modified by the variable loading of the oceanic tide, which produces additional vertical crustal movements of typical amplitude 0.05 m, as discussed below. Such movements are sensed by gravimeters, the amplitudes of the change in gravity associated with the last two figures being 45 and 15 μGals, respectively. The tilt and strain of the Earth's crust associated with these components of the Earth tide are also of interest, but these cannot be pursued in the present context.

Consider now the effect of the body tide on Laplace's tidal equations for the ocean ((4.2) and (4.9)), remembering that the water elevation ζ is strictly the *relative* elevation of the surface with respect to the local Earth's crust if theory is to be compared with conventional measurements (*). The equation of continuity (4.2) is unaltered but since the horizontal gradient of pressure is strictly referred to *geocentric* elevation $(\zeta + \zeta_e)$ and the potential is increased by the factor k_2, the right-hand side of (4.9) should now read :

$$-g \text{ grad } [\zeta + \zeta_e - (1 + k_2)\bar{\zeta}] = -g \text{ grad } [\zeta - (1 + k_2 - h_2)\bar{\zeta}]. \qquad (6.7)$$

Thus, from mere consideration of the body tide, the oceanic tide-generating potential should be multiplied by $(1 + k_2 - h_2) = 0.69$ in solving Laplace's equations. This adjustment was first applied practically by GRACE (1930), who attempted to estimate the elastic constant from an analysis of the tides in the Red Sea. It was also used in the calculations of MUNK *et al.* (1970) and of CARTWRIGHT (1971). Surprisingly, it seems to have been ignored by all computers of global tides considered in § 6 prior to 1972. Presumably the constraints to fit boundary data and the adjustment of arbitrary parameters have to some extent compensated for the error.

HENDERSHOTT (1972), however, went further than the body-tide adjustment (6.7) ; he allowed also for the 'loading tide'. This is the additional yielding of the Earth's crust due to the direct loading of the ocean tide itself. The deformation of the sea floor due to elastic yielding to a point load $\zeta\delta(\theta - \theta', \lambda - \lambda')$ at (θ', λ') can be expressed as :

$$\zeta_e'(\theta, \lambda) = \zeta G(\phi) \qquad (6.8)$$

where G is a function only of the angular distance ϕ between (θ, λ) and

(*) Satellite altimeters introduce the possibility of directly measuring geocentric tidal elevations, $(\zeta + \zeta_e + \zeta_e')$, see §9.

(θ', λ'). G depends on the elastic structure of the Earth, but FARRELL (1972) has shown that several plausible models based on seismic observations yield approximately the same function, at least for values of ϕ greater than a few degrees. By applying certain harmonic integrals to $G(\phi)$ one can equivalently express the Earth deformation to a spherical harmonic in ocean elevation of degree n, namely ζ_n as :

$$\zeta_{en}' = h_n' \alpha_n \zeta_n \tag{6.9}$$

where α_n is a normalized density ratio (water/earth), and h_n' is a 'loading Love number' (LAMB, 1932). The combination of loading harmonic and corresponding crustal deformation also increase the effective potential by :

$$(1 + k_n') g \alpha_n \zeta_n \tag{6.10}$$

The quantities h_n', k_n', α_n have been evaluated to high order and may be found listed for example in HENDERSHOTT (1972). Since positive sea elevation depresses the crust in the main, h_n' and k_n' are negative.

With both body tide and loading included, the equation for horizontal acceleration now becomes :

$$\partial \mathbf{v}/\partial t + 2\mathbf{\Omega} \wedge \mathbf{v} = -g \operatorname{grad} [\zeta - (1 + k_2 - h_2) \bar{\zeta} + \sum_n (1 + k_n' - h_n') \alpha_n \zeta_n] \tag{6.11}$$

Since the expansion of ζ into spherical harmonic coefficients ζ_n implies an integral process, this is an integro-differential equation. HENDERSHOTT (1972) attempted to solve it by iteration, starting with the straightforward solution which ignores the loading tide (as represented in figure 9), using this to evaluate the spherical harmonic series ζ_n, then re-solving (6.11) with this value of ζ_n. The loading tide has amplitudes of several centimetres so the correction is not entirely trivial. Unfortunately, HENDERSHOTT found that the second solution to (4.2) and (6.11) differs considerably from the first, suggesting that an iterative procedure on the same lines would not converge. At a symposium on 'Tidal Interactions' held during the IUGG General Assembly at Grenoble in 1975, HENDERSHOTT reported some further investigations of iterative procedures for solving these equations but without a wholly satisfactory result (*).

Thus, no solution has yet been found to the equations which most completely represent the physical mechanism of the global oceanic tides. All we have is a number of differing solutions to sets of equations which only partially represent the physical mechanism. In § 9 we shall consider other possible approaches for the future, involving a more coordinated approach to computation combined with direct measurement of tides in the open sea. While still under the heading 'Earth tides', mention should also be made of some recent attempts by KUO & JACHENS (1977) to deduce the ocean tides in limited areas entirely from land-based measurements of the loading tide. Given sufficiently wide coverage of accurate measurements of tidal gravity and tilt over the world's continents, the loading tide may be deduced by subtraction of the known body tide. In principle it may be possible to deduce $\zeta(\theta, \lambda, t)$ from $\zeta_e'(\theta, \lambda, t)$ by inverting an integral equation derived

(*) PEKERIS (1977, lecture) has recently solved (6.11) when unconstrained by boundary data.

from (6.8), and a similar one involving crustal tilt. There are, however, problems of precision of measurement, especially tilt (BAKER & LENNON, 1973) and the influence of near-shore tidal systems. Some of the results reported by KUO & JACHENS (1977) for the north-eastern Pacific appear promising, but the possibility of global coverage is highly controversial.

7. POWER BUDGETS AND INTERNAL TIDES

In the previous section much was made of the way friction has been represented in various global tidal models. This is not merely a matter of computational technique : ultimately it has a bearing on a fundamental geophysical quantity, the total rate of dissipation of energy by the oceanic tides. In this and the following section we shall consider present knowledge of total dissipation and its bearing on the celestial mechanics of the Earth-Moon system.

Viewing the oceans as a mechanical entity, the first energetic quantity which deserves attention is the total (mean) energy of the tidal system, evidently 'constant' on the time scale of a century or so (*). Calling this Eo, we have :

$$\text{Eo} = \text{Potential Energy (PEo)} + \text{Kinetic Energy (KEo)}$$

where

$$\text{PEo} = \iint \frac{1}{2} \rho g < \zeta^2 + 2\zeta_e(\zeta + h) > \mathrm{d}S$$

$$\text{KEo} = \iint \frac{1}{2} \rho h < |\mathbf{v}|^2 > \mathrm{d}S$$

Here the symbols $<\ldots>$ denote time averages of the enclosed quantities and the integrals are supposed taken over the surface of the ocean. Curiously few authors have carried out these integrals. HENDERSHOTT's (1972) values for the M_2 tide, namely 2.1, 5.2, 7.3×10^{17} J for PEo, KEo and Eo respectively, must be taken as the standard estimates (calculations based on the equilibrium tide give about an order of magnitude less).

More important are the rates at which the gravitational forces feed energy into the tidal system and at which energy is lost from it through friction. These must of course be equal but we have to distinguish at least three types of estimate of power transfer which unfortunately do not agree, owing to the inadequacy of our knowledge of the mechanical system. These are represented by P_1, P_2 and P_3 in figure 10.

P_1 is the power input from the Moon and the Sun and from the Earth-tidal movement of the sea bed. The input from the Earth tide is ignored by all authors except HENDERSHOTT (1972), who shows that it contributes about 30 % of the total ; authors of rigid-Earth calculations may, however, compensate by some artificial means of dissipation. It may be shown to be expressible as :

$$P_1 = \iint \rho < \dot{\zeta} V + \nabla \cdot (h\mathbf{v}) + g\zeta\dot{\zeta}_e > \mathrm{d}S$$

(*) CARTWRIGHT (1973) found the amplitude of M_2 at Brest to have decreased by 1 % per century over about 240 years.

For the M_2 tide, HENDERSHOTT (1972) derived $P_1 = 3.0 \times 10^{12}$ W, while from simpler but roughly equivalent expressions MUNK & MACDONALD (1960) derived 3.2 from DIETRICH's (1944) cotidal map, PEKERIS & ACCAD (1969) derived 6.3 and ZAHEL (1977) 3.8×10^{12} W. We should not place too much weight on PEKERIS & ACCAD's figure since their model lacks realism, so a fair average for P_1 is $3.5 \pm 0.5 \times 10^{12}$ W from the M_2 tide only. An estimate including M_2, S_2 and other tidal constituents is $5 \pm 1 \times 10^{12}$ W. The M_2 tide thus has its mean energy replaced in $E_0/P_1 = (7 \times 10^{17})/(3.5 \times 10^{12}) = 2.0 \times 10^5$ s $= 2.3$ d. The corresponding 'Q factor' for the ocean tides is $Q = \omega E_0/P_1 = 28$.

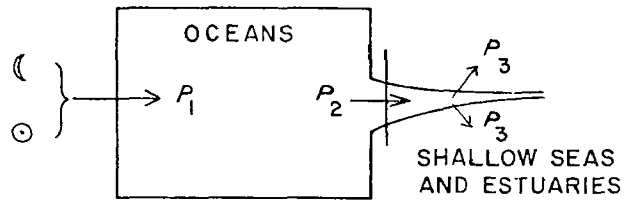

FIG. 10. — Diagram of power input to the oceans from Moon and Sun and output across the continental shelf boundaries to the shallow-water dissipation areas.

At this point, mention should be made of a loose relationship between the ocean's Q factor and the 'age' of the tide. Age is a 19th century term for the time lag of the peak of spring tides after the peak of the generating forces at Full and New Moons. In most parts of the ocean it is 1-2 d, although a few places have much larger or negative ages. It is easily seen that positive age implies an increase in phase lag with frequency from the lunar M_2 constituent to the higher frequency solar S_2 constituent. PROUDMAN (1941) showed analytically that energy dissipation in coastal seas was necessary to produce predominantly positive age in a hemispherical frictionless ocean. GARRETT & MUNK (1971) showed how the phase gradient could be roughly related to the response curve of the normal modes of the ocean near resonance, the gradient being proportional to Q. They deduced from a worldwide survey of tidal ages that Q must be greater than 15, an interesting result to obtain from such simple observations. Using an estimate for P_1 somewhat lower than that quoted in the previous paragraph, and a value of E_0 for the equilibrium tide of 3×10^{16} J, they also deduced that the spatial coupling between the tidal forcing function and the dominant normal mode must be rather weak, a fact which is qualitatively evident from figures 8 or 9. WEBB (1937 b) is also interested by the Q/age relationship and points out that the parts of the oceans where the tidal age is large are also those where dissipation is known to be concentrated (MILLER, 1966) and in some cases local resonance is suspected (e.g. Gulf of Patagonia, Coral Sea, Hudson Bay). Because of this localized nature of the strongly resonating areas, WEBB criticizes GARRETT & MUNK's (1971) use of an overall Q factor. He prefers to discuss the mechanics of ocean tides in terms of 'decay times', roughly equal to the age, and the time needed for the energy fed into mid-ocean to be transported to the dissipating areas of resonance. But this is a controversial subject to which it is difficult to apply precise arguments.

Returning again to figure 10, P_3 and P_2 are respectively the power dissipated directly by friction on the bottom of the shallow seas and the power transmitted across their boundaries with the ocean. They are given by :

$$P_3 = -\iint < F . \bar{v} > \mathrm{d}S \qquad P_2 = \int \rho g h < \bar{v}_n \zeta > \mathrm{d}s$$

where F is the frictional stress, integrated over shallow seas, and $\bar{v}_n$ is the inward normal component of mean velocity across their oceanic boundaries. Since $F = - c\rho \mid \bar{v} \mid^2/h$, the integral for P_3 involves the cube of the current speed which is certainly not known to any precision over most of the world's seas. TAYLOR (1919), who was the first to examine these relations seriously, showed that P_2 and P_3 were fairly comparable in the isolated case of the Irish Sea, for which the total P_2, principally from the southern entrance, was 6.4×10^{10} W, for Spring tides. Reduced to the power dissipated by the M_2 tide alone, this comes to 5×10^{10} W, about a hundredth of the total astronomical input to the oceans.

Soon after TAYLOR's (1919) paper, two geophysicists independently estimated the total dissipation in all the world's shallow seas — essentially by P_3 calculations using the known areas and depths of such seas and what figures were available for the strength of their tidal currents. JEFFREYS (1920) obtained 1.1 and HEISKANEN (1921) 1.9×10^{12} W for the M_2 tide. But the dependence of these estimates on the cube of roughly known current speeds made them appear dubious. MILLER (1966) repeated the calculation using the more reliable P_2 integral and up-to-date information on the amplitudes and phases of currents and elevations, and obtained 1.7×10^{12} W. He showed that about two-thirds of this total comes from the Bering Sea, the Okhotsk Sea, the seas north of Australia, the seas surrounding the British Isles, the Patagonian Shelf, and Hudson Bay, in order of importance, while the narrow shelves surrounding most of Africa and along the west coast of the Americas south of Canada dissipate hardly any power at all. (JEFFREYS and HEISKANEN had named mostly the same important seas as MILLER, but differed greatly in their individual contributions).

MILLER's estimate for P_2 seems fairly reliable, although the data on which the dissipation rates for individual seas are based are very variable in quality and quantity. Figure 11 shows a detailed distribution of the P_3 dissipation over the various parts of the seas surrounding the British Isles and the corresponding P_2 power transfers from a recent computational model by FLATHER (1975). The tidal elevations and currents are known in great detail over most of this sea area, and the major dissipation figures are corroborated by other estimates. The total P_2 input is 20.5×10^{10} W, whereas MILLER (1966) assumed 17.3, no great difference. On the other hand, MILLER's figure of 13×10^{10} W for the Patagonian Shelf was based on very sparse current data. WEBB (1976) suggests by a simplified model that the Patagonian Shelf may be a very strong absorber of tidal energy. This region therefore seems to merit a thorough investigation based on measurement and computation. The result may (or may not) add significantly to MILLER's total P_2, and incidentally may shed some light on the problem of the existence or non-existence of a M_2 amphidrome in the South Atlantic Ocean (cf. figures 8 and 9).

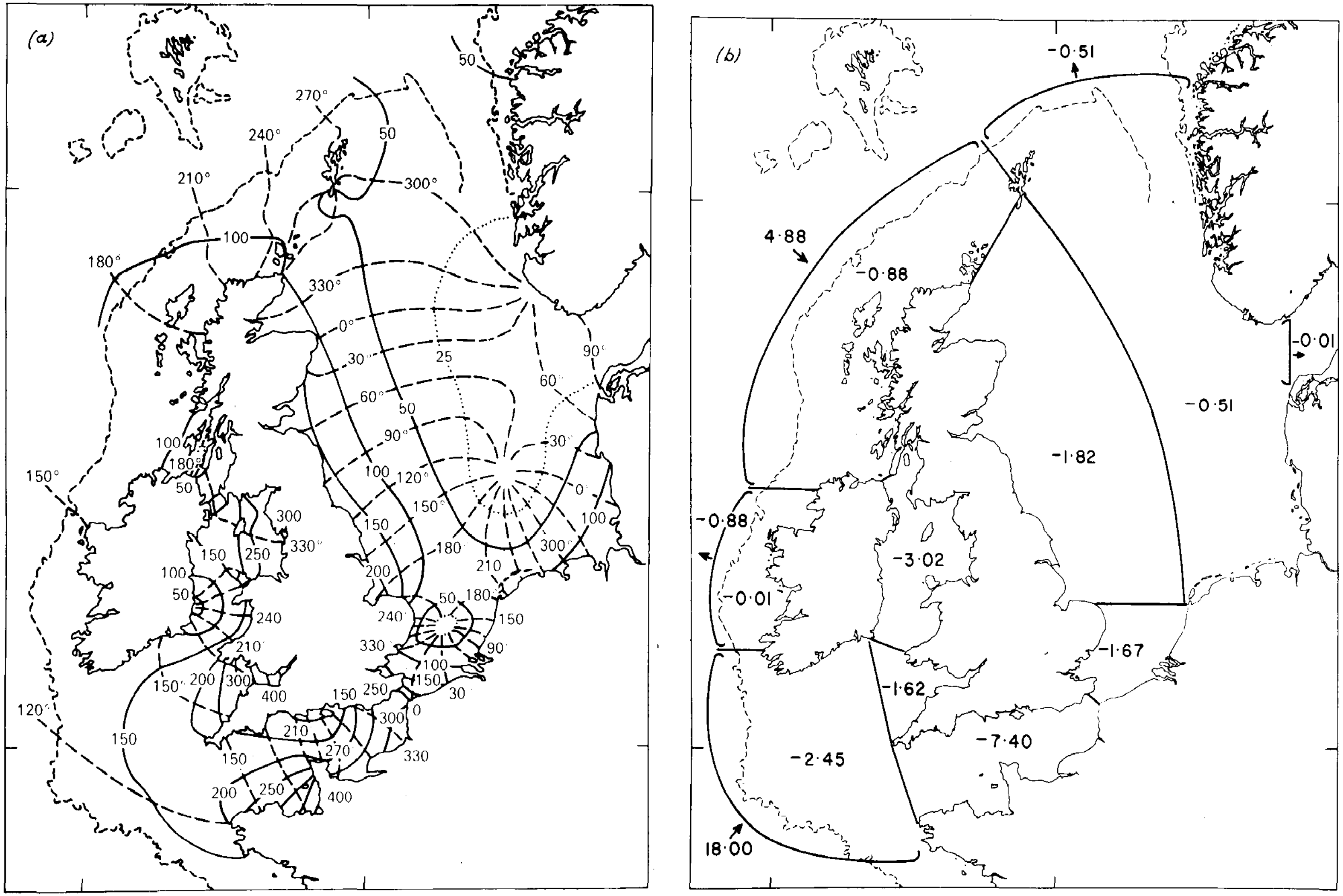

FIG. 11. — *(a)* Cotidal map for the M_2 tide in the shelf seas surrounding the British Isles based on measured data along the open oceanic boundary, zero normal flow at the coast, and suitable frictional laws in the interior. *(b)* Rate of flow of energy into the area across the oceanic boundaries and frictional losses in the interior of the same model (from FLATHER 1975).

At all events, the salient point in all this is that the current estimate of about 1.7×10^{12} W for shallow-sea dissipation falls short of the estimated input power, P_1, from the gravitational forces by 50 %. This is a very puzzling gap in the power budget, for which neither oceanographers nor solid-Earth geophysicists can at present suggest a likely explanation. From a certain point of view it may be seen as the principal motivation for continuing the study of oceanic tides. The problem has been appreciated for several years and was discussed at length by MUNK & MACDONALD (1960) and by MUNK (1968). Their discussions are, however, based on an estimate of P_1 of only 2.7×10^{12} W derived from observations of the angular deceleration of the Moon's orbit. Recent advances by astronomers (see § 8) now suggest a figure for P_1 which is even larger than the estimate of 3.5×10^{12} W suggested earlier in this section. The gap in the budget is therefore even more serious than when discussed by MUNK and his colleagues.

Where else is tidal energy being dissipated ? There are of course many hundreds of coral atolls and other island features where large tidal currents are set up, which could cause dissipation on a localized scale. MILLER (1966) considered these in a general way but did not think they could contribute significantly. He also estimated that the tidal currents of the order of 1 cm s^{-1} in the deep oceans would contribute less than 10^9 W despite their large area. JEFFREYS (1968) seems to be alone in considering the tides as setting up a small perturbation to ordinary sea waves which is dissipated as those waves break on a coastline in the usual way. No rigorous calculations have been carried out, but intuitively it would seem to be a minor effect. Another possible sink is in the internal viscosity of the ocean-loading Earth tide (for dissipation in the Earth as a whole the viscous dissipation in the body tide also has to be included, but the results for both are small). This is discussed by MUNK & MACDONALD (1960) and by JEFFREYS (1976 and earlier editions). From arguments based on the Q of the 14-month 'Chandler wobble' of the North Pole, MUNK & MACDONALD (1960) deduce $0.3\text{-}0.6 \times 10^{12}$ W from the total Earth tide, which is about 100 times greater than JEFFREYS' estimate. Allowing 0.3×10^{12} W for the loading tide may bring the ocean dissipation up to 2×10^{12} W, but this is still a long way short of the accepted P_1 figure. There does, however, remain one plausible but as yet uncertain mechanism, the conversion of ordinary ocean tides to 'internal tides' in the density layering of the ocean. An account of this mechanism requires a separate subsection.

7.1 Internal tides

Continuous measurements of temperature at fixed depths in the interior of the ocean show wave motions of large vertical amplitude (tens of metres) known as internal waves. These waves cover a wide spectrum of frequencies, mostly between the frequencies of ordinary surface waves and the tides. But peaks do occur at the tidal frequencies, characterizing 'internal tides'. Records of current in the deep ocean also show tidal components much larger than the few cm s^{-1} associated with the barotropic tide, and these too are due to the internal tides. Internal tides generally have much lower signal/noise ratio than the ordinary barotropic or surface tides, and their phases are not so closely locked to the tide-generating potential. The random element in their phases, which can also be described as 'line broadening', is caused by the dependence of internal tidal mechanics on the density structure of the ocean which varies with the large-scale structure of oceanic currents. A comprehensive review of the internal tides has recently been made by WUNSCH (1975). Here I review briefly only those features which are relevant to the global tidal energetics.

The basic dynamical equations for internal tides have already been given in § 4 in connection with Laplace's tidal equations. In the simplified case of a constant (exponential) density gradient, and hence constant buoyancy-frequency N and constant depth h, it may be simply shown that the vertical dependence of w and the other variables of the internal waves are defined by the functions :

$$\sin N\gamma_n(z+h) \qquad \gamma_n \simeq n\pi/Nh \qquad n = 1, 2, 3, \ldots$$

which, it will be seen, give zero vertical velocity at the bottom and the surface of the sea, and $n-1$ reversals of sign in between. In fact, it is found that internal tides, unlike internal waves of high frequency, are concentrated in the first few vertical modes (low values of n). Further, the horizontal velocities and pressure gradients are governed by equations identical in form to the barotropic (Laplace's) tidal equations but with equivalent depths $d_n = (hN/n\pi)^2/g$, which are very much smaller than the actual depth h. For example, with typical values $N = 0.002\ \mathrm{s}^{-1}$, $h = 4\,000$ m, we have $d_1 = 0.64$ m, $d_2 = 0.16$ m. While a surface tide of 'Kelvin' type travels with a natural speed $(gh)^{1/2} = 200\ \mathrm{m\ s}^{-1}$, the first mode of the internal tide of similar form travels with speed $(gd_1)^{1/2} = 2.5\ \mathrm{m\ s}^{-1}$.

We thus have at first sight a complete mismatching in both vertical waveform and speed of the internal tides on one hand, the surface tide and the tide-generating stress on the other, both the latter being essentially constant over the water depth. From this point of view it is hard to see how the observed internal tides are generated at all. However, it is now well established that the irregularities in the ocean bottom permit a transference of energy from the surface tides to the internal tides, and hence providing a source of generation for the latter and of energy loss to the former. The gradient of the bottom imparts a vertical component on the otherwise uniformly horizontal motion of the surface tide. This deflects the surfaces of constant density and hence internal waves are generated of tidal period, but with wavelength determined by the bottom topography.

In mathematical terms, suppose for simplicity that the bottom irregularity is small compared with the general depth of water and that it is uniform in the y direction (geographically arbitrary). If $\bar{u}\exp(-i\omega t)$ is the horizontal velocity due to the surface tide, assumed uniform on the length scale of the irregularities, and $\bar{h}$ is the mean depth, then $\bar{u}\exp(-i\omega t)\,\partial h/\partial x$ must equal $w(-\bar{h})$, the vertical velocity at the bottom. Dropping the time factor, the last expression may be expanded in a Fourier series in x :

$$\bar{u}\,\partial h/\partial x = \int_{-\infty}^{\infty} A(k)\exp(ikx)\,\mathrm{d}k$$

The vertical velocity field $w(x, z)$, which essentially defines the internal tide, is then given by :

$$w(x,z) = \int_{-\infty}^{\infty} A(k) \frac{\sin [Nkz(\omega^2 - 4\Omega'^2)^{-1/2}]}{\sin [-Nk\bar{h}(\omega^2 - 4\Omega'^2)^{-1/2}]} \exp (ikx) \, dk$$

This integral is dominated by the values of the bottom spectrum $A(k)$ for which :

$$k^2 = (\omega^2 - 4\Omega'^2)/gd_n$$

d_n being the equivalent depth for the internal wave of nth-order mode in a constant depth $\bar{h}$, as previously defined.

The full linearized theory of internal tide generation by bottom scattering was first expounded by Cox & SANDSTROM (1962), who calculated $A(k)$ for typical sections of the Atlantic Ocean. By a similar calculation, MUNK (1966) estimated the rate of energy transfer from the M_2 tide into the first 20 internal modes in the global ocean to be 0.5×10^{12} W. This is by no means negligible, but it is only some 30 % of the gap in the power budget discussed in the main part of § 7.

The restriction to small irregularities compared with $\bar{h}$ in the theory discussed above eliminates many major features such as seamounts and the edges of continental shelves. Strong local generation of internal tides near such features is confirmed by measurement, but theoretical assessment of their rate of transfer is more difficult. The most amenable theories for finite topography (RATTRAY *et al.*, 1969, BAINES, 1974) treat the internal motion not as a superposition of modes with continuous vertical waveforms but as a series of beams of concentrated energy whose direction in the vertical may be calculated for a given density structure and frequency by the method of characteristics. Energy transfer occurs most intensively at those points of the topography where its slope coincides with the direction of a characteristic. However, the beam theory is strictly two-dimensional (horizontal and vertical) and it is not clear how a general three-dimensional shelf edge will reflect. Observational evidence for the existence of such beams is controversial (WUNSCH, 1975), and a modal decomposition is usually more meaningful.

What is needed from the point of view of energetics is either a tractable means of calculating the energy transfer to internal motions at all the shelf boundaries and other large topographical features in the ocean, or else an estimate of the rate at which observed internal tides lose their energy by turbulent viscosity. The transfer rate was estimated by WUNSCH & HENDRY (1972) at the intensively measured ' site D ' off the New England shelf, but worldwide extrapolation of the result gave only about 10^9 W. SCHOTT (1977) also arrives at an insignificantly small figure compared to the known loss to the surface tide. The weakness of any such extrapolation is the sensitivity of any tractable solution to the detailed density structure. SANDERSTROM (1976) has, however, recently suggested that the shelf profile is not a sensitive factor and that a worldwide calculation based only on figures for stratification and using vertical-wall shelf edges would be viable.

Calculations of the dissipation rate of internal tides are even more controversial, and probably involve the full non-linear theory of interactions between the whole continuum of internal waves in the ocean. Some approaches are discussed briefly by WUNSCH (1975) ; on the whole they seem to point again to a rather low figure, of the order of 10^{-1} of the required power loss.

Yet another possible mechanism for internal tidal generation is at the critical latitudes where a tidal frequency equals the inertial frequency $2\Omega'$. Here, theory tells us that the wavelengths of internal tides become 'infinite' and the possibility of direct coupling with the surface tide becomes more plausible. MILES (1974) has further shown that such coupling takes place at the critical latitudes through the terms involving the horizontal component Ω'' of the Earth's rotation, usually neglected in the full dynamical equations for the tides. This possibility requires further investigation, but if it were important we should expect significant generation of diurnal internal tides near 30° latitude, at least in the Pacific Ocean where the diurnal surface tides are fairly large. However, observations of internal tides in such regions do not differ greatly from observations elsewhere. The problem remains open, but without any hopeful pointers towards a likely sink of the required magnitude.

8. THE MOON'S ORBIT AND THE EARTH'S ROTATION

The emphasis on energy dissipation in §§ 6 and 7 may seem exaggerated to the uninvolved reader in view of the more obvious controversial features of tidal definition in particular oceans. Nevertheless, we have to pursue the implications of global tidal friction still further in order to explain its importance to our understanding of the history of the Moon's orbit and of astronomical time. In the briefest terms, the *average* bulge of the tide (strictly, its $P_2(\cos\alpha)$ harmonic) is retarded by terrestrial friction so that its mean axis points some tens of degrees east of the Moon (*). The gravitational forces acting on this displaced second-harmonic bulge constitute a couple in opposite sense to the Earth's rotation, so the day lengthens. But the total angular momentum of the Earth-Moon system must be constant, so the decrease in the Earth's rotation is accompanied by an increase in the momentum of the Moon's orbit. From Kepler's third law this can only be brought about by a steady increase in the Moon's mean distance combined with a related *deceleration* in the mean angular motion of its longitude.

The deceleration of the Moon's longitude, originally estimated at about 10″ century^{-2}, was first detected by Halley in comparing records of ancient eclipses with contemporary observations of the Moon's orbit. About a century later Laplace, in his one negative contribution to tidal theory, thought he had explained it in terms of planetary perturbations. Adams showed that Laplace's calculations were inaccurate and that the full calculation gave only 5″ century^{-2}, thus re-opening the suggestion that tidal friction was an important factor, first made apparently by the philosopher

(*) The usual diagram of this situation (MUNK & MACDONALD 1960, p 199, JEFFREYS 1976, p 316) can give the misleading impression of a slightly modified equilibrium tide theory.

Kant. The role of tidal friction in the evolution of the lunar orbit was taken up seriously by Kelvin and more notably by G.H. Darwin ; details of its magnitude have been a subject of active discussion to the present day (references to the earlier literature may be found in MUNK & MACDONALD (1960) — see also MUNK (1968) for a somewhat later assessment).

As mentioned in § 7, the subject has taken a new turn in the 1970s with the agreement among several independent astronomers on a radically increased figure for the longitudinal deceleration of about 40″ century^{-2}. Discussions earlier than 1970 usually accept a figure of 22″.4 century^{-2} derived by SPENCER JONES in 1939 from an apparently exhaustive study of 'modern' (1670-1930) astronomical observations of the Moon. SPENCER JONES' figure could not be reconciled with records of solar eclipses going back to 1000 BC, but there was so much scatter and uncertainty in interpreting the ancient tablets that this was not regarded as important. It would be beyond the scope of this review and the author's experience to give an extended account of the astronomy, but a brief summary of the main concepts which are involved seems worthwhile.

If T represents astronomical time in centuries from a certain epoch of origin, the mean longitude of the Moon (that is, the observed longitude with removal of all periodic terms) can be expressed as :

$$L_M = l_0 + l_1 T + (l_2 + \frac{1}{2}\dot{n})\, T^2 + l_3 T^3 \qquad (8.1)$$

where the l_n are established partly by observation and partly (in particular l_2 and l_3) by calculation from the Newtonian theory of rigid gravitational bodies. In the absence of planetary and tidal perturbations, only the uniform motion described by the first two terms would apply. Planetary perturbations supply the small accelerative terms l_2 and l_3. $\dot{n}$, to use the invariable notation of the literature, is the non-Newtonian acceleration (numerically negative) caused by tidal deformation with friction, and is one of the quantities which concern us. However, if T is expressed in the so-called 'Universal Time' system, as is effectively the case for all observations before very recent times, T itself is subject to the rotation of the Earth, which also accelerates negatively.

The difference between T and the strictly uniform 'Ephemeris Time', which is based on the Earth's orbit round the Sun, may be expressed as :

$$\Delta T = r_0 + r_1 T + \frac{1}{2}\dot{\Omega} T^2 + f(T) \qquad (8.2)$$

where the coefficients are here entirely based on observation of stellar transits, and $f(T)$ is an apparently random function with variations in a time scale of less than a century. As in (8.1), the first two terms describe the uniform motion which was unquestioned until the late 19th century, while the last two terms, whose causes are still only partially understood, represent the mean acceleration and its fluctuations observed in modern precise measurements and by their comparison with ancient records. As will be shown later, $\dot{\Omega}$ is partly related to $\dot{n}$ by conservation of angular momentum : the two quantities are rather difficult to separate from each other from purely astronomical observations. The effect of the Earth's acceleration $\dot{\Omega}$ on estimates of $\dot{n}$ when L_M is measured from, say, times of occultations of stars by the Moon may be removed by comparing with observations of the longitude of the Sun or of an inner planet, which are affected in a similar way, proportionally to the mean motions of the Moon and Sun respectively. But solar eclipses of mediaeval and ancient times were not accurately timed : when correctly identified, the only usable information from them is the geographical position of the place of observation. This depends on both $\dot{n}$ and $\dot{\Omega}$.

MULLER & STEPHENSON (1975) seem to have cleared up a long history of research and controversy about interpretation of historical solar eclipses in which the most notable authority had previously been Fotheringham. They introduced newly translated Chinese records to the Babylonian and European eclipses used by others, and applied rigorous tests for reliability before any record would be included in calculations. Out of more than 1000 historical accounts of eclipses, only 28 passed these tests, ranging from 1375 BC at Ugarib to some observations organized by Halley himself in 1715. Adding the 'modern' observations of lunar phenomena from 1650 to 1970, MULLER & STEPHENSON evaluated all the variables r_0, r_1, $\dot{\Omega}$, $\dot{n}$ independently, with the final result :

$$\dot{n} = -37.5 \pm 5'' \text{ century}^{-2} \qquad \dot{\Omega} = -91.6 \pm 10 \text{ s century}^{-2},$$

where the tolerances quoted are actual bounds to the data, not standard deviations. The figure for $\dot{\Omega}$ can also be expressed as an increase in the length of the day by 2.4 ± 0.3 ms century^{-1}. NEWTON (1970) obtained 2.1 ± 0.3, re-interpreted by MULLER & STEPHENSON as 2.3 ± 0.3. The figures for $\dot{n}$, the acceleration in lunar longitude, compare well with — 41.5 ± 4.3 (NEWTON, 1970 ; *ancient eclipses*), — 38 ± 8 (OESTERWINTER & COHEN, 1972; 20th century data), and other modern results.

The corresponding value of $\dot{a}$, the rate of change of the semi-major axis of the Moon's orbit, is, by differentiating Kepler's law :

$$\dot{a} = -2a\dot{n}/3n = 0.058 \text{ m yr}^{-1} \tag{8.3}$$

Theories of the implied state of affairs some 10^9 yr ago, when the Moon may have been drastically closer to the Earth and the tides enormous, are outside the scope of this review. They are discussed by MUNK (1968) but with somewhat different numerical parameters.

How far do the new estimates of the angular accelerations tie up with oceanographic estimates of the oceanic tides discussed in earlier sections ? LAMBECK (1975) has outlined a direct method of calculating $\dot{a}$ from the perturbing gravitational potential corresponding to a given global cotidal map. The tidal elevation for a given harmonic constituent, s, is expanded in spherical harmonics (as in the elastic Earth problem discussed in § 6.1) to give :

$$\zeta_s(\theta, \lambda) = \sum_{m,n} P_n{}^m(\cos\theta)\, C_{m,n}{}^{(s)} \cos(\omega_s t + m\lambda - \epsilon_{m,n}{}^{(s)}) \tag{8.4}$$

By using an expression due to KAULA (1964) for the perturbing potential due to ζ_s, applying it to the dynamical equations for the orbit, and extracting terms of zero frequency, the secular rate of increase of the semi-major axis can be expressed in the form :

$$\dot{a} = Ka^{-5/2}(1 + k_2') \sum_s A_s(a, e, i)\, C_{m,n}{}^{(s)} \sin \epsilon_{m,n}{}^{(s)} \tag{8.5}$$

where K is a combination of physical constants, k_2' is the loading Love number (§ 6.1) for the Earth tide, and A_8 is a rational function of a, and the eccentricity e and inclination i of the orbit. Expressions similar in form to (8.5) can also be written for de/dt and di/dt. For the semi-diurnal tides, only the second-order harmonic $m = n = 2$ need be considered. M_2 is the principal contributor to $\dot{a}$, but the function A_8 and analogous functions E_8 and I_8 in the expressions for de/dt, di/dt vary considerably for different major tidal constituents, so that for example N_2 has the major influence on e, and K_2 on di/dt.

LAMBECK (1975) applied these calculations to the cotidal maps of HENDERSHOTT (1972), PEKERIS & ACCAD (1969) and BOGDANOV & MAGARIK (1967), with the following numerical results for $\dot{n}$, derived from $\dot{a}$ through equations (8.3) :

	$C^{(s)}$	$\epsilon^{(s)}$	$\dot{n}$
PEKERIS & ACCAD (1969)	0.044 m	70°	– 33″ century^{-2}
HENDERSHOTT (1972)	0.051 m	46°	– 29″ century^{-2}
BOGDANOV & MAGARIK (1967)	0.043 m	48°	– 26″ century^{-2}
Average $\dot{n}$ for M_2			– 29″ century^{-2}
Addition for S_2, etc			– 6″ century^{-2}
Total			– 35″ century^{-2}(*)

The values listed as $C^{(s)}$, $\varepsilon^{(s)}$ are the arguments in equation (8.4) for M_2, $m = n = 2$. The agreement between the three values of $\dot{n}$ from three very different looking cotidal maps is remarkable, suggesting that the details of such maps are not really very important in their effect on the second harmonic. Equally remarkable is the very good agreement between the final result and MULLER & STEPHENSON's estimate for $\dot{n}$ quoted earlier, derived independently from entirely different data and procedures. The agreement greatly strengthens one's confidence in both schemes of reasoning.

Having established that the observed acceleration in longitude is accounted for by the observed ocean tides, we may proceed to calculate the corresponding rate of change of the Earth's rotation by using the momentum equation. The angular momentum of the Moon's orbit resolved in the plane of the Earth's equator is :

$$H_M = ME(M + E)^{-1} a^2 n(1 - e^2)^{1/2} \cos i \qquad (8.6)$$

where M and E are the masses of Moon and Earth respectively (to use the notation of § 2). A similar expression H_E gives the momentum of the Earth's orbit about the Sun, and we must have :

$$H_M = H_E + C\Omega = \text{constant} \qquad (8.7)$$

(*) K. LAMBECK (1977, *private communication*) says that his figures and the astronomical estimate for $\dot{n}$ have since been reduced to about 28″ century–2. P_4 (see 8.8) is affected proportionately.

where C is the Earth's principal moment of inertia. Differentiating (8.7) with respect to time and taking C as constant (an assumption which is nowadays seriously questioned), we get a fairly simple relationship between $\dot{\Omega}$ and the time derivates of a (or n), e and i, which can be calculated in terms of the tides as explained above. LAMBECK (1975) thus obtained the following components of $-\dot{\Omega}$, expressed here in units of ms d^{-1} century^{-1} :

Mean of three solutions for M_2	2.4
Estimates for N_2, L_2, O_1, K_1	0.4
From BOGDANOV & MAGARIK (1967) for S_2	0.5
Atmospheric S_2 tide	— 0.1
Total increase in length of day per century	3.2 ms

The negative increment for the atmospheric tide reflects the fact that its phase *lead* of 2 h causes a slight acceleration in the Earth's rotation. However, the important fact which arises is that the final figure for the tidal effect, 3.2 ms, is significantly greater than the value 2.4 ± 0.3 ms observed by MULLER & STEPHENSON (1975) or the similar figures from NEWTON (1970). This implies some geophysical factor which is positively increasing the Earth's rotation, causing the day to decrease by some 0.8 ms century^{-1}, in opposition to the stronger retarding effect of the tides.

The non-tidal acceleration, as it is called, has been appreciated for several years, even when much lower figures were accepted for $\dot{n}$ and $\dot{\Omega}$. It is extensively discussed by MUNK & MACDONALD (1960) and by other authors since then. Several explanations have been suggested, too diverse to summarize here, but no one cause has yet been positively identified. Most probably, the acceleration is due to a steady decrease in the moment of inertia C, caused by some change in the distribution of the Earth's mass. MUNK & MACDONALD rule out the transfer of water between the oceans and polar ice, pointing out that there are large fluctuations in the length of day on a much shorter time scale, which are clearly not correlated with the known variations in mean sea level. The mass transfer process may be occurring in the atmosphere or in the liquid core (a popular hunting ground for theorists). We cannot proceed further with this manifestly non-tidal subject here.

Finally, we may calculate the rate of energy dissipation corresponding to the tidally induced part of $\dot{\Omega}$. The rate of loss of rotational energy due to both lunar and solar tides is $-C\Omega\dot{\Omega}$, but the Moon's orbit gains rotational energy to give a small correction. The correction may most simply be derived by observing that $C\dot{\Omega}$ must equal the total tidal couple acting on the Earth, and its rate of working is the product of this couple and the angular velocity of the Earth relative to the Moon, i.e. the total dissipation rate is :

$$P_4 = -C\dot{\Omega}(\Omega - n) \quad (8.8)$$

The tidal lengthening of the day of 3.2 ms century^{-1} is equivalent to $\dot{\Omega} = -8.9 \times 10^{-22}$ rad s^{-2}. With $(\Omega - n) = 70 \times 10^{-6}$ rad s^{-1}, and $C = 81 \times 10^{36}$ kg m^2, (8.8) gives $P_4 = 5.0 \times 10^{12}$ W. Alternatively, if we restrict the calculations to those terms which are due to the Moon only, with a proportional allowance for the lunar part of K_1, we get $\dot{\Omega}_M = -7.6 \times 10^{-22}$, $P_4 = 4.3 \times 10^{12}$ W. This last figure is comparable with the estimate of the work done by the Moon on the tides as derived by direct calculation (§ 7), namely $P_1 = 3.5 \times 10^{12}$ W, with an uncertainly of about 1×10^{12}. This may be taken to confirm the validity of the rather roundabout chain of arguments used and suggests that no important factor has been omitted. The two major gaps in our understanding of this geophysical system are the cause of the non-tidal acceleration of the Earth and the missing sink of energy in the oceanic tidal dynamics.

9. NEW RESEARCH TECHNIQUES AND THE FUTURE

I have now surveyed the principal areas of our knowledge about the oceanic tides and identified its shortcomings. It is appropriate to conclude with some speculations about the techniques which will probably be used in the future towards overcoming the present difficulties. This is not easy because, as I stated in the Introduction, the subject has entered a period of recession, following a wave of renewed interest in the 1965-1975 decade. The previous recession in oceanic tidal research, for which the definitive review paper was DOODSON (1958), marked a feeling of exhaustion of the field of analytical tidal solutions, while the hope for the future was in numerical methods. Now numerical methods have been extensively explored and have been found to lack physical realism in one way or another. Coupling with the Earth tide has been shown to be an indispensable factor, raising the computational problem into a still higher class of difficulty. We need to know more about tidal dissipation, internal tides, and the actual definition of the surface tide in regions remote from tho land. These call for a vigorous programme of tidal measurement on a worldwide scale. In my view, the next phase of progress will only come after such a programme.

At a few points in this review I have mentioned the recent facility for recording tides in the open sea — the 'pelagic tide recorder' — without describing any outstanding results from its use. In fact, the development of the pelagic tide recorder in the 1960s was a major 'breakthrough' in oceanographic technology, desired for decades by previous generations of researchers. It has not yet figured importantly in this review because the theoreticians have advanced too rapidly to allow any interaction with the technologists. Occasionally, a researcher computing the global tides has pointed to a few spots on the globe where some measurements would be useful for verification, usually zones of locally maximum amplitude. But none of the few laboratories equipped with the new tidal technology (in England, France and the USA) have felt justified in sending a ship thousands of miles for this express purpose. Rather, they have deployed their instruments where logistically feasible (usually not too far from home), or where research cruises were directed for other objectives. The experiments have sometimes provided interesting papers in their own right, but without any obvious impact on the global tidal scene (e.g. COLLAR & CARTWRIGHT, 1972 ; IRISH & SNODGRASS, 1972 ; FILLOUX, 1973 ; ZETLER *et al*, 1975). Could a more globally oriented programme of measurements be mounted ? To answer this we must first consider what a pelagic tide recorder consists of and how it is deployed.

The principle is to record the variations of pressure at a capsule sitting on the sea bed for a period of the order of one month. A precision of 1 mm equivalent head of water at a depth of some kilometres is less than one part per million, while the capsule itself has to withstand an ambient pressure of the order 5×10^6 kg m^{-2}. Pressure sensors are inevitably sensitive also to temperature, which contains tidal variations, so this also has to be recorded and compensated for. Mechanical creep causes the pressure signal to drift, requiring another correction. Fused quartz crystals have the best all-round properties but are expensive. Good results have also been obtained from cheaper but carefully engineered metallic strain gauges. Some designs use a compressed nitrogen backing pressure, applied when the capsule reaches full depth and then held 'constant', so that the sensor has only to deal with the small dynamic range of the tide itself, but this requires knowledge of the sensitivity of the compressed gas to temperature variations. The pressure is usually recorded as a total count of vibrations of a variable oscillator over a fixed period, of the order of $10^2 - 10^3$ s, and so a timing mechanism precise to about 1 in 10^6 is also required. Low power consumption is obviously desirable, and the battery power supply, computing circuits and data logger, all inside the pressurized capsule (usually a metal sphere) have to be robust and reliable at temperatures near 0 °C. Another important feature is the acoustic transponding system, used for interrogation, homing-in and releasing of ballast when the capsule is finally recalled to the surface.

Altogether, these requirements have been compared with those of an instrumented space satellite. The capsules are expensive and have to be tended by specialized scientists. Not least in cost is the ship itself, which either has to make two return journeys between home-port and recording site or have another research programme in the same sea area to occupy the month or so between laying and recovering the capsule(s). Obviously this activity is possible only to a scientifically developed nation with an interest in tidal research. The small number of deployments is therefore not surprising.

I have not quoted any references in the last two paragraphs because most of the instruments which have been developed since the first experiments in the early 1960s are summarized in an account of an intercalibration experiment (Unesco, 1975) which was organized by the international SCOR Working Group No 27 in late 1973 in the North Biscay area. Seven capsules designed in various laboratories in Canada, France, UK and USA were laid in close proximity from the British *RRS Discovery* and their results compared. Out of a great variety of techniques and independent calibrations, estimates of the M_2 tide were within ± 1 % in amplitude and ± 1° in phase. The experiment also brought out at least two other interesting facts. Nine different methods of analysing one-month tidal records, individually favoured by various authorities, were compared in application to a single record : their results differed by more than the various instruments. Tidal analysis is not such a perfected art as some authorities would have us believe. The other interesting fact was that four out of the seven capsules submitted were restricted to shelf-sea depths (200 m or less). This emphasizes the somewhat easier technique required for shallow-water work, and a tendency for institutes concerned with tides to restrict their activity to their national continental shelf seas.

The intercalibration experiment marked the culmination of activity of SCOR Working Group No 27. It disbanded shortly after the publication. During its ten years of office it did much to encourage development of pelagic instruments and methods of analysis, and generally to re-awaken interest in ocean tides. Under its influence, more than 600 station-days of pelagic tidal records were made at places in the Atlantic, Pacific and Southern oceans where the tides were previously unknown. Yet it failed to stimulate a genuinely worldwide activity or to organize any campaign of measurement coordinated with computer models. Pelagic tide recording remains the pursuit of three or four independent national groups, all in the northern hemisphere, and some expressly confined to national waters. The reasons are basically expense, slowness of the method to achieve significant results, and a shortage of oceanographers who are really dedicated to the study of tides.

The British (IOS) programme now seems to be the most active and deserves some description, since it represents the only attempt to simulate part of a plan suggested to SCOR Working Group No 27 by HANSEN (1966) of Hamburg (a pioneer of numerical tidal modelling). The plan is to surround a sizeable area of ocean with a chain of tidal measurements, then compute the tides in the interior of the area by one of the usual methods. By extending the chain across to important land boundaries, it can be slowly extended to cover whole oceans and eventually the globe. Concentration on a relatively small well-measured area enables one to test the various formulations and choose an optimum method in a way inaccessible to global tidal modellers. Processes at the shelf edge and internal tidal generation can be directly studied. The area chosen by IOS is bounded by the western European coastline and the mid-Atlantic Ridge between the latitudes of Iceland and the Azores, and measurements at about 500 km separation are now nearly complete. In addition, the tides along the edge of the continental shelf from Brittany to Norway have been measured with associated currents at 100 km separation. The latter measurements have already provided the necessary input to a shelf-sea model (figure 11) and the direct evaluation of power loss over this important oceanic boundary. Tidal models corresponding to the entire oceanic area are being prepared at IOS. The completion of such an area may encourage others to extend it to the west as far as Canada and the USA, while the possibility of isolating the whole of the North Atlantic Ocean by measurements between the Guinea Coast and northern Brazil is now within reach.

Even so, the possibility of parcelling out all the world's oceans by such combinations of pelagic measuring technology and computation looks rather remote. It would undoubtedly require generous cooperation from oceanographic institutes in the southern hemisphere and the countries bordering the Pacific Ocean. Perhaps equally important is to obtain reliable measurements of the major dissipating regions such as the Okhotsk Sea and the shelf sea off Argentina. In these days of extended national maritime boundaries this has to be left largely to the national authorities themselves and the hope that they are interested in tides.

The difficulties in the above scenario make one turn to other technologies than oceanographic which might be stretched to give useful measurements of ocean tides. The possibility of using widespread measurements of Earth-tidal gravity was mentioned in § 6.1 — the inverse loading problem. This may be reliable for areas of ocean well surrounded by points of land and islands, but the large tidal amplitudes localized in shallow seas tend to distort the loading picture. Large areas of the southern oceans are sparsely furnished with islands. If one has to put gravimeters on the ocean floor, one may as well use direct tide recorders as discussed previously.

A totally different technique, of tantalizing promise for the future, is the high-precision altimetry of the sea surface from Earth satellites. After several years' focusing on meteorology and Earth imagery, instrumented satellites are becoming more oriented towards geophysical and geodetic measurements, particularly those designed in France and the USA. One of the latest developments in the USA is a compressed-pulse vertical radar which can time the reflection from the sea surface to nanosecond or better accuracy. With reasonably known corrections for atmospheric transmission properties and bias from surface waves, it can essentially measure the height of separation to 0.1 — 0.5 m precision. The basic principles of its use are discussed in GREENWOOD *et al* (1969). Preliminary results from the radar altimeter mounted in the Skylab satellite are very promising (McGOOGAN *et al*, 1975) ; improved models are now in use in GEOS-3 and planned for SEASAT-A in 1978.

For the results to be useful for tidal and other studies, however, the absolute geocentric height of the orbit must also be known to decimetre accuracy. This is the more critical limitation at present. For accurate calculations of the orbit, the spacecraft must be well above the atmosphere to avoid drag and the smaller scale variations of the Earth's gravitational field, but too great a height would strain the accuracy of the altimeter and increase the area of its 'target' on the ocean surface. 800 km above the mean surface is typical of current plans. In order to fix the orbit to the required precision, laser-ranging stations are set up at strategic points on the Earth's surface, but it is not yet known whether this precision will extend more than 1 000 km or so from the ranging stations. At present the laser-ranging stations are concentrated near the USA, but a consortium of European laboratories is also planning a high-precision tracking exercise on their side of the Atlantic Ocean.

Given the altimeter data and adequate ranging — and there are political difficulties even here — extraction of tidal information is still far from straightforward. Ideally, one would like estimates of the height of each given portion of ocean surface, say a 500 km square, at 100 different times, effectively at random. But satellite tracks are never precisely repeated, and the mean geocentric height of the sea surface can vary spatially by several metres over such an area due to the shape of the geoid (KING-HELE, 1975). Indeed, the prime purpose of satellite altimetry to many scientists is the improved determination of the geoid, with the tides as a mere 'noise factor' to be removed by averaging. In any case, evaluation of the tidal signals will almost certainly have to be done in conjunction with computer models which aim to simulate the tides by the methods discussed earlier. Certainly, the disentanglement of the complex tissue of data in time, latitude and longitude which will arise from these altimeters will require the cooperation of experts from a range of scientific disciplines. It should also be noted that the altimeter detects the *total* tide, the sum of the oceanic tide as usually defined, the body tide, and the loading tide of the solid Earth. Separation of these components is a secondary problem (CARTWRIGHT, 1977).

Despite these difficulties, the prospect of nearly global coverage without the expense and logistics of research ships makes satellite altimetry, in the author's opinion, the main hope for the eventual solution of the world's cotidal map. Meanwhile a continuing programme of ship-based oceanographic research should concentrate on the outstanding physical problems associated with tidal dissipation.

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Postscript (February 1978)

The re-publication of this article by the International Hydrographic Organisation allows me to note some important developments during 1977.

According to recent lectures by the respective authors, both C. PEKERIS and W. ZAHEL have independently solved the global tidal equations (4.2), (6.11) which allow for elastic crustal yielding and self-attraction. See also GORDEEV *et al* (1977). The iteration processes apparently converge in the absence of constraints from boundary data, whereas they diverged in HENDERSHOTT's (1972) formulation. ZAHEL, using a 4° grid, obtains significantly closer agreement with observed coastal data than in his previous solution with the same grid size for a rigid Earth, (ZAHEL, 1970), but the general pattern remains qualitatively similar to that of Figure 8.

K. LAMBECK (1977) has published an extensive new review of tidal dissipation in the Earth-Moon system, demonstrating even better agreement between astronomically based estimates and those based on computed cotidal maps than in the figures quoted in this article. A typical agreed figure for the dissipation at the M_2 frequency is $3.20 \pm 0.15 \times 10^{12}$ W, a little lower than as quoted above. LAMBECK (1977) suggests that the Bering Sea has been grossly over-rated as a major sink of energy, and this is confirmed independently by a computer model of that sea by SÜNDERMANN (1977). Removing most of the contribution from the Bering Sea reduces MILLER's (1966) estimate of M_2 bottom-frictional losses from 1.7 to 1.5×10^{12} Watts, thus widening the gap still further.

As additional confirmation of the higher figure for total tidal dissipation, LAMBECK (1977) discusses the important recent estimates from the perturbations of satellite orbits (CAZENAVE, DAILLET & LAMBECK, 1977). The tidal masses distort the gravitational field to give periodic perturbations to the inclination and nodal longitude of artificial satellites, with amplitudes of order 0″.05. These can now be detected with some accuracy, and give in effect independent estimates of $C_{2,2}$ and $\varepsilon_{2,2}$ in (8.4), namely 0.031 m and 123° respectively. These and the equivalent M_2 dissipation of 2.7×10^{12} W are lower than the figures from cotidal maps, but there is some uncertainty associated with the influence of the $C_{4,2}$ harmonic on the satellite results.

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