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SOME NOTES ON SIMPLIFYING THE HSWC METHOD OF TIDAL ANALYSIS

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SUMMARY

The method of least squares, as used in the Harmonic Shallow Water Correction (HSWC) technique by AMIN (1977), is adapted under restrictions of fixed data length and specified constituents, imposed by DOODSON's (1957) method. It is shown that the technique provides a quick and simple method of computing HSWC coefficients. Also some simple graphic explanation of the criteria for selection of data length, essential to resolve constituents having given speeds, is provided.

INTRODUCTION

The HSWC method of analysis and prediction of tides was developed by DOODSON to overcome the inadequacies of the harmonic method of analysis which then (1950's) could not cope with the large number of shallow water constituents generated by tidal interaction and bottom friction. The HSWC method is designed to compute an improved quality of predicted high and low water turning points. The method assumes that the M_2 constituent is wholly dominant, and therefore high and low waters will occur close to the maxima and minima of the M_2 tide.

In principle, if

$$\delta = \eta - \eta_p$$

where η is the level of observed tide (high or low water) and η_p is the associated predicted level based on a few principal constituents from the same, or a close by location, is computed for a shallow tide, then the series δ can be analysed to provide terms which can be used in combination with η_p to obtain substantially improved predictions. The series of residuals (or differences) δ are functions of residuals of standard constituents, which are grouped together to form a small number of constituents known as

HSWC constituents (see Table 1), brought about by the effects of aliasing on the time interval of half a lunar day. The method lends itself to treating separately both the heights and times of high or low water.

TABLE 1

HSWC constituent names are as given by DOODSON

No.	Speed (deg/HLD)	HSWC cons. names	Constituents
1	0.0	C (00)	M_4 , M_6 , M_8 . . . .
2	0.5100967	C (01)	Sa , MA_2 , Ma_2 . . . .
3	1.0201934	C (02)	Ssa , MKS_2 , MSK_2 , OP_2 . . . .
4	5.8565756	C (11)	SN_4 , MSN_6 , $2MSN_8$, SNM_2 , $M\nu_4$ . . . .
5	6.7614611	C (13)	N_2 , L_2 , $2MN_2$, MN_4 , ML_4 , $2MN_6$, $3MN_8$
+ 6	12.1079635		T_2 . . . .
7	12.6180366	C (25)	S_2 , $2MS_2$, MS_4 , $2MS_6$, $3MS_8$ . . . .
+ 8	13.1281333		R_2 . . . .
9	13.6382313	C (27)	$2MK_2$, MK_4 , $2MK_6$ . . . .
10	18.4746122	C (36)	$2SN_6$, $2SMN_8$, $M\nu S_2$ . . . .
11	19.3794977	C (38)	MSN_2 , MNS_2 , $2MSN_4$, $2MNS_4$ . . . .
12	25.2360732	C (50)	$2SM_2$, S_4 , $2SM_6$, $2(MS)_8$, $3M2S_2$ . . . . . .
13	26.2562679	C (52)	SK_4 , MSK_6 , $2MSK_8$, SKM_2 , $3M(SK)_2$ . . .
+ 14	27.2764626		$2KM_2$, K_4 . . . .
15	159.5426537	C'(40)	OO_1 , KO_3 . . . .
16	160.5628484	C'(38)	SO_1 , SK_3 . . . .
17	161.5830418	C'(36)	SP_3 . . . .
18	166.4194239	C'(27)	J_1 , Q_1 , MJ_3 , MQ_3 . . . .
19	173.1808850	C'(13)	K_1 , O_1 , MK_3 , MO_3 , $2MK_3$, $2MO_3$ . . . . .
* 20	173.6909817	C'(12)	S_1 , MS_1 . . . .
21	174.2010784	C'(11)	P_1 , MP_1 , SO_3 . . . .
+ * 22	180.0		M_1 , M_3 , M_5 . . . .

+ These constituents were not included by DOODSON.
* These constituents are not considered in the matrix computations given here.

AMIN [1] in his further development of DOODSON's [2] HSWC technique has shown that each set of differences δ can be represented in the form of a discrete series.

$$\delta_r = \sum_k \left\{ H_k \cos (W_k + \omega_k r - \Psi_k) + n(r) \right\} \tag{1}$$

where summation is over all residual lines of the spectrum, and

H is the amplitude
W is initial phase lag
ψ an arbitrary phase lag
ω speed in degrees/half lunar day, and
n(r) is noise in the series.

$r = 0, 1, 2, \ldots N$ relate to the HSWC constituents. The computation of the initial phase lag W has been fully described by AMIN [1].

Equation (1) can be rewritten as:

$$\delta_r = \sum_k (a_k \cos \omega_k r + b_k \sin \omega_k r) + n(r) \tag{2}$$

where :

$$a_k = H_k \cos(\omega_k - \psi_k)$$
$$b_k = -H_k \sin(\omega_k - \psi_k)$$

METHOD OF LEAST SQUARES

Solution of the redundant system of equation (2) for all a_k, b_k can be obtained by using the method of least squares in a normal fashion, by minimising the expression

$$\sum_0^N \left[\delta_r - \sum_{k=1}^{M} (a_k \cos \omega_k r + b_k \sin \omega_k r) \right]^2 \tag{3}$$

In this manner the above $N+1$ expressions are differentiated partially with respect to the coefficients a_k, b_k and equated to zero to obtain a linear set of 2M equations which can be written :

$$\mathbf{PX} = \mathbf{Y} \tag{4}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} \mathbf{Y}^c \\ \mathbf{Y}^s \end{bmatrix}$$

A, **B**, $\mathbf{Y}^c$ and $\mathbf{Y}^s$ being vectors of length M where

$$y_i^c = \sum_{r=0}^{N} \delta_r \cos \omega_i r \text{ and : } y_i^s = \sum_{r=0}^{N} \delta_r \sin \omega_i r \tag{5}$$

The elements of the matrix **P** in equation (4) are simply

$$p_{ij} = \sum_{r=0}^{N} \cos \omega_i r \cos \omega_j r \quad 1 \leqslant i \leqslant M \,.\, 1 \leqslant j \leqslant M \tag{6.1}$$

$$\sum_{r=0}^{N} \cos \omega_i r \sin \omega_j r \quad 1 \leqslant i \leqslant M \,.\, M < j \leqslant 2M \tag{6.2}$$

$$\sum_{r=0}^{N} \sin \omega_i r \cos \omega_j r \quad M < i \leqslant 2M \,.\, 1 \leqslant j \leqslant M \tag{6.3}$$

$$\sum_{r=0}^{N} \sin \omega_i r \sin \omega_j r \quad M < i \leqslant 2M \,.\, M < j \leqslant 2M \tag{6.4}$$

It is readily seen that matrix **P** is symmetric. Considerable simplification of the system of equations (4) can be made if the central point of the time series is chosen as the time origin, such that $r = -N/2, -N/2+1, \ldots 0 \ldots N/2$ whereby the matrix **P** reduces to :

$$p_{ij} = 1 + 2 \sum_{r=1}^{N/2} \cos \omega_i r \cos \omega_j r \qquad 1 \leqslant i \leqslant M \,.\, 1 \leqslant j \leqslant M \tag{7.1}$$

$$= 0 \qquad 1 \leqslant i \leqslant M \,.\, M < j \leqslant 2M \tag{7.2}$$

$$= 0 \qquad M < i \leqslant 2M \,.\, 1 \leqslant j \leqslant M \tag{7.3}$$

$$= 2 \sum_{r=1}^{N/2} \sin \omega_i r \sin \omega_j r \qquad M < i \leqslant 2M \,.\, M < j \leqslant 2M \tag{7.4}$$

and is simply represented as :

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & S \end{array}\right] \tag{8}$$

The matrix is symmetrically partitioned, and (7.1) to (7.4) show that the square sub matrices **C** and **S** are also symmetric.

A solution to equation (4) can be written as :

$$\mathbf{A} = \mathbf{C}^{-1} \mathbf{Y}^c \tag{9.1}$$

$$\mathbf{B} = \mathbf{S}^{-1} \mathbf{Y}^s \tag{9.2}$$

with the further simplification that :

$$y_i^c = \delta_0 + \sum_{r=1}^{N/2} F_r \cos \omega_i r \qquad F_r = \delta_r + \delta_{-r} \tag{10.1}$$

$$y_i^s = \sum_{r=1}^{N/2} G_r \sin \omega_i r \qquad G_r = \delta_r - \delta_{-r} \tag{10.2}$$

SOLUTION OF EQUATIONS

The stability of equations (9.1) and (9.2) depends upon the conditioning of matrices **C** and **S** and any perturbations in vectors $\mathbf{Y}^c$ and $\mathbf{Y}^s$ arising from non tidal effects and noise. Equations (7.1) and (7.4) show that the diagonal elements of **C** and **S** increase with r, and the off-diagonal elements oscillate in the range :

$$\left[0, \; \sum_{r=0}^{r^*} \{ \cos (\omega_i - \omega_j) r + \cos (\omega_i + \omega_j) r \} \right] \tag{11}$$

where r* is some optimal value.

Referring back to matrix **P**, in the final form (8) it is easy to understand the development of its elements from the graphical representation in figure 1.

For a continuous function (viz. δ in (2)) the matrix elements p_{ij} are simply given as the sum of the areas enclosed by the function and the time axis, with associated proper signs. If the function is discrete, as in the case discussed here, then the magnitude of the elements is given by the sum of values of the function at discrete points of given interval. In

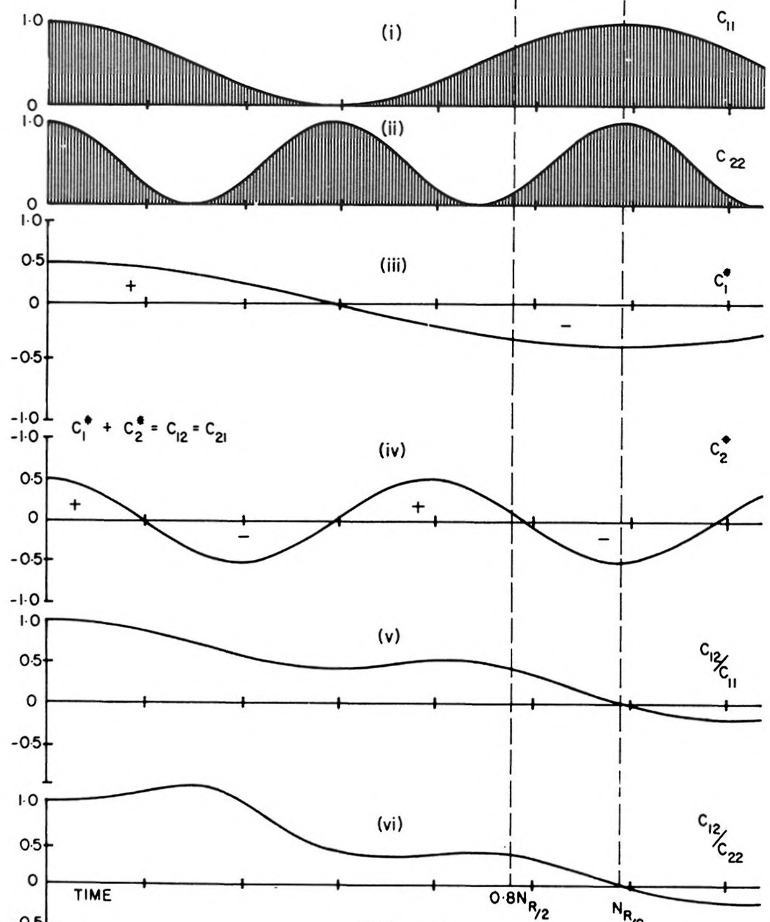

FIG. 1 A.

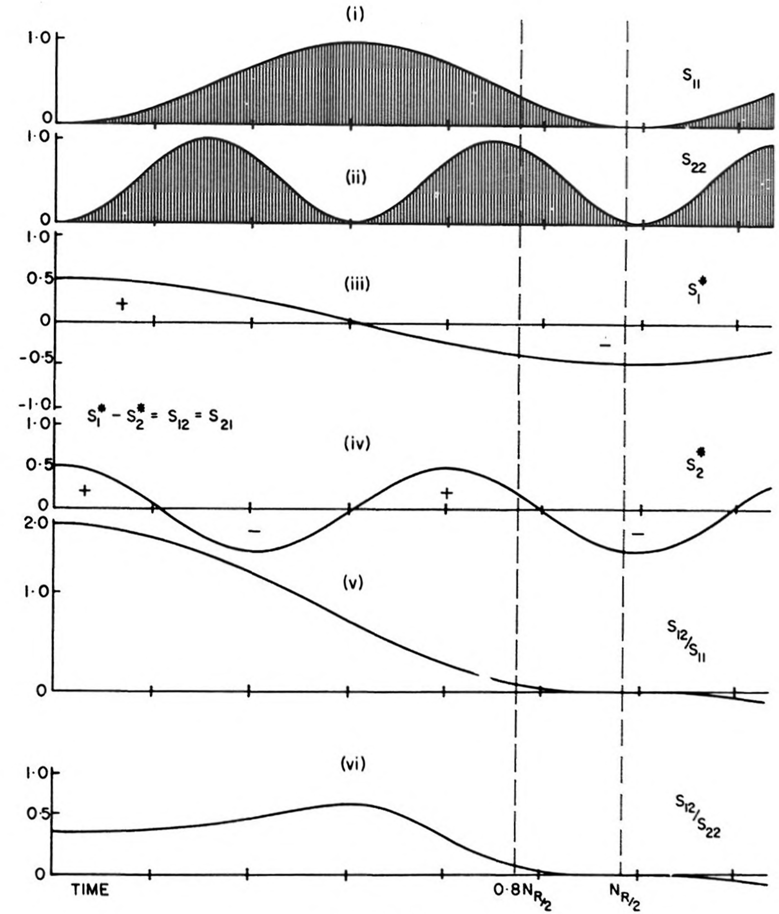

FIG. 1 B.

FIG. 1 A. — Development of elements for a (2×2) matrix **C** with variable span of data and origin chosen at the centre. Because of symmetry only the right hand side of element functions is shown. The magnitude of the elements (C_{ij}, $i = 1,2$; $j = 1,2$) is given by the area, with due sign, enclosed by the function and time axis.

(i) (ii) show the diagonal elements C_{11}, C_{22}.
(iii) (iv) show the separate components of the off–diagonal elements C_{12} and C_{21} ($C_{12} = C_{21}$).
(v) (vi) gives the ratio of off–diagonal element to the respective diagonal elements C_{11}, C_{22} thus determining the conditioning of the matrix.

FIG. 1 B. — Same as figure 1 A except that the associated (2×2) matrix **S** is considered.

general $(\omega_i - \omega_j) \leqslant (\omega_i + \omega_j)$ and the contribution from the second term under the summation sign in (11) is negligible compared with that of the first term. With a suitable choice of $N = N_R$ according to Rayleigh's criteria, the off-diagonal elements diminish in size and the matrix **P** becomes highly diagonal dominant, and also positive definite. This provides a sufficient condition for direct and iterative methods of solution of (4) to be convergent. With increased N, any error in **Y** will be reduced to improve the solution **X**.

GODIN's (1970) criterion of $N = 0.8\,N_R$ is such that the matrix **P** is diagonally dominant, but the off-diagonal elements can be of significant magnitude (see figure 1A, (v) (vi)). This makes application of only direct methods, for solution of the system of linear equations, safe and simple because the process of positioning of pivotal elements is eliminated and the chance of developing round-off errors is substantially reduced. However because the system is not necessarily positive definite, the convergence of any iterative scheme of solution is not guaranteed. In the format presented here it is sufficient to compute the inverse of the final simple matrix form **P**, indicated in (8), since this will remain invariant for the further solution of any HSWC coefficients. Two such inverted matrices for 365 days (N = 705) and 355 days (N = 685) are given in the Appendix.

CONCLUSION

Although the method as adapted and discussed here does not converge exactly to DOODSON's method, it is a parallel process. Once the inverse of **P** (viz. **C** and **S**) is computed the vector **Y** (viz. $\mathbf{Y}^c$ and $\mathbf{Y}^s$) is found and the coefficient vectors **A**, **B** are simply obtained by multiplication (viz. $\mathbf{C}^{-1}\,\mathbf{A}$ and $\mathbf{S}^{-1}\,\mathbf{B}$). The overall process is simple, but requires marginally more arithmetic procedures than the DOODSON method. It can easily be adapted for hand calculators with the advantage that it can be programmed for small desk computers with only a few instructions. With a selected span of data (365 or 355 days) and a given number of coefficients (M = 20), the inverted **P** matrix and its sub matrices (**C** and **S**) are diagonally dominant. Thus any constituent (coefficient a_k or b_k) which is expected to be small can be left unresolved by deleting corresponding rows and columns from $\mathbf{C}^{-1}$ and $\mathbf{S}^{-1}$, as well as associated elements of $\mathbf{Y}^c$ and $\mathbf{Y}^s$, without adverse effects on the solution. Although the time series length can be selected according to GODIN's criterion ($N = 0.8\,N_R$) it will be suitable only when data are of good quality, otherwise any error in **Y** is transmitted into the system and may have adverse effects in the solution procedure.

ACKNOWLEDGEMENTS

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(See Appendix on page 174)

APPENDIX

The inverse of symmetric matrices **C** and **S** for analysis of 705 and 685 observations shown as element ($x10^3$) of upper triangular matrices. The constituents considered are as indicated in Table 1.

1003	-9	9	59	37	21	-20	20	-19	-11	1	13	13	12	-3	-3	-3	1	1	1
	2016	-18	-120	-75	-42	41	-40	38	22	-2	-26	-25	-24	6	-6	6	-1	-3	-3
		2016	123	76	42	-41	40	-39	-22	2	26	25	24	-6	-6	-6	1	3	3
			2037	230	-28	25	-22	20	-9	-12	-1	-1	-1	0	0	0	0	0	0
				2014	-65	59	-55	51	8	-12	-19	-18	-17	4	4	4	-1	-2	-2
					2021	-22	22	-22	-52	-26	24	23	22	-4	-4	-4	1	2	2
						2020	-22	22	57	29	-25	-23	-22	4	4	4	-1	-2	-2
							2019	-21	-63	-33	25	24	22	-4	-4	-4	1	2	2
								2019	69	37	-25	-24	-22	4	4	4	-1	-2	-2
									2029	242	45	39	35	-5	-5	-5	1	2	2
										2037	67	58	51	-6	-6	-6	1	3	3
											2002	4	3	1	1	1	0	0	0
												2001	3	1	1	1	0	0	0
													2000	1	1	1	0	0	0
														1998	0	0	38	-7	-5
															1998	0	46	-7	-6
																1998	58	-8	-6
																	2011	35	30
																		1982	-17
																			1979

C^{-1}

705 observations

1004	-68	67	25	-29	3	0	-1	4	8	-6	0	1	4	3	3	3	-2	3	3
	2077	-135	-51	39	-7	0	3	-8	-17	12	1	-3	-8	-7	-7	-6	5	-6	-6
		2076	54	-36	8	-2	-2	7	17	-11	-1	3	8	6	6	6	-5	6	6
			2021	285	5	-11	16	-20	-12	17	0	-5	-10	-6	-6	-6	5	-6	-6
				2062	-27	17	-9	1	-24	8	4	-1	-6	-6	-6	-6	5	-6	-6
					2003	-63	64	-65	-24	24	0	-6	-11	-6	-6	-6	5	-6	-6
						2036	-66	67	30	-22	-1	5	10	6	6	6	-5	6	6
							2009	-69	-37	19	3	-3	-9	-6	-6	-6	5	-6	-6
								2011	45	-15	-5	1	8	6	6	6	-5	6	6
									2036	286	-15	-22	-26	-6	-6	-5	5	-5	-5
										2048	20	6	-4	-7	-7	-6	5	-6	-6
											2003	63	63	6	6	6	-5	6	6
												2006	65	6	6	6	-5	6	6
													2008	6	6	6	-5	6	6
														2008	65	62	7	7	9
															2005	62	20	0	3
																2002	38	-7	-4
																	2012	-30	-32
																		2006	60
																			1996

C^{-1}

685 observations

705000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1998	0	-11	-7	-1	1	-1	1	1	0	-1	-1	-1	0	0	0	0	0	0
		1998	22	13	3	-2	2	-2	-2	-1	1	1	1	0	0	0	0	0	0
			2046	271	-52	48	-44	41	-35	-41	3	4	4	-1	-2	-2	-6	5	5
				2066	-64	58	-53	48	-20	-31	-3	-2	-1	-1	-1	-1	-4	4	4
					1993	5	-5	4	-57	-43	14	13	12	-1	-1	-1	-4	4	4
						1993	4	-4	62	46	-15	-14	-13	1	1	1	4	-4	-4
							1994	4	-68	-50	16	14	14	-1	-1	-1	-4	4	4
								1995	75	54	-16	-15	-14	1	1	1	4	-4	-4
									2048	254	30	25	21	1	1	1	3	-3	-3
										2036	51	42	36	0	0	0	0	0	0
											2005	7	7	-1	-2	-2	-6	5	5
												2005	7	-1	-1	-2	-6	5	5
													2004	-1	-1	-2	-6	5	5
														2006	8	9	55	-31	-30
															2006	9	64	-33	-32
																2008	76	-35	-34
																	2000	28	23
																		2022	25
																			2025

S^{-1}

705 observations

685000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1941	-1	-15	-6	5	5	-5	4	-3	-3	2	2	2	0	0	0	0	0	0
		1943	30	13	11	-10	10	-9	7	6	-5	-4	-4	0	0	0	0	0	0
			2067	289	26	-27	29	-29	9	20	-11	-12	-12	0	0	0	1	0	0
				2024	-36	30	-24	19	-21	-11	11	9	7	0	0	0	0	0	0
					2010	-68	66	-64	-11	19	-3	-6	-8	0	0	0	0	0	0
						2008	-65	63	19	-15	0	3	5	0	0	0	0	0	0
							2005	-63	-28	10	3	0	-2	0	0	0	0	0	0
								2003	38	-4	-7	-3	0	0	0	0	0	0	0
									2051	289	-23	-27	-29	0	0	0	1	0	0
										2036	24	13	5	0	0	0	0	0	0
											2005	62	59	0	0	0	0	0	0
												2002	59	0	0	0	0	0	0
													2000	0	0	0	0	0	0
														1999	60	60	21	0	6
															2003	63	32	-2	4
																2007	46	-5	2
																	1991	-11	-18
																		1998	61
																			2008

S^{-1}

685 observations