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ON THE SHALLOW-WATER HARMONIC TIDAL CONSTITUENTS

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ABSTRACT

This is an investigation of the physical law governing the interaction of the astronomical tidal constituents which generates the shallow-water tidal constituents, for the simple case of a narrow channel. The solution for both progressive and standing waves, up to the fifth order, is presented. The fluid is supposed non-viscous, and the flow frictionless.

1. INTRODUCTION

The problem of generation of shallow-water constituents is a difficult one. Many scientists have tried to find the law of interaction between purely astronomical constituents in shallow water, but the known simple solutions, based on hydrodynamical equations, yield very few terms (AMIN, 1977). It is interesting to note that DOODSON's simple geometrical solution permitted him to use all the shallow water constituents included in his analysis method. The relative amplitudes of these constituents, derived according to DOODSON's own theory on propagation, led him to such a choice (DOODSON & WARBURG, 1941 and DOODSON, 1947).

Since DOODSON started with a formula expressing the celerity of the progressive tidal wave, the present author had the idea of also starting with the celerity, but deriving the theory analytically (FRANCO, 1956). However, theory to be useful, must start with an accurate formula for the celerity. Such a formula was derived by De SAINT VENANT in 1871 (LEVY, 1898). Let me quote what LEVY said : "... We emphasized, as a result of both the first and the second approximation, that the speed of the current in a considered section of the channels depends upon the water height only. It is logical to wonder if this law is accurate or approximate. The solution given by De SAINT VENANT shows that the latter is the answer to the question".

De Saint Venant arrived at his solution by considering a rectilinear channel with constant depth and indefinite length, and ignoring friction. He started with the Lagrangian equations, and also found the celerity of the wave travelling through the channel. It is surprising that such a solution is not in more common use among oceanographers. The derivation of the celerity expression due to De Saint Venant was presented by Levy (*op. cit.*) in his interesting and rare book. Since this paper is based on the celerity, it seems useful to give here a new derivation of the above-mentioned expression, based on Euler equations, for water motion in a channel, as envisaged by De Saint Venant. Such an equation for a channel in which there is no lateral velocity is :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \zeta}{\partial x} \tag{1a}$$

and it must be associated with the equation of continuity which is expressed by :

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} = -(h+\zeta)\frac{\partial u}{\partial x} \tag{1b}$$

where in equations (1a) and (1b) :

ζ = height of the tide referred to mean sea level;
u = velocity of the tidal current along the channel;
t = time;
h = mean depth of the channel;
g = acceleration due to gravity.

Since the length of the channel is not considered, De Saint Venant assumed that it would be sufficient to adopt :

$$u = A\sqrt{g(h+\zeta) + C}$$

as a particular solution to equations (1a) and (1b), in such a way that A and C are the constants to be computed. If we start from a still condition we must have $\zeta = 0$ for $u = 0$. Thus :

$$C = -A\sqrt{gh}$$

? ? $C = -gh$

and consequently :

$$u = A\,[g(h+\zeta)]^{\frac{1}{2}} - A\,(gh)^{\frac{1}{2}} \tag{1c}$$

The derivatives of u with respect to x and t are :

$$\frac{\partial u}{\partial x} = \frac{Ag}{2}\,[g(h+\zeta)]^{-\frac{1}{2}}\,\frac{\partial \zeta}{\partial x} \tag{1d}$$

$$\frac{\partial u}{\partial t} = \frac{Ag}{2}\,[g(h+\zeta)]^{-\frac{1}{2}}\,\frac{\partial \zeta}{\partial t} \tag{1e}$$

From equations (1b), (1c) and (1d) we obtain :

$$\frac{\partial \zeta}{\partial t} = \left\{ -\frac{3A}{2}\,[g(h+\zeta)]^{\frac{1}{2}} + A\,(gh)^{\frac{1}{2}} \right\}\frac{\partial \zeta}{\partial x} \tag{1f}$$

Substituting for $\partial\zeta/\partial t$ into equation (1e), and introducing the resulting $\partial u/\partial t$ as well as equations (1c) and (1d) into equation (1a) we find :

$$\left(-\frac{A^2}{4}+1\right)\frac{\partial\zeta}{\partial x}=0$$

Since $\partial\zeta/\partial x \neq 0$, except for maxima and minima, we must equate its coefficient with zero, in order to solve for A. The result for such a solution is $A = \pm 2$. This positive value expresses conveniently the physical conditions; thus from equation (1c) we have :

$$u = 2\,[g(h+\zeta)]^{\frac{1}{2}} - 2\,(gh)^{\frac{1}{2}} \tag{1g}$$

We derive also from equation (1f) :

$$\frac{\partial x}{\partial t} = -3\,[\,g\,(h+\zeta)]^{\frac{1}{2}} + 2\,(gh)^{\frac{1}{2}}$$

If we consider that $x = 0$ at the entrance of the channel and that it corresponds to instant t, then a point of the wave surface at distance ζ from the mean sea level will arrive at a section of the channel at distance x from the entrance at instant t_1. Thus we can integrate the above expression from 0 to x or from t to t_1 which gives :

$$x = \{-3\,[g(h+\zeta)]^{\frac{1}{2}} + 2(gh)^{\frac{1}{2}}\}\,(t_1 - t)$$

or if we put :

$$c = 3\,[g\,(h+\zeta)]^{\frac{1}{2}} - 2\,(gh)^{\frac{1}{2}} \tag{1h}$$

it follows that

$$t_1 - t = -x/c$$

which means that the height ζ will arrive at section x with a time lag x/c. Thus, c is the wave celerity. It is interesting to note that both the celerity c and the velocity of the current u are functions solely of ζ. It can be seen that the particular solution of De Saint Venant is a rigorous one for the assumed physical conditions.

Further developments need to have 1/c expanded according to the increasing powers of ζ/h, up to the fifth power. Thus it is convenient to write equation (1h) as :

$$c = (gh)^{\frac{1}{2}}\;[3\,(1+\zeta/h)^{\frac{1}{2}} - 2] \tag{1i}$$

If we make

$$\sqrt{gh} = c_o \tag{1j}$$

the desired development yields :

$$1/c = \left(1 - \frac{3}{2}\rho + \frac{21}{8}\rho^2 - \frac{75}{16}\rho^3 + \frac{1\,077}{128}\rho^4 - \frac{3\,877}{256}\rho^5\right)\Big/ c_o \tag{1k}$$

where

$$\rho = \zeta/h \tag{1l}$$

The test for convergence showed that we must have $\rho < 0.555$ (recurring).

If the tidal range is 10 m in a channel with $h = 20$ m, from equation (1l) we have $\rho = 0.25$. Hence, in the usual applications, ζ seldom reaches the value of ρ given.

2. TIDAL HEIGHT IN SHALLOW WATER

The complex expression for tidal height may be taken as :

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp(i\,\omega_j\, t) \tag{2a}$$

where Q is the number of constituents involved, and

$$a_j = \delta_j R_j \exp(-ir_j) \quad \delta_j = \begin{cases} 1/2 & j \neq 0 \\ 1 & j = 0 \end{cases} \tag{2b}$$

R_j and r_j being respectively the amplitude and phase of the constituent of order j. The following conditions must be observed :

$$\begin{aligned} r_j > 0\,,\omega_j > 0 \quad &\text{for} \quad j > 0 \\ r_j = 0\,,\omega_j = 0 \quad &\text{for} \quad j = 0 \\ r_j < 0\,,\omega_j < 0 \quad &\text{for} \quad j < 0 \end{aligned} \tag{2c}$$

Equation (2a) represents the height of a point on the progressive wave profile travelling along the channel referred to the mean level ($a_o - 0$). Thus if c is the celerity at that point, it can be seen that the same height will occur at distance x from the channel entrance with a time lag x/c. Hence, from equation (2a), for such a point we have :

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp[i\omega_j(t - x/c)] \tag{2d}$$

But we can write equation (1k) as follows :

$$1/c = 1/c_o + \Delta \tag{2e}$$

where Δ is the algebraic sum of all the ρ^n terms. Thus from equations (2d) and (2e) we have :

$$\zeta(t) = \sum_{j=-Q}^{O} a_j \exp[i\omega_j(t - x/c_o)] \exp[-i\omega_j x\Delta] \tag{2f}$$

The second exponential in equation (2f) was expanded according to the powers of $\omega_j\Delta$. Afterwards, Δ was replaced by the sum of the terms in ρ taken from equation (1h). For powers of ρ up to the fifth, the result is :

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp[i\omega_j(t - x/c_o)] \left[1 + \sum_{n=1}^{5} (A'_n + iB'_n)\rho^n\right] \tag{2g}$$

But equation (1l) shows that ρ is a function of ζ, and another expansion (FRANCO, 1976) is necessary to eliminate ζ from the second member, by expressing ρ in terms of

$$\rho_o = \zeta_o/h \tag{2h}$$

where

$$\zeta_o(t) = \sum_{j=-Q}^{Q} a_j \exp[i\omega_j(t - x/c_o)] \tag{2i}$$

Then we find :

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp[i\omega_j(t - x/c_o)] \left[1 + \sum_{n=1}^{5} (Z_{nj}/h^n)\zeta_o^n(t)\right] \tag{2j}$$

where

$$Z_{nj}/h^n = A_{nj} + iB_{nj} = Z'_{nj} \tag{2k}$$

A_{nj} and B_{nj} being as shown in table 2-I. It follows from equations (2i) and (2j) that :

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp[i\omega_j(t - x/c_o)] +$$

$$+ \sum_{n=1}^{5} \left\{ \underbrace{\sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} \cdots \sum_{p=-Q}^{Q}}_{n+1} Z'_{nj}(a_j a_k \ldots a_p) \exp[i(\omega_j + \omega_k + \ldots + \omega_p)(t - x/c_o)] \right\} \tag{2l}$$

This is the expression for the law of interaction between astronomical constituents in a shallow-water channel.

Table 2-I

n	A_{nj}	B_{nj}
1	0	$\frac{3\omega_j x}{2c_o h}$
2	$-\frac{27\,\omega_j x^2}{8\,c_o^2 h^2}$	$-\frac{21\,\omega_j x}{8\,c_o h^2}$
3	$\frac{63\,\omega_j x^2}{4\,c_o^2 h^3}$	$\frac{75\,\omega_j x}{16\,c_o h^3} - \frac{144\,\omega_j^3 x^3}{16\,c_o^3 h^3}$
4	$-\frac{6\,705\,\omega_j^2 x^2}{128\,c_o^2 h^4} + \frac{3\,375\,\omega_j^4 x^4}{128\,c_o^4 h^4}$	$-\frac{1\,077\,\omega_j x}{128\,c_o h^4} + \frac{4\,725\,\omega_j^3 x^3}{64\,c_o^3 h^4}$
5	$\frac{38\,286\,\omega_j^2 x^2}{256\,c_o^2 h^5} - \frac{40\,824\,\omega_j^4 x^4}{128\,c_o^4 h^5}$	$\frac{3\,873\,\omega_j x}{256\,c_o h^5} - \frac{96\,228\,\omega_j^3 x^3}{256\,c_o^3 h^5} + \frac{6\,561\,\omega_j^5 x^5}{80\,c_o^5 h^5}$

According to equation (2b) :

$$a_j a_k \ldots a_p = \frac{1}{2}(2\,\delta_j\,\delta_k \ldots \delta_p)(R_j R_k \ldots R_p) \exp[-i(r_j + r_k + \ldots + r_p)]$$

Consequently, since $\delta_j = \delta_k = \ldots = \delta_p = 1/2$ for j. k, ... p $\neq$ 0, and since no interaction exists for $j = k = \ldots = p = 0$, it follows that :

$$a_j a_k \ldots a_p = \frac{1}{2}(2^{-n} R_j R_k \ldots R_p) \exp[-i(r_j + r_k + \ldots + r_p)] \tag{2m}$$

which is the part of the complex amplitude of the compound constituent that is independent from coefficient Z'_{nj}. This expression shows that if

$$R_j = f_j H_j$$

and if

$$r_j = \alpha_j + u_j$$

f_j and u_j being respectively the lunar nodal factor and angle, then

$$a_j k_j \ldots a_p = F_{jk\ldots p} \left\{ \frac{1}{2}(2^{-n} H_j H_k \ldots H_p) \exp\left[-i(\alpha_j + \alpha_k + \ldots \alpha_p)\right] \right\} \qquad (2n)$$

where

$$F_{jk\ldots p} = (f_j f_k \ldots f_p) \exp\left[-i(u_j + u_k + \ldots + u_p)\right] \qquad (2o)$$

is the complex nodal factor. The signs of u_j, u_k, ... u_p are the same as those of j, k, ... p, respectively.

From equations (2l) and (2m) we take :

$$\zeta(t) = \sum_{j=-Q}^{Q} a_j \exp\left[i\omega_j (t - x/c_o)\right] + \sum_{n=1}^{5} \left\{ \underbrace{\sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} \ldots \sum_{p=Q}^{Q}}_{n+1} Z'_{nj} \frac{1}{2}(2^{-n} R_j R_k \ldots R_p) \exp\left[i(\omega_j + \omega_k + \ldots \omega_p)(t - x/c_o) - i(r_j + r_k + \ldots r_p)\right] \right\} \qquad (2p)$$

This expression shows that many shallow-water constituents are generated, with amplitudes proportional to :

$$R_{jk\ldots p} = R_j R_k \ldots R_p / 2^n \qquad (2q)$$

their frequencies being

$$\omega_{jk\ell\ldots} = \omega_j + \omega_k + \omega_\ell \qquad (2r)$$

where ω_q and q have the same sign.

Since the angular frequency of M_2 is approximately equal to 1/2 rad/hour it is possible to make $|\omega_j| = s/4$, where s = 1 for the diurnal constituents, and s = 2 for the semidiurnal constituents. Such an approximation is used for $n \geqslant 2$ only. The multiple summations of equation (2p) show that according to equation (2r) we can have as many constituents with the same frequency as the number of possible permutations of ω_j, ω_k, etc. If we number the constituents as shown in table 2-II and desire to generate, say, the third order constituent $MSNK_4$, the frequency of this constituent will be :

$$\omega_{1234} = \omega_1 + \omega_2 + \omega_3 - \omega_4$$

where j = 1, k = 2, l = 3 and m = − 4. The same frequency will however be generated if

$$\omega_{4321} = -\omega_4 + \omega_3 + \omega_2 + \omega_1$$

where j = − 4, k = 3, l = 2 and m = 1. The final constituents will be

$2^{-3}R_1R_2R_3R_4$ multiplied by the algebraic sum of Z'_{nj} shown in table 2-I, for $n = 3$ and $|\omega_j| = s/4$, where $s = 2$.

Table 2-II

Comp.	j k ℓ . .	Linear part of the astronomic argument					Frequency in radians per hour	Relative coefficient
		τ	s	h	p	90°	$\omega_{j,k,\ell\ldots}$	$H_{j,k,\ell\ldots}$
M_2	1	2	0	0	0	0	0.5058680	0.90805
S_2	2	2	2	− 2	0	0	0.5235988	0.42248
N_2	3	2	− 1	0	1	0	0.4963669	0.17380
K_2	4	2	2	0	0	0	0.5250323	0.11495
ν_2	5	2	− 1	2	− 1	0	0.4976385	0.03301
L_2	6	2	1	0	− 1	2	0.5153692	0.02567
K_1	7	1	1	0	0	1	0.2625162	0.53009
O_1	8	1	− 1	0	0	− 1	0.2433519	0.37689
P_1	9	1	1	− 2	0	− 1	0.2610826	0.17546
Q_1	10	1	− 2	0	1	− 1	0.2338507	0.07214
J_1	11	1	2	0	− 1	1	0.2720713	0.02963
M_1	12	1	0	0	− 1	1	0.2530150	0.02963

In order to draw up tables covering constituents up to the fifth order, formulae based on combinatorial analysis were derived so as to facilitate the computation of the relative coefficients. Table 2-III is a sample of a more complete table.

The author (Franco, 1976) has studied the interaction which generates shallow-water constituents M_2, S_2, etc., for the second order terms. These constituents are large enough to modify the astronomical M_2, S_2, etc., and it is important to note that the nodal factors may not be the same.

3. STANDING WAVES

Let us suppose a narrow rectangular basin (fig. 1). If we imagine that an oscillation is generated at O, that such an oscillation propagates in a manner similar to a progressive wave, and that it travels up to F where it is reflected, then we may represent the resulting oscillation at O by :

$$\zeta_o(t) = \sum_{j=-Q}^{Q} a_j \exp(i\omega_j t) + \sum_{j=-Q}^{Q} a_j \exp[i\omega_j (t - 2L/c)]$$

Table 2-III

Constituents of 4th order

Constituents	Astronomical arg. τ	s	h	p	1°	Angular freq. (°/hour)	Relative coefficient: Formula	Relative coefficient: Value	Complex modal coef.	Doodson's number	Phase combination
General coefficient $\mid A_{41} + i3B_{41}/5 \mid$											
$4M\overline{K}_6$	6	−2	0	0	0	85.8542795	$5H_1^4H_4/16$	0.024	$F_1^4F_4^*$	635,555	$4M_2 - K_2$
$4M\overline{S}_6$	6	−2	2	0	0	85.9364168	$5H_1^4H_2/16$	0.090	F_1^4	637,555	$4M_2 - S_2$
$2MSN\overline{K}_6$	6	−1	−2	1	0	86.3258006	$60H_1^2H_2H_3H_4/16$	0.026	$F_1^2F_3F_4^*$	643,655	$2M_2 + S_2 + N_2 - K_2$
$3MS\overline{K}_6$	6	0	−2	0	0	86.8701753	$30H_1^3H_2H_4/16$	0.068	$F_1^3F_4^*$	653,555	$3M_2 + S_2 - K_2$
$3MK\overline{S}_6$	6	0	2	0	0	87.0344500	$30H_1^3H_2H_4/16$	0.068	$F_1^3F_4$	657,555	$3M_2 + K_2 - S_2$
$4M\overline{N}_6$	6	1	0	−1	0	87.4966873	$5H_1^4H_3/16$	0.037	$F_1^4F_3$	665,455	$4M_2 - N_2$
$2M2S\overline{K}_6$	6	2	−4	0	0	87.8860711	$30H_1^2H_2^2H_4/16$	0.032	$F_1^2F_4^*$	671,555	$2M_2 + 2S_2 - K_2$
$3MS\overline{N}_6$	6	3	−2	−1	0	88.5125832	$30H_1^3H_2H_3/16$	0.103	$F_1^3F_3^*$	683,455	$3M_2 + S_2 - N_2$
General coefficient $\mid A_{41} + A_{47}/4 + i(B_{41} + B_{47}/4) \mid$											
$2M2NO_9$	9	−3	0	2	270	128.7907030	$24H_1^2H_3^2H_8/16$	0.014	$F_1^2F_3^2F_8$	925,755	$2M_2 + 2N_2 + O_1$
$3MNO_9$	9	−2	0	1	270	129.3350778	$H_1^3H_3H_8$	0.049	$F_1^3F_3F_8$	935,655	$3M_2 + N_2 + O_1$
$4HO_9$	9	−1	0	0	270	129.8794525	$4H_1^4H_8/16$	0.064	$F_1^4F_8$	945.555	$4M_2 + O_1$
$2M2NK'_9$	9	−1	0	2	90	129.8887360	$24H_1^2H_3^2H_7/16$	0.020	$F_1^4F_{12}$	945,755	$2M_2 + 2N_2 + K_1$
$3MNP_9$	9	0	−2	1	90	130.3509736	$H_1^3H_3H_9$	0.023	$F_1^3F_3F_9$	953,655	$3M_2 + N_2 + P_1$
M_9	9	0	0	1	90	130.4331108	$4H_1^4H_{12}/16$	0.005	$F_1^2F_3F_7$	955,655	$4M_2 + M_1$
$3MNK'_9$	9	0	0	1	90	130.4301108	$H_1^3H_3H_7$	0.069	$F_1^3F_3F_8$	955,655	$3M_2 + N_2 + K_1$
$3MSO_9$	9	1	−2	0	270	130.8953483	$H_1^3H_2H_8$	0.119	$F_1^3F_8$	963,555	$3M_2 + S_2 + O_1$

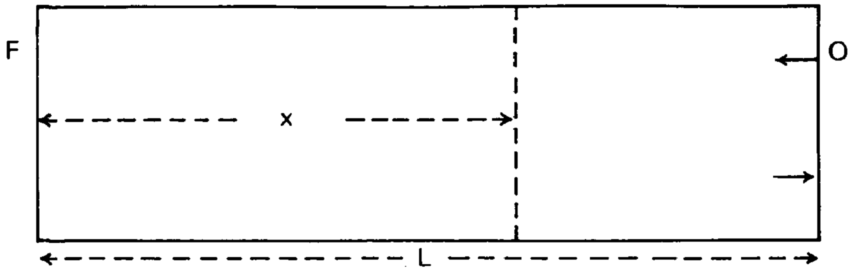

FIG. 1. — Rectangular basin.

If we develop the second summation in exactly the same way as described in section 2, we obtain :

$$\zeta_o(t) = \sum_{j=-Q}^{Q} 2a_j \exp[i\omega_j(t - L/c_o)] \cos(\omega_j L/c_o) + \sum_{n=1}^{5} \left\{ \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} \ldots \sum_{p=-Q}^{Q} Z'_{nj} a_j a_k \ldots a_p \exp[i(\omega_j + \omega_k + \ldots + \omega_p)(t - 2L/c_o)] \right\}$$

The first term in this expression corresponds to a typical standing wave for x = L. The second term will be reflected at O and will generate a new shallow-water progressive wave which will be treated as the main term, to give :

$$\zeta_o(t) = \sum_{j=-Q}^{Q} 2a_j \exp[i\,\omega_j(t - L/c_o)] \cos(\omega_j L/c_o) + \sum_{n=1}^{5} \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} \ldots \sum_{p=-Q}^{Q} 2Z'_{nj} a_j a_k \ldots a_p \exp[i(\omega_j + \omega_k + \ldots \omega_p(t - 3L/c_o)] \cos[\omega_j + \omega_k + + \omega_p) L/c_o] \quad (3a)$$

as the new shallow-water constituents. Such an iterative process should continue until the dynamical equilibrium is reached. However, since there are fewer shallow-water constituents than astronomical constituents, a second iteration may perhaps suffice.

To have a general expression for any section of the basin, it will suffice to change L into x in the cosine argument of equation (3a).

4. TESTING THE THEORY

In order to evaluate the theoretical convergence of the above development, the case of the port of Santana (154 km from the open sea, along the Amazon river) was considered. The general coefficients (table 2-III) are of the form $|X + iY|$. They were thus computed (table 4-I) for $h = 18$ m, $|\omega_j| = 1/2$, $1/4$ and $c_o = 3.6\sqrt{gh}$ km/h.

Table 4-I

General coefficients

c.p.d.	n				
	1	2	3	4	5
1	0.13413				
2	0.13413	0.02734		0.00072	
3	0.13413	0.02126			
4	0.13413		0.00745		0.00093
5		0.02970	0.00925	0.00158	
6		0.02997		0.00176	
7			0.00913		0.00087
8			0.00868		
9				0.00293	
10				0.00285	
11					0.00100
12					0.00093

A least squares analysis in the frequency domain for a 2^{14} hours span (FRANCO, 1978) showed a good convergence, since constituents of order higher than 3[d] were negligible. In order to have an idea of this convergence, we computed M_4/M_6 and M_4/M_8 using the amplitudes from the analysis and those given by the theoretical coefficients :

$$\frac{\frac{1}{2}M_2^2}{\frac{1}{n}M_2^n} \times \frac{C_1}{C_n}$$

where $C_1 = 0.13413$ (as shown in table 4-I for $n = 1$) and C_n are the other coefficients taken from the same table for each species. For $M_2 = 1.149$ m, $M_4 = 0.213$ m, $M_6 = 0.034$ m and $M_8 = 0.011$ m, we find :

M_4/M_6	Theoretical : 5.8	Observed : 6.3
M_4/M_8	Theoretical : 23.4	Observed : 19.4

The above figures show that in this river situation there is fairly good agreement between observed and theoretical convergence. Thus it is seen that at Santana friction is not the main agent in the generation of shallow-water constituents.

Another test was made in order to verify the consistency of the relative values. If we represent the amplitudes of the constituents by their usual symbols, we obtain the following results :

$M_2 = 1{,}149$	$M_4 = (M_2^2/2) = 0{,}213$
$S_2 = 0{,}268$	$MS_4 = (M_2S_2) = 0{,}101$
$N_2 = 0{,}239$	$MN_4 = (M_2N_2) = 0{,}089$

K_2	$= 0{,}076$	$MK_4 = (M_2K_2)$	$= 0{,}028$
$M_6 (M_2^3/4)$	$= 0{,}034$	$M_8 (M_2^4/8)$	$= 0{,}011$
$2MS_6 (3M_2^2S_2/4$	$= 0{,}021$	$3MS_8 (4M_2^3S_2/8)$	$= 0{,}010$
$2MN_6 (3M_2^2N_2/4$	$= 0{,}025$	$3MN_8 (4M_2^3N_2/8)$	$= 0{,}008$
$2MK_6 (3M_2^2N_2/4)$	$= 0{,}004$	$2M2N_8 (6M_2^2S_2^2/8)$	$= 0{,}005$
$MSN (6M_2S_2N_2/4)$	$= 0{,}013$	$2M2S_8 (6M_2^2N_2^2/8)$	$= 0{,}008$

Utilizing the formulae between brackets which are taken from the theory and the values from the analysis, the following relationships are obtained.

	M_4/MS_4	M_4/MN_4	M_4/MK_4	
Theory	2.1	2.4	7.6	
Analysis	2.1	2.4	7.6	
	$M_6/2MS_6$	$M_6/2MN_6$	$M_6/2MK_6$	M_6/MSN_6
Theory	1.4	1.6	5.0	3.4
Analysis	1.5	1.4	8.5	2.6
	$M_8/3MS_8$	$M_8/3MN_8$	$M_8/2M2S_8$	$M_8/2M2N_8$
Theory	1.1	1.2	3.0	3.9
Analysis	1.1	1.3	2.2	1.3

It is seen from these values that there is agreement between theory and analysis for the case of the larger constituents. However, it should be noted that the constituents T_2 and ν_2 have amplitudes of 5.8 cm and 6.9 cm respectively, thus not much less than 7.6 cm which is the amplitude of K_2. Constituents $2MT_6$, MKN_6, $2M2K_8$ and $2M2\nu_8$ should therefore be taken into account in order to improve results. Similarly, the sixth diurnal constituents should be improved by the introduction of shallow-water constituents containing T_2 and ν_2.

It should be noted that the sixth and eighth diurnal constituents are small and have amplitudes less than some of the residuals computed in the analysis.

5. CONCLUSION

The present theory goes a long way to explain the generation of shallow-water constituents, but is far from being complete. Other approaches to include friction are considered necessary. Some research has already been done along these lines (DRONKERS, 1964; GALLAGHER and MUNK, 1971), but the results are only provisional. It seems likely that hydrodynamical numerical models will provide the best approach. However, even in this case the problem of linearizing hydrodynamical equations is a difficult one.

Accordingly it would appear useful to choose shallow-water constituents based on the principles presented in this paper.

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