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NOTE ON AMPHIDROMIC SYSTEMS

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ABSTRACT

Some simple theoretical relations are given for the water motion around amphidromic points in the sea. The relations are based on the local approximation of the sea surface as the superposition of two sloping *plane* surfaces each uniformly rotating around a vertical axis, one in anti-clockwise direction and the other in clockwise direction. The simple relationship between current velocities and elevations is only valid in co-oscillating tides. The theoretical relations were tested for a few actual situations, with satisfactory results.

List of symbols

(Only real quantities have been used)

r, θ	horizontal polar coordinates around a given point in the sea, θ increasing in anti-clockwise direction.
x, y	horizontal orthogonal coordinates. $x = r \cos \theta$, $y = r \sin \theta$.
u, v	components of velocity (depth-averaged) in x and y directions, respectively.
u_m, v_m	defined by eq. (7).
t	time.
ζ	water surface elevation with respect to its time average.
ζ_m	tidal amplitude, positive or zero.
A_0	a constant, positive or zero (dimension : length).
A, B	constants, positive or zero (dimensionless, i.e. slopes).
ω	angular frequency of tidal constituent considered, positive.
f	coriolis parameter (or "inertial frequency") $= 2\,\Omega \sin \Phi$ where Ω is angular velocity of earth $= 7.29 \times 10^{-5} s^{-1}$, Φ is geographic latitude, positive if North, negative if South.

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α = f/ω frequency ratio.
γ = $(A - B)/(A + B)$, $-1 \leqslant \gamma \leqslant 1$.
δ = v_m/u_m, with sign.
Φ_0, Φ_1, Φ_{-1}, ... phase angles.
g constant of gravity.
H water depth.

INTRODUCTION

Points in the sea where the (vertical) amplitude of a certain tidal constituent vanishes are called amphidromes (also : amphidrom*ies*) or amphidromic points. This name was first introduced by R.A. HARRIS (1904) [4]; it is derived from the (ancient) Greek words amphi (around) and dromas (running), and expresses the property that the tidal wave in the neighbourhood of such a point is always found to rotate around that point. In the German literature one also finds the words "Drehtiden" and "Drehwellen".

According to the many tidal analyses made, such amphidromes occur in various places in the open oceans, as well as in marginal seas. A good review, mainly on the M_2-tides in marginal seas, was given in the book of DEFANT (1961)[1]. A recent compilation for the oceans and for 7 tidal constituents, based on extensive numerical calculations supplemented by tidal observations of coastal and island stations, was published by SCHWIDERSKI (1978-1981)[13][14].

As far as known, *all* observed or analysed amphidromes in marginal seas with "co-oscillating tides" in the northern hemisphere have anti-clockwise rotation of the co-tidal lines. It may be expected that in marginal seas in the southern hemisphere this rotation will always be in a clockwise direction but no examples are known to the author. In the open oceans north of the equator 4 out of 6 amphidromes rotate anti-clockwise (2 others are very close to the equator) and south of the equator 4 out of 7 amphidromes rotate clockwise.

A special type of amphidrome occurs in relatively small enclosed (or almost enclosed) seas, like the Black Sea or the Caspian Sea. Such amphidromes can be readily explained from the direct action of the tidal forces : the eigenperiod of these seas is much smaller than the tidal period and the water levels approximate the "equilibrium tide"; see DOODSON and WARBURG (1941) [3], DEFANT (1961) [1].

More generally, the tidal force locally plays a subordinate role in most cases, especially so in marginal seas ("co-oscillating tides"). The occurrence and the properties of amphidromes, therefore, are usually explained by considering one of various models : two crossing standing waves with equal period differing in phase (for instance 90°), see, e.g., THORADE (1931) [15], KRAUSS (1973) [7], DIETRICH *et al.* (1975) [2]; or one standing wave affected by the gyration of the earth, see, e.g., DOODSON and WARBURG (1941) [3]; or two superposed Kelvin waves of equal period travelling in opposite directions, see, e.g., THORADE (1931) [15], DEFANT (1961) [1], KRAUSS (1973) [7], DIETRICH *et al.* (1975) [2]. In nature, however, pure standing waves, pure Kelvin waves, or other theoretical wave modes do not exist.

It is the purpose of this note to demonstrate that the typical properties of the various kinds of amphidromic systems can be described in a more general and unified way, by considering the following simple general approximate expression for the water surface elevation of a tidal consituent in and near *any* point in the sea off the coast, in terms of suitably defined horizontal polar coordinates around that point :

$$\zeta(r,\theta,t) = A_0 \cos(\omega t - \Phi_0) + Ar \cos(\omega t - \theta) + Br \cos(\omega t + \theta) \tag{1}$$

In fact, the right hand side of (1) represents the first few terms of a Taylor-like series in powers of the radius r expressing any non-singular solution of the linearized equation for frictionless long waves in water of constant depth (up to and including the first power of r). This is explained in the appendix. If A_0 is taken as zero, the origin ($r = 0$) is an amphidrome by definition.

The expression (1) has been simplified as much as possible, by choosing the orientation of the line $\theta = 0$ such that the terms with A and B are in phase there, and by choosing the zero point of the time scale such that both these terms have their maxima at $t = 0$. For another choice of the line $\theta = 0$, e.g. along the W-E direction, and for another time scale, two phase angles should be included in the arguments of the cosines in these two terms.

The terms with A and B represent two sloping *plane* surfaces both rotating around a vertical axis through the origin, the first one (with A) anti-clockwise, or positive (by the usual definition), the second one (with B) clockwise, or negative. It will be clear that (1) represents the plane that is tangential in the origin to the actual sea surface.

Note that expression (1), after introduction of local orthogonal coordinates x, y, with x along the line $\theta = 0$ and y along the line $\theta = 90°$, can be written in the form :

$$\zeta(x,y,t) = A_0 \cos(\omega t - \Phi_0) + (A + B)x \cos \omega t + (A - B)y \sin \omega t \tag{1a}$$

This expression is fully equivalent with (1). The last two terms in (1a) are also a local approximation of two crossing standing waves differing 90° in phase.

In the following we consider the case $A_0 = 0$, unless stated otherwise.

CO-TIDAL LINES

High water (or low water) will be given by $\partial\zeta/\partial t = 0$, or in a point close to the amphidrome, from (1) with $A_0 = 0$, by :

$$A \sin(\omega t - \theta) + B \sin(\omega t + \theta) = 0,$$

or from (1a) with $A_0 = 0$ by :

$$(A + B)x \sin \omega t - (A - B)y \cos \omega t = 0.$$

From this, defining

$$\gamma \underset{=}{\text{def}} (A - B)/(A + B), \quad -1 \leqslant \gamma \leqslant 1 \tag{2}$$

we find (unless $\cos\omega t$ or $\cos\theta$ is zero) :

$$\tan \theta_h = \gamma^{-1} \tan \omega t,$$

or

$$\theta_h(t) = \text{arc tan}(\gamma^{-1} \tan \omega t) \tag{3}$$

where the subscript h stands for high water.

The radius r has vanished : the co-tidal lines, in this approximation, are straight lines. If $B < A$, $\gamma > 0$, the tidal wave rotates anti-clockwise and the amphidrome is usually called positive. If $B > A$, $\gamma < 0$, the wave rotates clockwise and the amphidrome is usually called negative. If B and A happen to be equal, $\gamma = 0$, expressions (1) and (1a) with $A_0 = 0$ degenerate into a standing wave with a nodal line along $\theta = 90°$ and $270°$, or the y-axis.

The angular speed with which the wave rotates varies when both A and B are non-zero. From (3) by differentiation :

$$\frac{d\theta_h}{dt} = \omega(\gamma^{-1} \cos^2\theta_h + \gamma \sin^2\theta_h) \tag{4}$$

This expression, which is proportional to the angular spacing between two neighbouring co-tidal lines with a fixed (small) time interval, varies periodically with frequency 2ω, between maxima, $\gamma^{-1}\omega$, for $\sin\theta_h = 0$ and minima, $\gamma\omega$, for $\cos\theta_h = 0$. This means that the co-tidal lines are most crowded near $\theta = 90°$ and $270°$.

CO-RANGE LINES

The tidal amplitude (or half the range), ζ_m, as function of r and θ near an amphidrome, is given by the maximum value of the sum of the last two terms of (1) or of (1a). We find :

$$\begin{aligned} \zeta_m^2 &= r^2\left[(A+B)^2\cos^2\theta + (A-B)^2\sin^2\theta\right] \\ &= (A+B)^2x^2 + (A-B)^2y^2 \end{aligned} \tag{5}$$

If ζ_m is kept constant this is the equation for an ellipse with its major principal axis along $\theta = 90°$, or the y-axis, magnitude $2\zeta_m|A-B|^{-1}$, and its minor principal axis along $\theta = 0$, or the x-axis, magnitude $2\zeta_m(A+B)^{-1}$. The axis ratio, with sign, of these co-range ellipses is thus given by $(A-B)/(A+B) = \gamma$, as defined above.

A comparison of (5) with (4), and (2), shows that the angular speed of the wave is inversely proportional to the square of the radius of a co-range ellipse .

TIDAL VELOCITIES

It is only in connection with the velocities that the coriolis parameter f comes into play. This parameter may be positive (northern hemisphere), negative (southern hemisphere), or zero (equator).

As will be explained in the appendix, the velocity field associated with the simple elevation field according to (1) or (1a) is independent of r and θ, or of x and y, to this order of approximation. It is then most convenient to use the orthogonal coordinates x and y, and the components of the velocity field are given by (see appendix):

$$\left.\begin{aligned} u &= -g\,[A(\omega+f)^{-1} + B(\omega-f)^{-1}]\sin\omega t \\ v &= g\,[A(\omega+f)^{-1} - B(\omega-f)^{-1}]\cos\omega t \end{aligned}\right\}\ \text{provided } \omega \neq f \tag{6}$$

The velocity components u and v are understood to be depth-averaged velocity components; all "baroclinic" effects are neglected throughout.

It should be stressed here that the relationship (6) between velocities and elevations is *only* valid for "co-oscillating tides", i.e., with negligible local tidal forcing.

It may also be noted that (6) holds in the neighbourhood of *any* point in a sea with "co-oscillating tide" as long as the approximations (1) and (1a) apply, i.e., with any vertical tidal amplitude A_0, since the term with A_0 in (1) or (1a) is of zero order in r and therefore is not associated with a current.

The terms with A in (6) represent a velocity vector with constant magnitude rotating anti-clockwise (positive) and the terms with B represent a velocity vector with constant magnitude rotating clockwise (negative). If both A and B are non-zero (positive) the end-point of the total velocity vector describes an ellipse with principal axes along x and y.

If either B or A is zero, eq. (6) shows that the end-point of the velocity vector describes a circle. This vector then rotates uniformly with the same speed as the water surface does, such that the velocity is directed along the zero line of the water surface (given by $y/x = -\cot\omega t$, or $= +\cot\omega t$, respectively – see (1a) with $A_0 = 0$). In a real sea, though, locations where B or A in (1) vanishes will *not* coincide in general with locations where A_0 vanishes. Points in the sea with uniformly rotating velocity vectors of constant magnitude (B or A in (1) being zero) were denoted by SAGER (1963) [11] as "current amphidromes" ("Stromamphidromien"). It will be obvious that in general there is no direct relation between these "current amphidromes" and the (vertical) amphidromes as considered in this note.

Returning to (6) in the general case with both A and B positive, if we define:

$$\left.\begin{aligned} u_m &\ \underline{\underline{\text{def}}}\ g\,[A(\omega+f)^{-1} + B\,(\omega-f)^{-1}] \\ \text{and } v_m &\ \underline{\underline{\text{def}}}\ g\,[A(\omega+f)^{-1} - B\,(\omega-f)^{-1}] \end{aligned}\right\} \tag{7}$$

(u_m and v_m may have either sign), we can abbreviate (6) to:

$$u = -u_m \sin\omega t, \quad v = v_m \cos\omega t \tag{6a}$$

Fig. 1. – Example of co-tidal lines and co-range ellipse (left upper corner) and velocity vectors near an amphidrome.

The "co-range ratio" γ (eq. (2)) was taken +0.8 in all cases, or $B/A = 0.111$.

The frequency ratio $\alpha = f/\omega$ was varied; the "velocity ratio" δ (eq. (9)) follows from eq. (10).

The figures 0, 45, 90, etc., give the phase ωt for high water and for the velocity, respectively.

Eq. (7) for $B/A = 0.111$ gives $u_m = gA\omega^{-1}[(1+\alpha)^{-1} + 0.111\,(1-\alpha)^{-1}]$ and $v_m = gA\omega^{-1}[(1+\alpha)^{-1} - 0.111\,(1-\alpha)^{-1}]$. The lengths of the arrows along the axes in the figure are proportional to the expressions inside the square brackets.

Remember that the local x and y axes were chosen such that high water at time zero occurs along the positive x-axis.

Note that values 1.1 and higher for α are possible for diurnal tides but not so for semi-diurnal tides.

→

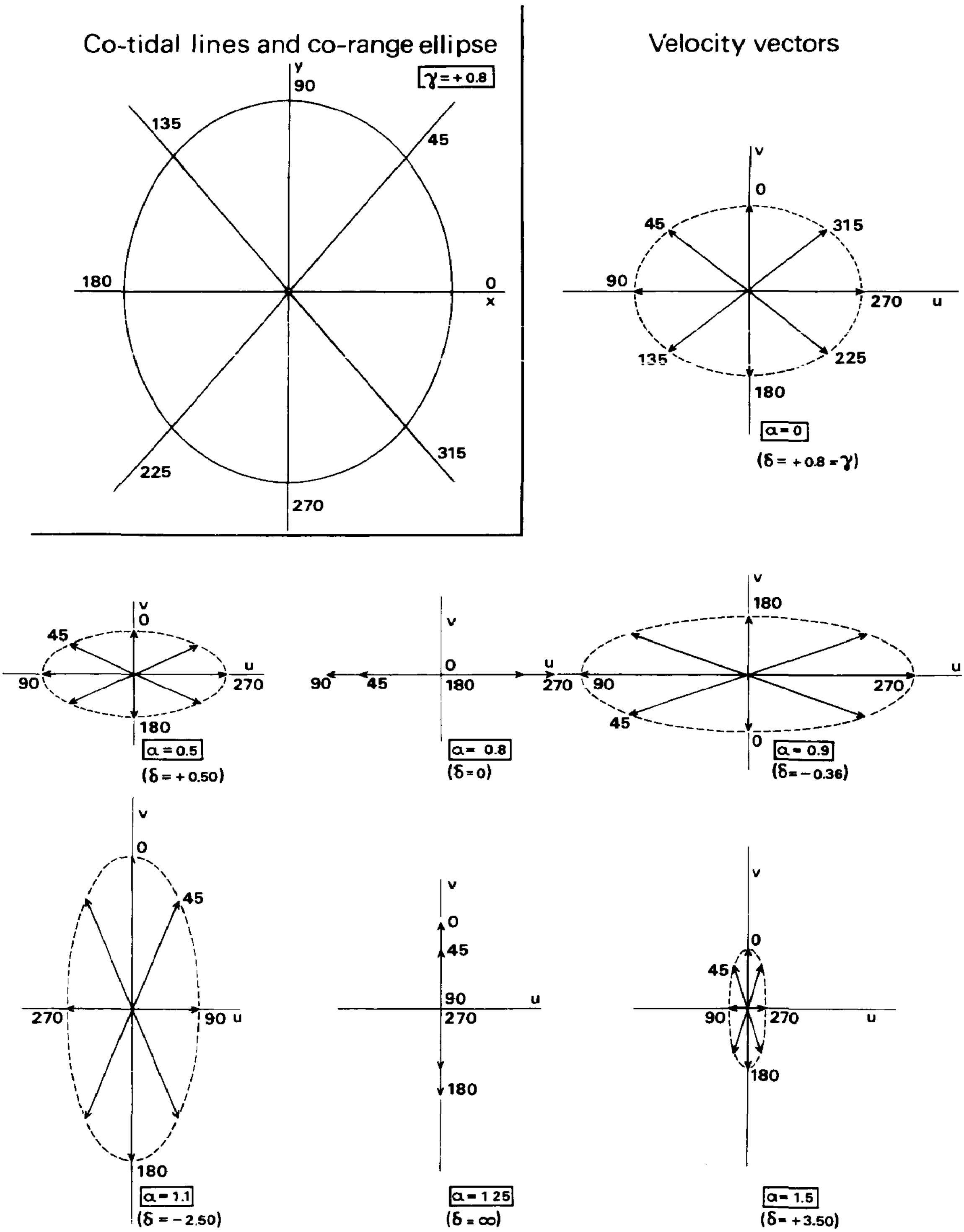
Co-tidal lines and co-range ellipse
γ = +0.8
Velocity vectors
α = 0
(δ = +0.8 = γ)
α = 0.5
(δ = +0.50)
α = 0.8
(δ = 0)
α = 0.9
(δ = −0.36)
α = 1.1
(δ = −2.50)
α = 1.25
(δ = ∞)
α = 1.5
(δ = +3.50)

Fig. 1

As to the sense of rotation of the velocity vector, if we now define ψ as the angle between this vector and the positive y − direction (measured in anti-clockwise direction) we have from (6a):

$$\tan \psi = -u/v = +(u_m/v_m) \tan \omega t \tag{8}$$

which is a formula analogous to formula (3) for the co-tidal lines, showing that we have anti-clockwise (positive) rotation if u_m and v_m have the same sign and clockwise (negative) rotation if u_m and v_m have different signs.

Figure 1 illustrates the large variation in current behaviour near a positive amphidrome if the value of the frequency ratio $f/\omega = \alpha$ is varied, even if the "co-range ratio" γ is kept constant.

It appears that in the northern hemisphere (f positive) amphidromes $(B < A)$ can be associated with both positive and negative senses of rotation of the velocity vector, but negative amphidromes $(B > A)$ can be associated only with negative senses of rotation of the velocity vector.

If $A(\omega + f)^{-1} = B(|\omega - f|)^{-1}$, the velocity vector does not rotate. This is the special case occurring theoretically near the nodal points in two superposed Kelvin waves travelling in opposite directions in a straight channel. The amphidromes in this case can be positive $(B < A)$ only if f is positive and negative $(B > A)$ only if f is negative.
dromes in this case can be positive $(B < A)$ only if f is positive and negative $(B > A)$ only if f is negative.

If we define the axis ratio for the velocity ellipse [see (6) and (7)] by:

$$\delta \underline{\underline{\mathrm{def}}}\ v_m/u_m \tag{9}$$

with sign, not necessarily $|\delta| \leqslant 1$, a theoretical relation between this ratio, the "co-range ratio" γ defined by (2) and the frequency ratio $\alpha = f/\omega$ can be readily found from (7):

$$\delta = (\gamma - \alpha)(1 - \alpha\gamma)^{-1} \tag{10}$$

or alternatively:

$$\gamma = (\alpha + \delta)(1 + \alpha\delta)^{-1} \tag{10a}$$

Equations (10) and (10a) have the property that, in order to have $|\gamma| < 1$, for $|\alpha| < 1$, $|\delta|$ should be < 1 and for $|\alpha| > 1$, $|\delta|$ should be > 1. This means that for $|\alpha| < 1$ the major axis of the co-range ellipses [y, according to (5)] coincides with the minor axis of the velocity ellipse [according to (9)]; for $|\alpha| > 1$ the major axis of both ellipses coincide. This is illustrated by figure 1.

Conclusion so far

According to the simple theory as given above, all relevant parameters of an amphidromic system in a co-oscillating tide, with given tidal frequency ω and cariolis parameter f, can be derived:

- either from the surface slopes A and B, given by the shape of the co-range ellipses together with the sense of rotation;
- or from the velocity amplitudes u_m and v_m, given by the shape of the local current ellipse together with its sense of rotation.

Such amphidromes have only two "degrees of freedom", apart from their orientation.

TABLE

	North Sea, Southern Bight	North Sea, west of Denmark	Arctic	Pacific, west of California
Latitude (degrees)	52.5 - 52.7	55.3 - 55.5	81 - 82	about. 27
Frequency ratio $\alpha = f/\omega$	0.825	0.845	1.025	0.471
Velocity ellipse, axis ratio $\delta = v_m/u_m$	− 0.10 (± 0.04)	− 0.25 (± 0.03)	− 0.2 or + 0.2 − 5 or + 5 (± 1)	0.0 (± 0.05)
Velocity, sense of rotation	same clockwise	same clockwise	undeterm.	same no rotation
Ratio γ according to (10a)	+ 0.79 (± 0.02)	+ 0.77 (± 0.01)	+ 0.98 or + 0.96 (± 0.00)	0.47 (± 0.04)
Ratio $\gamma = (A - B)/(A + B)$ from co-range ellipses	+ 0,7 (± 0.1)	+ 0.7 (± 0.2)?	+ 0.95 (± 0.05)	0.57 (± 0.01)
Orientation of major axis of velocity ellipse (degrees)	010 (± 5)	100 (± 10)	020 (± 5)	145 (± 3)
Orientation of major axis of co-range ellipses (degrees)	040 (± 30)	010 (± 20)	undeterm.	050 (± 5)
Maximum velocity, $\lvert u_m \rvert$ or $\lvert v_m \rvert$ (cm s^{-1})	86 (± 3%)	28 (± 10%)	1.0 (± 20%)	2.4 (± 10%)
From this and δ: "Mean velocity amplitude", $1/2 \cdot \lvert u_m + v_m \rvert$ (cm s^{-1})	39 (± 6%)	10.5 (± 12%)	0.4 or 0.6 (± 24%)	1.2 (± 12%)
Mean slope amplitude A (x10^{-6})	10.8 (± 10%)	2 (± 30%)	0.045 (± 10%)	0.29 (± 5%)
From this : $gA/(\omega + f)$ (cm s^{-1})	41 (± 10%)	8 (± 30%)	0.16 (± 10%)	1.4 (± 5%)
Maximum slope of moon's equilibrium tide (x10^{-6}) (see Appendix)	0.052	0.049	0.013	0.077

Examples of amphidromes

In an attempt to test the theory in really occurring situations I have analysed a few amphidromic systems for which both elevation and current data could be found in published charts. The current velocities were supposed to approximate the depth-averaged velocities.

Three tests were made for each amphidrome :

1) the velocity ratio $\delta = v_m/u_m$ was read, or computed, from current charts and the validity of equation (10a) was tested by comparing the ratio γ from α and δ according to (10a) and the axis ratio γ as derived from the co-range ellipses;
2) the relative orientation of the major axes of velocity and co-range ellipses was checked – this should be 90° if $\omega > |f|$ and 0° if $\omega < |f|$;
3) the relation between surface slopes and velocities was tested by comparing the "*mean* velocity amplitude", $1/2 \cdot |u_m + v_m|$, as found from the current data, and the quantity $gA/(\omega + f)$ derived from the *mean* slope amplitude A [see (7) and (6a)].

The results have been compiled in the table below. All examples refer to positive amphidromes (thus $A > B$, γ positive) in the northern hemisphere (f positive) of the M_2 constituent (angular frequency $\omega = 1.4052 \times 10^{-4} s^{-1}$). Orientations are given in degrees clockwise from north. Estimated standard errors are given in parentheses.

Explanations and comments on the table :

North Sea

Sources : U.K. Ministry of Agriculture, Fisheries and Food (1981)[16], SAGER and SAMMLER (1968)[12]. Origin of data : partly direct observations, partly numerical calculations, perhaps partly some intuition of the chart maker. Most of the data refers to mean spring tide, which means that at least the S_2 constituent is included, for which the frequency ratio α is about 3 percent lower than that for M_2. This small complication is ignored here.

The results of the tests are satisfactory to good, except for the relative orientation of the ellipses for the Southern Bight amphidrome. The value 010° for the velocity ellipse should be nearly correct. Now, the co-*tidal* lines in the charts appear to be most crowded along the directions east and west, indicating that the major axis of the co-range ellipses has about this orientation. This would be in accordance with the orientation 010° for the velocity ellipse.

For the second amphidrome, the co-range lines in the charts hardly allow a reasonable estimation of A, B or γ, due to the low tidal surface slopes in the area. Apparently the test for this case is less conclusive but there are, at least, no inconsistencies.

Arctic

Sources : KOWALIK and UNTERSTEINER (1978)[5], KOWALIK (1979)[6]. Origin of data : numerical calculation for the whole Arctic, with mesh size 75 km.

The frequency ratio in this case is close to unity, which means that the "co-range ratio" γ, close to unity too, is very insensitive to large variations in the velocity ratio δ [as can be directly seen from (10a)]. There is some disagreement for the velocity amplitudes but the reading from the chart was very inaccurate.

The direct tidal forcing was neglected in the calculations in [5][6], but the table shows that the tide-generating force near the amphidrome is about 30 percent of the surface slope force. This means that in reality the Arctic M_2-tide cannot be considered as merely co-oscillating.

Pacific

Sources : MUNK, SNODGRASS and WIMBUSH (1970)[10] and various later reviews on ocean tides. Origin of data : analytical calculation for a simplified topography, with tidal forcing.

The results of two of the tests are good but the agreement between both γ-values is only moderate. Maybe this is due to the relative magnitude of the direct tidal force, about 25 percent of the surface slope force.

The general conclusions of the analysis of the examples presented in the table are 1) that the linear approximation (1), (1a) of the water elevation around an amphidrome is a fair one over an area including, at least, the nearest drawn co-range contour, or contours, around the amphidrome; and 2) that the theory based on this approximation is well confirmed, at least for co-oscillating tides.

FINAL REMARKS

The explanation of the pronounced preference for anti-clockwise rotation of the co-tidal lines in amphidromes in marginal seas for the northern hemisphere is obviously the fact that the tidal energy enters these seas mainly as a wave more or less *resembling* a Kelvin wave, the entering wave being modified and distorted within the marginal sea by diffraction, reflection and dissipation. But also in the open oceans, where there is tidal forcing (which, for semi-diurnal tides, by itself would lead to clockwise rotating amphidromes in the northern hemisphere), there is a preference for anti-clockwise amphidromes in the northern hemisphere, probably caused by the fact that also here the propagation of tidal energy mainly occurs in a Kelvin-wave-like fashion.

APPENDIX

The simplified equations for co-oscillating tidal motion, or for free long wave motion in general, in orthogonal coordinates x and y (if x is east, y is north) are the equations of motion in x- and y-direction and the mass balance equation. They can be written as follows (see list of symbols at the beginning of this note).

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \zeta}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \zeta}{\partial y} = 0 \qquad \text{(A1)}$$

$$\frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) + \frac{\partial \zeta}{\partial t} = 0$$

Assumptions and simplifications made are : constant water density, curvature of earth neglected, constant coriolis parameter f and gravity constant g, all quadratic terms neglected (implying, inter alia, $|\zeta| << H$), friction neglected, depth H varying slowly with position.

If we eliminate the velocity components u and v from (A1) we can obtain a linear differential equation for the elevation ζ alone, which reads, with H taken constant :

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} - \frac{1}{gH}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\zeta = 0 \qquad \text{(A2)}$$

or after introduction of polar coordinates r, θ:

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} - \frac{1}{gH}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\zeta = 0 \qquad \text{(A2a)}$$

This differential equation has solutions, bounded for $r = 0$, of the form, see, e.g., LAMB (1945) [8] :

$$\left.\begin{aligned} &\zeta = a_m J_m(Kr).\cos(m\theta - \omega t - \Phi_m) \text{ if } \omega^2 > f^2, \\ &\qquad K(\text{pos}) \underline{\underline{\text{def}}} (\omega^2 - f^2)^{\frac{1}{2}} (gH)^{-\frac{1}{2}} \\ &\zeta = a_m I_m(K'r).\cos(m\theta - \omega t - \Phi_m) \text{ if } \omega^2 < f^2, \\ &\qquad K'(\text{pos}) \underline{\underline{\text{def}}} (f^2 - \omega^2)^{\frac{1}{2}} (gH)^{-\frac{1}{2}} \end{aligned}\right\} \qquad \text{(A3)}$$

where m is a positive or negative integer or zero, ω is an angular frequency, to be chosen freely provided it differs from f, the coefficients a_m and the phase angles Φ_m are constants, J_m is the Bessel function of the first kind and order m, I_m is the modified Bessel function of the first kind and order m.

We avoid the use of complex notations, so that all quantities are real.

The function $J_m(\xi)$ can be expanded in a power series in ξ (see e.g. McLACHLAN, 1934 [9]; for positive m :

$$J_m(\xi) = (m'.2^m)^{-1}/\zeta^m\left(1 - \frac{\xi^2}{4(m+1)} + \frac{\xi^4}{32(m+1)(m+2)} - \dots\dots\right) \qquad \text{(A4)}$$

and for negative m :

$$J_m(\xi) = (-1)^m J_{|m|}(\xi) \tag{A4a}$$

The modified Bessel functions $I_m(\xi)$ similarly, for positive m :

$$I_m(\xi) = (m!\,2^m)^{-1}\,\xi^m\left(1 + \frac{\xi^2}{4(m+1)} + \frac{\xi^4}{32(m+1)(m+2)} + \ldots\ldots\right) \tag{A5}$$

and for negative m :

$$I_m(\xi) = I_{|m|}(\xi) \tag{A5a}$$

Substituting m = 0, 1 and − 1, respectively, in (A4 ... A5a) we get :

$$J_0(\xi) = 1 - \frac{1}{4}\xi^2 + \frac{1}{32}\xi^4 - \ldots$$

$$I_0(\xi) = 1 + \frac{1}{4}\xi^2 + \frac{1}{32}\xi^4 + \ldots$$

$$J_1(\xi) = -J_{-1}(\xi) = \frac{1}{2}\xi - \frac{1}{8}\xi^3 + \ldots$$

$$I_1(\xi) = I_{-1}(\xi) = \frac{1}{2}\xi + \frac{1}{8}\xi^3 + \ldots$$

The fact that the leading terms in the development of $J_m(\xi)$ and $I_m(\xi)$ are those with ξ^m suggest that *any* solution of the wave equation (A2a), non-singular near a given point, can be expanded, after introduction of local polar coordinates, in a series with terms as given by (A3), for $\omega^2 > f^2$:

$$\left.\begin{aligned}\zeta = {} & C_0J_0(Kr)\,.\cos(\omega t - \Phi_0) + \\ & + J_1(Kr)\{C_1\cos(\omega t - \theta - \Phi_1) + C_{-1}\cos(\omega t + \theta - \Phi_{-1})\} + \\ & + J_2(Kr)\{C_2\cos(\omega t - 2\theta - \Phi_2) + C_{-2}\cos(\omega t + 2\theta - \Phi_{-2})\} \\ & + \ldots\end{aligned}\right\} \tag{A6}$$

or a similar series if $\omega^2 < f^2$, where the C's are non-negative constants.

A proof of the existence of a series expansion similar to (A6) for any non-singular solution of the *three*dimensional wave equation with prescribed frequency ω was given in 1903 by WHITTAKER, see e.g. WATSON (1952) [17] or WHITTAKER and WATSON (1935) [18]). (In this threedimensional case, spherical harmonics appear in the series, instead of trigonometrical functions). That proof starts from the general solution of the wave equation expressed as a superposition of an infinite number of plane waves propagating in all directions, and it can readily be extended to *any* number of dimensions.

A rather simpler proof for the general non-singular solution of the wave equation in two dimensions with prescribed frequency is possible if we start from an old theorem of C. NEUMANN (1867), see again [17], [18], stating that an arbitrary (complex) function f(z), analytic in a domain of the complex z-plane including the origin, can be expanded in a series of the form :

$$f(z) = \alpha_0 J_0(z) + \alpha_1 J_1(z) + \alpha_2 J_2(z) + \ldots$$

where α_0, α_1, α_2... are independent of z.

The series (A6) can be written as a power series in Kr (or K'r), with coefficients that are dependent on θ. Taking only the terms of zero and first power in Kr (or K'r), we are left with an expression like (1). The coefficients A and B in (1) then are

$$\frac{1}{2}KC_1 \text{ and } \frac{1}{2}KC_{-1}.$$

As explained in the main text, the phase angles Φ_1 and Φ_{-1} have been eliminated in (1) by an adequate choice of the line $\theta = 0$ and of the zero point of the time scale.

Note that the simple expressions (1) and (1a) do *not* exactly satisfy the differential equations (A2) or (A2a). Expression (1a) might be considered as a local approximation of any non-singular solution, varying sinusoidally with t, of many other linear second order partial differential equations.

The x and y components of the current velocity u and v can be found from the first two equations of (A1) by eliminating v or u from them. This gives :

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = -g\left(\frac{\partial^2 \zeta}{\partial x \partial t} + f\frac{\partial \zeta}{\partial y}\right)$$
$$\frac{\partial^2 v}{\partial t^2} + f^2 v = g\left(f\frac{\partial \zeta}{\partial x} - \frac{\partial^2 \zeta}{\partial y \partial t}\right) \tag{A7}$$

Since it may be presumed that u and v, like ζ, vary sinusoidally with ωt we can replace $\frac{\partial^2}{\partial t^2}$ by $-\omega^2$. Then using (1a) we obtain equation (6).

It may be interesting to comment on the typical difference, according to (6), between the velocities associated with the A-term and with the B-term in (1). For $B = 0$ and f positive the current is always directed such that the water surface is higher to the right ("quasi-geostrophic" motion); then the acceleration of the water particles (directed to the left) is given by the *difference* between the surface slope force $(-g\nabla\zeta)$ and the coriolis force. For $A = 0$ however, with $\omega > f > 0$, the current is always directed such that the water surface is higher to the left; then the acceleration (directed to the right in this case) is given by the *sum* of the surface slope force and the coriolis force. Since, with ω given, the acceleration is proportional to the velocity, a given surface slope amplitude (A or B) in the second case is associated with a larger velocity than it is in the first case. This may have to do with the observed preference for positive amphidromes in the northern hemisphere.

For completeness a small remark should be made on the case in which the tidal frequence ω and the local inertial frequency f (or $-f$) coincide. Equation (6) indicates that in this case amphidromes are possible but only those with $B = 0$ in the northern hemisphere and only those with $A = 0$ in the southern hemisphere.

Up to here the local *tide-generating force* was not taken into account. If this is done, ζ in the first two equations of (A1) (but *not* in the third equation of (A1)) should be replaced by $\zeta - \overline{\zeta}$ where $\overline{\zeta}$(x,y,t) is the (known) equilibrium tide. Equation (A2) and the solutions (A3) and (A6) no longer apply. Since the gradient of $\overline{\zeta}$ varies only slowly with x and y, $\overline{\zeta}$ may be well approximated by a linear function oscillating in time, like ζ, but with two phase angles included and with co-range ellipses having their principal axes along the directions W-E and S-N. Thus linear approximations for ζ, $\overline{\zeta}$ as well as for $\zeta - \overline{\zeta}$ would be appropriate and the simple relations between co-tidal lines and co-range lines near amphidromes remain valid. But clearly the magnitude and orientation of the principal axes of the co-range ellipses of $\zeta - \overline{\zeta}$ would differ from those of ζ. Since the velocities are directly related to $\zeta - \overline{\zeta}$ (by (A7) with ζ replaced by $\zeta - \overline{\zeta}$), equation (6) is no more valid, nor are the relations (10), (10a): the relationship between velocities and surface slopes becomes much more complicated. The relations (6), (10), (10a) can be expected to hold only provided the gradients of the equilibrium tide are small as compared to the surface gradients, i.e., for co-oscillating tides.

The maximum slope of the moon's semi-diurnal equilibrium tide in the table was computed from: gradient of $\overline{\zeta}$ (in W-E direction) = 55 cm $(6\ 370\ \text{km})^{-1} \cos \Phi = 0.086 \times 10^{-6} \cos \Phi$, where Φ is the latitude.

The question remains *how* good are the approximations (1) and (6) in an actual situation with co-oscillating tides. There are different causes of deviations: 1) the neglect in (1) of the terms with second and higher powers of r occurring in the series (A6); 2) local variations in the depth H; 3) the neglect of various terms in the simplified equations (A1).

The order of magnitude of the error made by neglecting in (1) the terms with r^2 can be estimated. From (A6), with $C_0 = 0$, neglecting C_{-1} and C_{-2}, we arrive at a *relative* error in $\zeta(r,\theta,t)$ of order $\frac{1}{4} KrC_2C_1^{-1}$, which is $\frac{1}{4}$ Kr if C_2 and C_1 are of the same order. K is kind of wave number, defined in (A3). Roughly, if $(\omega^2 - f^2)^{\frac{1}{2}}$ is of order $10^{-4}s^{-1}$, with $g \approx 10$ m s^{-2}, the relative error becomes of order $10^{-5}\ rH^{-\frac{1}{2}}m^{\frac{1}{2}}$. For H = 100 m and r = 100 km this is 0.1, which is still rather small. The errors caused by depth variations are probably more important.

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