

## **THE TIDE IN RIVERS**

by Gabriel GODIN(\*)

---

### **Abstract**

Some features of the tide in rivers are described and they are explained in simple terms.

### **INTRODUCTION**

The tide in a river consists of waves which propagate into it from the ocean and which are distorted by friction and by the discharge of fresh water. Observations show that, as it progresses upstream, the time interval between low water (LW) and high water (HW) shortens so that the duration of ebb increases steadily. In extreme cases the level rises abruptly in some portions of the river and a shock wave develops, the bore. The tide in a river has other characteristics which are not as well known :

- 1) The semidiurnal component of the tide progresses upstream more rapidly than the diurnal component and it is more affected by friction
- 2) HW progresses more rapidly than LW
- 3) LW progresses faster in the estuary, while it is slowed down in the upstream region, during neap tides. The reverse holds during spring tides
- 4) An increased discharge of fresh water diminishes the range of the tide, it increases the velocity of LW in the upstream portion of the river while it may enhance the range in some parts of the estuary
- 5) Slow semimonthly and monthly oscillations in the level are induced in the upstream region by the succession of neap and spring tides downstream
- 6) The tide in the upstream region cannot be reproduced by a one dimensional model.

(\*) Visiting Professor : Centro de Investigacion Cientifica y de Educacion Superior de Ensenada (CICESE), Ensenada B.C. Mexico (12 Oct. 83 - 11 May 84).

Home address : 2936 Arles Mews, Mississauga, Ont., Canada L5N 2N2.

## THE EQUATIONS OF HYDRODYNAMICS

Most of these facts can be explained from elementary hydrodynamical considerations. We show in Fig. 1 the profile of a river : its bottom has a local slope  $I$  while its surface, which may be quite steep in the upstream region, becomes level in the estuary. When the river level is measured from a geodetic reference level  $Z_0$ , its depth  $D$  may be expressed as  $D = H + h$ , where  $H$  is the depth of the bottom from  $Z_0$  and  $h$ , the elevation of the surface from the same reference level.  $h$  may be expressed as  $h = h_0 + h_1$ ;  $h_1$  is the excess displacement of the surface from a plane whose slope is determined by the discharge and by the bottom rugosity (see eqn. (3)). The equations of hydrodynamics which apply to a one dimensional canal of varying width  $B$  and depth  $D$  are (DRONKERS, 1964) :

$$g \frac{\partial D}{\partial x} + \frac{\partial u}{\partial t} = - \frac{u|u|g}{C^2 D} - u \frac{\partial u}{\partial x} \quad (1)$$

$$\frac{\partial(BD)}{\partial t} = - \frac{\partial}{\partial x} (BDu) \quad (2)$$

$\partial u / \partial t + u \partial u / \partial x$  represents the total acceleration  $du/dt$ ; the term  $u \partial u / \partial x$  is called the convective term and is important at points where there occur large changes of velocity over short distances.  $g \partial D / \partial x$  is the hydrostatic pressure gradient necessary to balance  $u \partial u / \partial x$  and the friction term  $gu|u|/C^2 D$ . The role of friction in rivers was discovered by CHÉZY in 1739 and the coefficient  $C$  is known as the Chézy coefficient. In the upstream portion of a river where the bed has regular depth and width, we have a balance between the slope of the surface and the stress of friction :

$$\frac{\partial D}{\partial x} = \frac{u_0^2}{C^2 D} \quad (3)$$

where we have written  $u = - u_0$ , the current due to the discharge, which is directed downstream. It is negative as we take the positive  $x$  direction upstream, in the

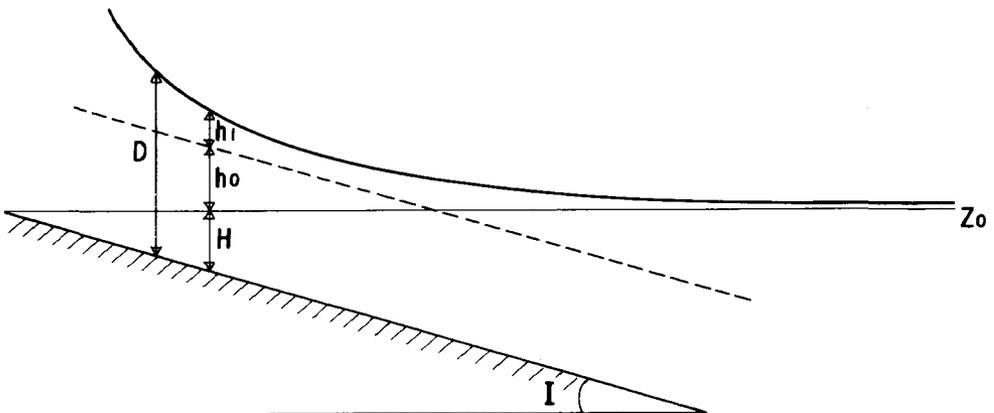


FIG. 1. — Position of the surface of a river with respect to various reference levels.

direction of the flood current. (3) indicates that the surface of the river slopes upstream; it may also be used to deduce the current  $u_0$  from the surface gradient detected by tide gauges :

$$u_0 = C \sqrt{D \partial D / \partial x} \quad (4)$$

C being the constant determining the current from the surface slope.

### Some special forms of the equations

Returning to (1) and (2) which apply to the portion of the river where tides are felt, it may seem impossible to extract simple solutions from them; this being due to the presence of the two terms in the right hand side of (1) which are nonlinear and which prevent the existence of separate solutions in the  $x$  and  $t$  variables. We go around this difficulty by looking at what happens in the estuary where the current  $u_0$  created by the discharge is very weak and in the upstream region where the same current  $u_0$  prevails over those created by the tidal motion. In both places we write the current  $u$  as  $u = -u_0 + u_1(t)$  where  $u_1$  is due to the tide. In the estuary  $u_1$  predominates so that  $u_1 \gg u_0$ . The  $u|u|$  term which determines the effect of friction becomes  $(-u_0 + u_1) |-u_0 + u_1|$  during flood and  $(-u_0 - u_1) |u_0 + u_1|$  during ebb; this can be approximated by  $u_1^2 - 2u_0u_1$  during flood and  $-(u_1^2 + 2u_0u_1)$  during ebb, the discharge current being in the same direction as the tidal current during ebb. The tide in a river is roughly modelled by a travelling wave : low water occurs near ebb and high water near flood. The effect of friction is increased during ebb and reduced during flood. LW should be more strongly affected by friction than HW. If the river discharge is increased, let us say from  $u_0$  to  $2u_0$ , the  $u|u|$  term will become  $u_1^2 - 4u_0u_1$  during flood and  $-(u_1^2 + 4u_0u_1)$  during ebb. The effect of friction on HW should be even more diminished when the fresh water discharge is increased.

Moving now to the upstream region, we have  $u_0 > u_1$ . We write  $D = H + h_0 + h_1(t)$  where  $h_0$  is the displacement of the surface maintained by the discharge and  $h_1$ , the oscillations in level due to the tide. In that portion of the river, if tides were absent, the position of equilibrium would be given by (4). Assuming that the tide creates slight oscillations about this position of equilibrium, neglecting terms in  $u_1^2$  and the convective term but retaining first order terms in the expansion of  $1/D$ , we obtain a set of equations for  $h_1$  and  $u_1$  (GODIN, 1983) :

$$\left( \frac{\partial}{\partial t} + \frac{2Ku_0}{H} \right) u_1 - u_0 \frac{\partial u_1}{\partial x} + g \frac{\partial h_1}{\partial x} + \frac{Ku_0^2}{H^2} h_1 = 0 \quad (5)$$

$$- \frac{\partial h_1}{\partial t} + u_0 \frac{\partial h_1}{\partial x} = H \frac{\partial u_1}{\partial x} \quad (6)$$

$$K = g/C^2 \text{ (dimensionless)}$$

### The upstream solution

(5) and (6) are linear in  $(h_1, u_1)$ ; we can search therefore for an oscillating solution  $e^{-ist}$  which represents a single component of the tide of frequency  $s$ . This

solution will depend on  $u_0$  and on the friction coefficient  $K = g/C^2$ . The  $x$  dependent solution of (5) and (6) for  $h_1$ , the surface displacement due to the tide, has the form :

$$h_1 \sim \exp \{ - [p + r(1 + a)] x - i[st - r(1 - a)x] \} \quad (7)$$

where  $a = 9Ku_0^3/16g'sH^2$ ,  $p = 3Ku_0^2/2g'H^2$ ,  $g' = g - u_0^2/H$ ,  $r = (2/H) \sqrt{sKu_0/g'}$ . We should not be blinded by the algebra involved. (7) has the form of a damped travelling wave :

$$h_1 \sim e^{-kx} e^{-i(st - mx)} \quad (8)$$

(8) represents the shape of the tidal wave as it travels in the upstream portion of the river. It diminishes steadily in amplitude because of the factor  $e^{-kx}$  and it takes a given time  $m \Delta x/s$  to reach a certain point. The damping is determined by  $[p + r(1 + a)]x$  while the delay depends on  $r(1 - a)x$ ;  $p$ ,  $r$  and  $a$  are created by friction through  $K$ . They also depend on the fresh water discharge in the following way :

$$r \sim \sqrt{u_0}/H \quad p \sim (u_0/H)^2 \quad a \sim u_0^3/H^2$$

$a$  is most sensitive to changes in the discharge and its existence can be traced to the change in depth between HW and LW. When the discharge increases,  $a$  increases more rapidly than  $r$ : there will be an increased damping of the tide but the delay will not increase as significantly because of  $(1 - a)$  in the phase term. There might even be an acceleration in the progress of the wave because of an increased discharge. We learn the following from (7) :

- 1) The tide is damped by the discharge in the upstream region of a river, a fact which is well known. But we see that the damping occurs through the interaction created by the friction term between the discharge current and the tidal current.
- 2) Its velocity of progress should not be as sensitive to changes in the discharge as its amplitude. We note that (7) represents a component such as the diurnal or the semidiurnal tide and does not apply to the crest of HW or LW.

Thinking of the latter, we know that at LW the depth is less than at HW; the phase of LW should be more reduced by an increased discharge so that its velocity of progress should increase upstream for a larger discharge.

The velocity of a tidal component, as given by (7) is :

$$c = \frac{s}{r(1 - a)} = \frac{H}{2(1 - a)} \sqrt{\frac{sg'}{Ku_0}} \quad (9)$$

The velocity of progress of a tidal component is proportional to  $\sqrt{s}$  for a fixed value of the discharge; since the diurnal frequency is half the semidiurnal one, we conclude that the diurnal tide should progress more slowly than the semidiurnal one in the upstream region of a river.  $r$  is also proportional to  $\sqrt{s}$  so that the damping of the diurnal tide should also be less effective.

Considering that a succession of spring and neap tides for constant discharge is equivalent to a decrease and an increase in  $u_0$  with respect to a constant tide signal, we surmise that LW should progress more rapidly in the estuary and less rapidly in the upstream region during neap tides.

Another feature peculiar to river tides is the long period tides, semimonthly and monthly, which affect the level upstream; these tides have the same frequency as some low frequency components of the tidal potential but they are due to the non linear interaction between the diurnal and semidiurnal components and not to the direct action of the tidal forces. The non linear terms in (1) are the friction term  $Ku|u|/D$  and the convective term  $u\partial u/\partial x$ . Although the friction is "quadratic" in the sense that it is proportional to  $u^2$ , the frictional term is an odd function and it may be closely approximated by an expression of the form :

$$Ku|u|/d \sim au + bu^3 \quad (10)$$

where  $a$ ,  $b$  are constants determined by the range of velocities  $u$  occurring in the portion of the river under scrutiny. The term  $au$ , once inserted into (1), corresponds to a damping of the tidal frequencies injected by the ocean into the river while the second term creates additional frequencies. Writing again  $u = -u_0 + u_1(t)$ , the cubing of  $u$  creates a term proportional to  $u_0u_1^2$ . Assuming that the tide is made up of two components,

$$u_1(t) = u_1 \cos s_1 t + u_2 \cos s_2 t ,$$

the  $u_0u_1^2$  term will create two new frequencies  $s_1 + s_2$  and  $s_1 - s_2$ . If the two original frequencies fall in the same band, let us say they are  $M_2$  and  $S_2$ , one of the new frequencies created will be  $M_2 - S_2 = MSf$ .  $MSf$  is a semimonthly tide present in the development of the tidal potential but the oscillation in level felt in a river is due to the frictional interaction of  $M_2$  and  $S_2$ . We could have considered the interaction of  $M_2$  and  $N_2$ , or  $K_1$  and  $O_1$ ; in this case, the low frequency created would be  $Mm$ . The convective term will create identical frequencies through the product  $u_1(t) \partial u_1(t)/\partial x$  which are due to rapid changes in the velocity of the tidal currents and which are independent of the state of the discharge. Both friction and convection are therefore responsible for creating slow oscillations in the level of a river through the interaction of the semidiurnal and diurnal components of the tide.

### Illustrations

We illustrate these theoretical findings by examples taken from nature. We show in Figs. 2 and 3 portions of the Fraser and Saint Lawrence rivers, in Canada, along with the position of some of the tide gauges operated in them. The state of the fresh water discharge is monitored constantly in these rivers and we have at our disposal abundant observational material. The Saint Lawrence has a well delineated estuary which terminates at Quebec. The Fraser river is a mountain torrent whose discharge becomes very large over a short time interval during the spring. It has a delta but it cannot be said to have an estuary as it debouches into Georgia Strait; the tide at Steveston which is located at its mouth is markedly affected by fluctuations in its discharge.

We check first on the existence of a difference in the speed of propagation of the diurnal and semidiurnal components of the tide into a river. To do this we calculate the cross spectrum between the tide predicted at a reference station at the mouth of the river, in our case, Steveston in the Fraser and Pointe au Père in the

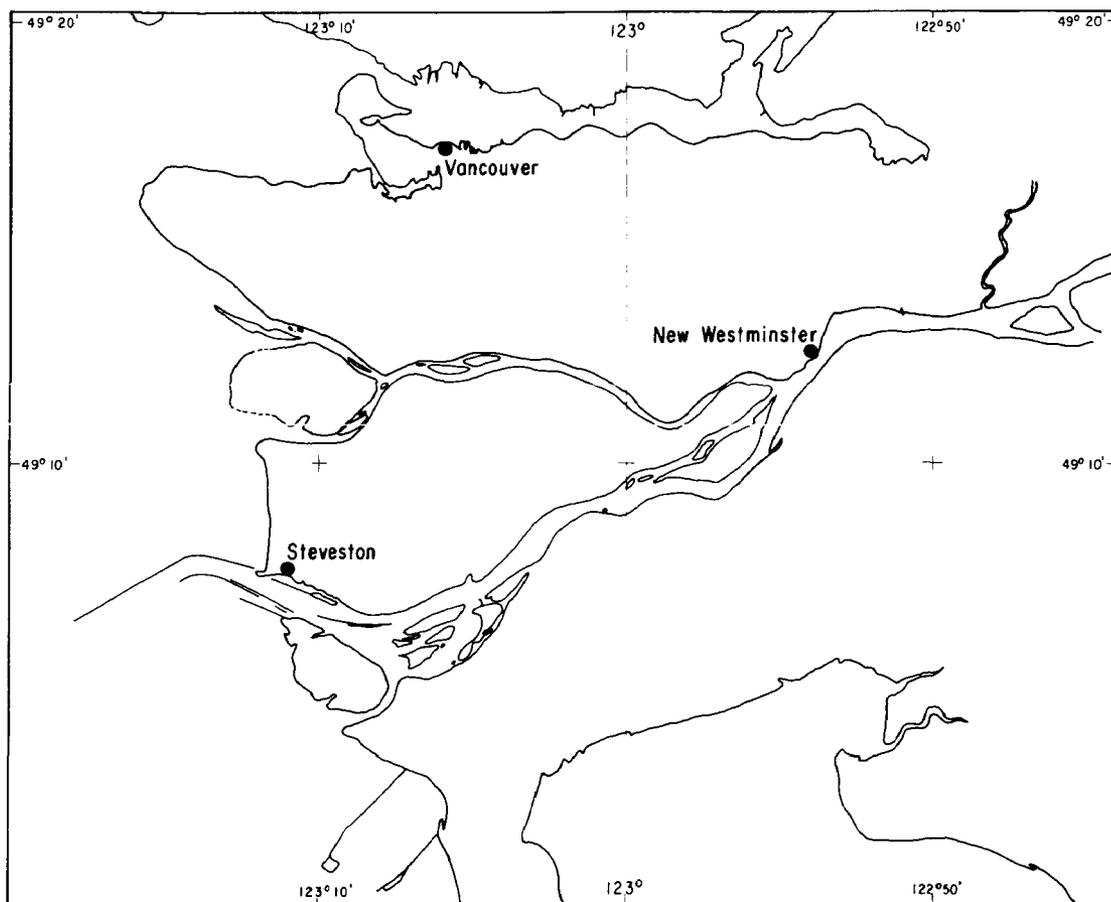


FIG. 2. — Lower portion of the Fraser river showing the position of some water level gauges.

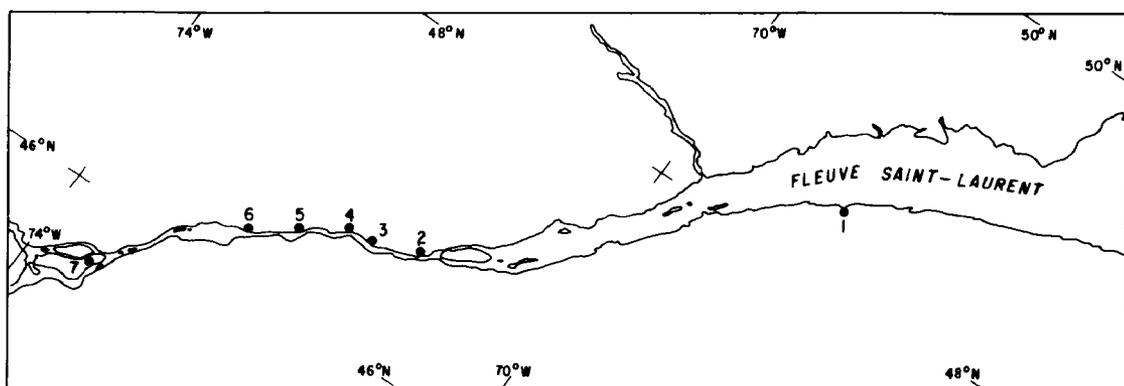


FIG. 3. — Saint Lawrence river. The numbers indicate the position of some water level gauges whose recordings have been utilized : 1. Pointe au Père; 2. Québec; 3. Portneuf; 4. Grondines; 5. Batiscan; 6. Trois-Rivières; 7. Montréal.

Saint Lawrence, and the hourly values of the water level observed at the upstream stations. This is done for various values of the discharge covering the years of observations, namely 1970 to 1979, and the average phase difference is calculated in the diurnal and semidiurnal bands. The results are shown in Table 1, the phases having been expressed in hours : the delay in the progress of the diurnal signal becomes quite obvious over large distances as well as its lesser damping.

**Table 1**

Mean time (hours) taken for the diurnal and semidiurnal components of the tide to reach some stations up river as well as their relative amplitude

STATION .....	FRASER RIVER		SAINT LAWRENCE RIVER							
	New Westminster		Québec		Portneuf		Batiscan		Trois-Rivières	
Distance* .....	25 km		291 km		348 km		390 km		410 km	
Band .....	T	R	T	R	T	R	T	R	T	R
Diurnal.....	1.17	0.563	5.37	1	7.74	0.790	10.54	0.272	12.64	0.083
Semidiurnal ....	1.05	0.552	4.33	1	6.30	0.773	8.68	0.145	10.02	0.032

T : Time necessary for the component to progress from Steveston or Pointe au Père.

R : Quotient of the amplitude at the upstream station to the one at Steveston or Québec.

\* This is the distance between Steveston or Pointe au Père and the upstream station.

The next check is on the velocity of progress of HW and LW. In order to do this, we pick the observed time of HW and LW from recordings (Water Level Books) for the years 1970, 1974, 1975 and 1976, selecting intervals during the summer months when the discharge was effectively constant; this is to eliminate the effect of fluctuations in discharge which affect the progress of HW and LW and to diminish the effect of storms and frontal passages which are less frequent during the summer months. We show in Table 2 the average time interval, in hours, necessary for HW and LW to reach points inside the river.

**Table 2**

Time (hours) taken for HW and LW to reach some points up river

STATION .....	FRASER RIVER		SAINT LAWRENCE RIVER		
	New Westminster (From Steveston)		Québec	Batiscan (From Pointe au Père)	Trois-Rivières
HW .....	0.86		4.40	7.92	8.80
LW .....	1.72		5.56	10.07	12.16

The next example is to illustrate the effect of changes in the discharge of the tide progressing upstream. We show in Fig. 4 the amplitude and the phase of the admittance, in the semidiurnal band, between the predicted tide at Vancouver and the hourly values of the water level observed at Steveston during intervals of near constant discharge. These intervals were chosen between 1970 and 1979 so as to

cover the whole range of discharge values. The figure takes the shape of a scatter diagram, each point giving the relative amplitude and phase of semidiurnal signal at Steveston as gauged by the reference signal (Vancouver). The discharge not being the only factor affecting the tide in rivers, we encounter an appreciable scatter of the sample points. We note that for low discharge values, both the amplitude and phase responses cluster in the horizontal, suggesting that the discharge is not very important when it is small. When it increases, the amplitude falls off sharply while the phase shows some sign of increasing. These facts reflect the features noted in solution (7). We continue our investigation more systematically by calculating the coefficient of correlation with the discharge value, for the amplitude and phase of the diurnal and semidiurnal signals for two stations in the Fraser river and four stations in the Saint Lawrence. The predicted tide used to compute the cross spectrum serves only as a ruler against which we gauge the response at the upstream station for a given value of the discharge. The results are shown in Table 3.

The correlation of the tidal amplitudes in the Fraser River with the discharge values is high and negative; the tide, as it enters the river, is increasingly damped by a larger discharge. We encounter no sharp correlations in the Saint Lawrence amplitudes where the range in discharge values is much narrower, the flow of water being controlled from the Great Lakes. Yet the tide is obviously damped by an increased discharge beyond Portneuf. The correlation of the phase lags with the discharge is low in both rivers; this agrees with the features of solution (7) and with the fact that the speeds of propagation of HW and LW react differently to changes in the discharge.

The next item we wish to check on is the influence of the discharge and of neap-spring cycles on the speed of propagation of HW and LW upstream. For this purpose, we use once again the Water Level Books and we give the results in Tables 4 and 5. In Table 4 we show the coefficient of correlation between the times necessary for HW and LW to reach points upstream against the value of the discharge. We show in Table 5 the mean times necessary for HW and LW to travel

**Table 3**

Coefficient of correlation between the relative response at the upstream station and the corresponding value of the discharge

STATION .....	FRASER RIVER*		SAINT LAWRENCE RIVER*			
	Steveston	New Westminster	Québec	Portneuf	Batiscan	Trois-Rivières
BAND	RELATIVE AMPLITUDE					
Diurnal .....	-0.81	-0.89	0.44	0.12	-0.66	-0.63
Semidiurnal	-0.81	-0.91	0.07	-0.25	-0.89	-0.63
	RELATIVE PHASE					
Diurnal .....	0.27	0.37	0.39	0.57	0.52	0.39
Semidiurnal	0.51	0.61	-0.22	0.18	0.46	0.11

\* Input : Predicted tide at Vancouver or Pointe au Père.

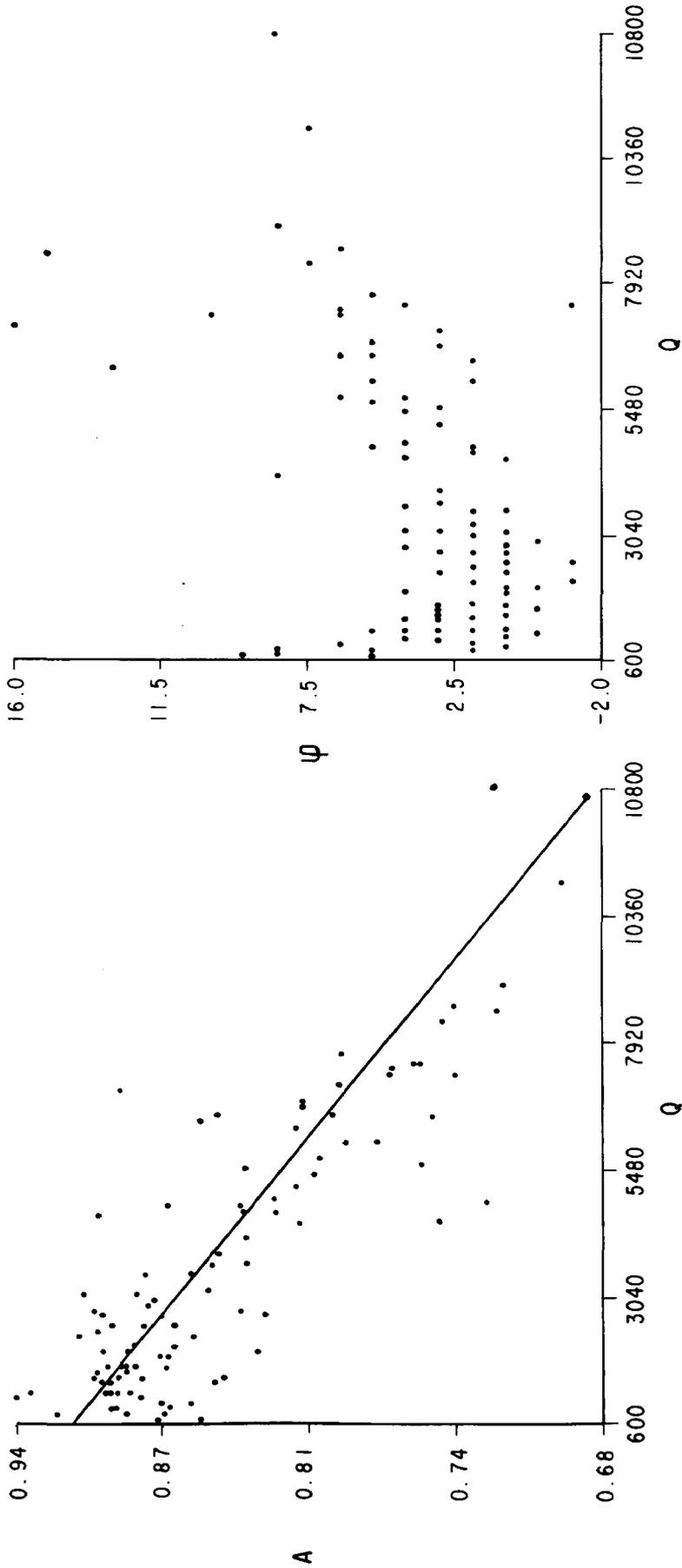


FIG. 4. — Relative amplitude and phase (with respect to the predicted tide at Vancouver) of the semi-diurnal component of the tide recorded at Steveston during intervals of near constant discharge in the Fraser river.  $Q$  in the abscissa gives the prevailing value of the discharge (meters<sup>2</sup>/second) during the interval.

from Pointe au Père to Québec, the estuary, and then from Québec to Trois-Rivières, the upstream region. It was not possible to do the same thing in the Fraser river because the tide there is mixed.

The correlations in Table 4 indicate that HW is delayed by an increased discharge in the Fraser river while LW is accelerated. In the Saint Lawrence the situation is not as clear but there is some indication that LW is delayed near Grondines and accelerated near Trois-Rivières.

**Table 4**

Coefficient of correlation between the time necessary for HW and LW to reach upstream points and the value of the discharge

STATION .....	FRASER RIVER	SAINT LAWRENCE RIVER		
	New Westminster (From Steveston)	Grondines	Batiscan (From Pointe au Père)	Trois-Rivières
HW .....	0.70	-0.18	0.64	0.17
LW .....	-0.51	0.51	0.07	-0.47

The times in Table 5 show that the HW progress is not affected by neap spring cycles in the Saint Lawrence estuary but that beyond Québec it is accelerated during spring tides. LW is accelerated in the estuary during neap tide and retarded in the upstream regions.

**Table 5**

Time (hours) taken from HW and LW to reach upstream points in the Saint Lawrence river during spring and neap tides

STATION .....	Québec (From Pointe au Père)	Trois-Rivières (From Québec)
HW Spring.....	4.39	4.04
HW Neap.....	4.42	4.53
LW Spring .....	5.81	6.26
LW Neap .....	5.30	6.94

We illustrate the existence of long period tides in the upstream regions of a river by showing in Fig. 5 the water level recorded at Québec, Trois-Rivières and Montréal during September 1976. We note the succession of spring and neap tides at Québec in the profile of HW's; at Trois-Rivières the semi-monthly oscillation exceeds the short period tide and in Montréal only the long period tide is noticeable.

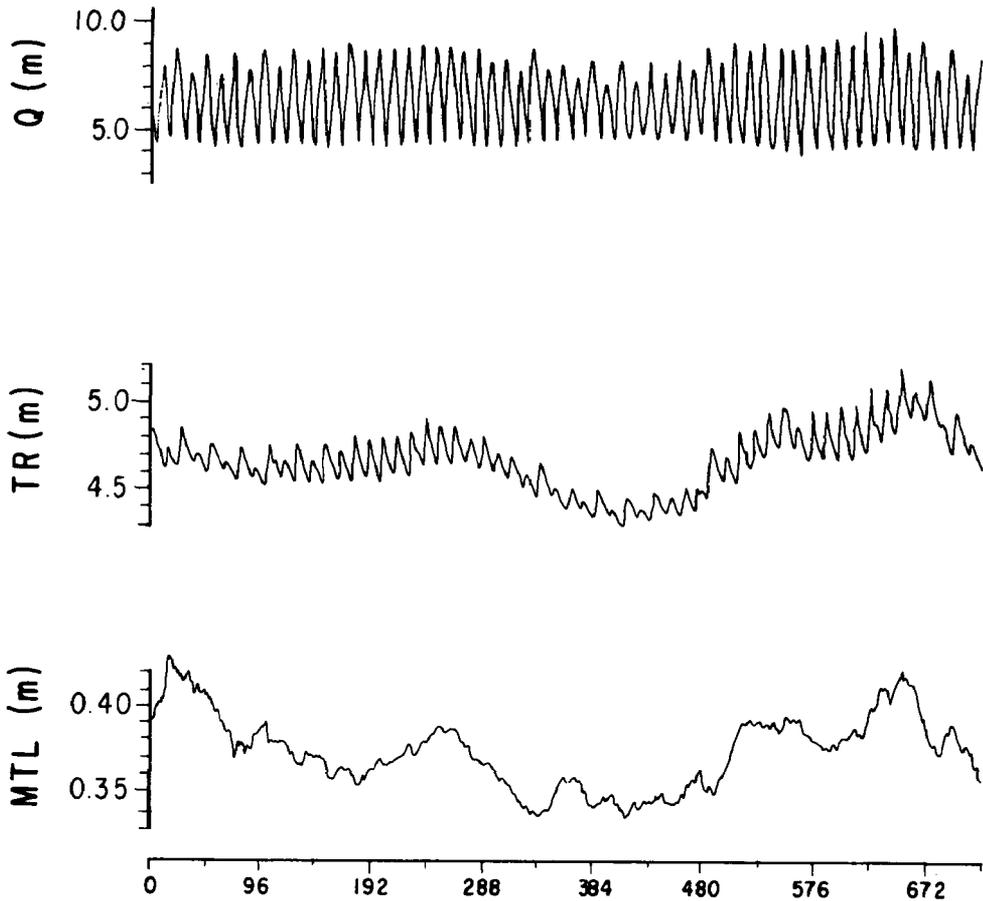


FIG. 5. — Water level recordings at Québec, Trois-Rivières and Montréal during September 1976. These show the development of long period oscillations in the upstream portion of the Saint Lawrence river.

### THE NUMERICAL MODELLING OF THE TIDE IN RIVERS

The profile of the tidal wave is given by a function of the form  $e^{-kx}$ ; the exponential function is a rapidly varying function and, in our case,  $k$  is determined by  $u_0$ , the current due to the discharge, and  $H$ , the depth. Our schematization consisted in representing the river by a one-dimensional channel of uniform depth across each section. In fact the depth varies across the section and the value of  $u_0$  corresponding to a given state of the discharge varies from point to point. The frictional dissipation is determined by  $u_0$  and is many orders of magnitude larger in the shallower sections than in the navigable part of the river. By using an average for the depth distribution across the section, one removes these zones of high dissipation; the model will not be able to reproduce the actual losses in that part of the river and, as a consequence, the tide observed. It is necessary to use a two-dimensional schematization of the river because of the non-linear nature of the frictional effects.

**REFERENCES**

- DRONKERS, J.J. (1964) : Tidal Computations in Rivers and Coastal Waters. North Holland Publishing Co., Amsterdam.
- GODIN, G. (1983) : Modification of a River's Tide by its Discharge. *Journal of Waterway, Port, Coastal and Ocean Engineering*. American Society of Civil Engineers, New York (In Press).
- Water Levels. Vol. 2. Tidal 1970, 1971; Vol. 2. Tidal, Parts 1 & 2, 1972, 1973, 1974; Tidal Highs and Lows, 1975, 1976, 1977, 1978, 1979. Sponsored by : 1970-1971 : Department of Energy, Mines & Resources; 1971-1975 : Department of the Environment; 1975-1978 : Fisheries and Environment; 1979 : Fisheries and Oceans, all located in Ottawa, Canada.