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MODULATION FACTORS IN TIDAL PREDICTION

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1. INTRODUCTION

The predicted tide h(t) is usually expressed by the following equation (DRONKERS, 1964, for example, or in any text on tides) :

$$h(t) = \sum_k A_k f_k(t) \cos(w_k t + u_k(t) - g_k) \qquad (1)$$

where w_k, A_k and g_k are the angular frequency, the amplitude and phase of the k^{th} constituent, and $f_k(t)$ and $u_k(t)$ are the modulation factors produced by its satellites.

For most of the constituents, the most significant period in the modulation factor is 18.6 years which is the period of revolution of the moon's node, and for a few constituents like L_2 the most important period is 4.4 years which is half of the period of the revolution of perigee of the lunar orbit. Because of the slow change in the modulation factors, they are usually assumed as a constant within a fixed interval T. This interval is usually taken as a month or longer. This paper derives the formula for computing the expected error in the tidal prediction caused by fixing the modulation factors within an interval T.

2. APPROXIMATION IN THE COMPUTATION OF THE EXPECTED ERROR

Let us assume that t′ is the time when the modulation factors are computed; therefore, the actual predicted tide is :

$$h'(t) = \sum_k A_k f_k(t') \cos(w_k t + u_k(t') - g_k) \qquad (2)$$

To simplify the algebraic expression in the following discussion, we consider only the largest satellite for each constituent.

Let us change the format of equations (1) and (2) as the following :

$$h(t) = \sum_k [A_k \cos(w_k t - g_k) + B_k \cos(w_k t + \Delta_k t - P_k)] \qquad (3)$$

$$h'(t) = \sum_k [A_k \cos(w_k t - g_k) + B_k \cos(w_k t + \Delta_k t' - P_k)] \qquad (4)$$

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where Δ_k, B_k and P_k are frequency difference, the amplitude and the phase of the satellite of the k^{th} constituent. In most practice, the ratio B_k/A_k is taken from the equilibrium tide, and the phase of P_k is assumed to be equal to g_k.

The error is defined as the difference between h(t) and h'(t), which is

$$e(t) = h'(t) - h(t) = \sum_k B_k [\cos (w_k t + \Delta_k t - P_k) - \cos (w_k t + \Delta_k t' - P_k)] \quad (5)$$

The expected error is computed from the time average of e(t) and $e^2(t)$, which is represented by $E[e(t)]_{T'}$ and $E[e^2(t)]_{T'}$ where T' is equal to 18.6 years. Assuming that $f_1(t')$ and $f_2(t)$ are functions of t' and t respectively, then the following relationship is true

$$E[f_1(t')f_2(t)]_{T'} = E[f_1 (t') E[f_2(t)]_T]_{T'} \quad (6)$$

because $f_1(t')$ is held as a constant during the interval of T. Since T is usually much longer than the period of any constituent discussed in this paper, we obtain that

$$E[\cos w_k t]_T \cong 0 \quad (7)$$

Furthermore T' is equal or close to the multiple of $2\pi/\Delta_k$, therefore

$$E[\cos \Delta_k t]_{T'} \cong 0 \quad (8)$$

To simplify the final solution, we also assume that

$$E[\cos(w_k - w_j)t]_T \cong 0 \quad (9)$$

This is true for most important constituents when T is in the order of several months. According to these assumptions we obtain :

$$E[e(t)]_{T'} \cong 0 \quad (10)$$

which indicates that the mean of h'(t) is the same as that of h(t), and therefore, the stepwise updating of the modulation factor will not produce any significant bias in the prediction.

3. EXPECTED ERROR

The average $e^2(t)$ can be obtained by a straightforward but lengthy algebraic manipulation. According to equations (6) to (9), one can realize that the only terms in the expansion of $e^2(t)$ which might not be zero are the following :

$$E[e^2(t)]_{T'} = \sum_k B_k^2 E[\cos^2 (w_k t + \Delta_k t - P_k) + \cos^2 (w_k t + \Delta_k t' - P_k) - 2\cos (w_k t + \Delta_k t - P_k) \cos (w_k t + \Delta_k t' - P_k)]_{T'} + (0) \quad (11)$$

Therefore, the $E[e^2(t)]_{T'}$ can be computed from the total sum of all the terms in (11). The average of each of the cosine square terms is 1/2, and

$$2 \cos (w_k t + \Delta_k t - P_k) \cos (w_k t + \Delta_k t - P_k) = \cos (2w_k t - 2P_k + \Delta_k (t + t')) + \cos (\Delta_k (t - t')) \quad (12)$$

The time average of the first term on the RHS of (12) is equal to zero, and

$$E[\cos \Delta_k (t - t')]_{T'} = E[\cos \Delta_k t' E[\cos \Delta_k t]_T]_{T'} + E[\sin \Delta_k t' E[\sin \Delta_k t]_T]_{T'} \quad (13)$$

The inner average during the time interval from t_m to t_{m+N} can be expressed as

$$E[\cos \Delta_k t]_T = [\sum_{n=m}^{m+N} \cos \Delta_k n\tau] / (N+1) = S(\Delta_k) \cos (m + N/2)\Delta\tau \qquad (14)$$

Where τ is the time interval of the prediction which is usually 1 hour and,

$$(N + 1)\tau = T \qquad (15)$$

$$S(\Delta_k) = \sin (\Delta_k\tau(N + 1)/2)/((N + 1) \sin (\Delta_k\tau/2)) \qquad (16)$$

by the same computation, we obtain

$$E[\sin\Delta_k t]_T = S(\Delta_k)\sin (m + N/2)\Delta\tau \qquad (17)$$

If the modulation is computed at the beginning of each time interval T, then

$$t' = n(N + 1)\tau \qquad (18)$$

and

$$m = n(N + 1) \qquad (19)$$

Therefore

$$\begin{aligned} E[\cos \Delta_k t' E[\cos \Delta_k t]_T]_{T'} &= S(\Delta_k) E[\cos \Delta_k t' \cos \Delta_k (t' + N\tau/2)]_{T'} \\ &= 1/2\, S(\Delta_k) \cos (\Delta_k N\tau/2) \end{aligned} \qquad (20)$$

and by the same derivation, we obtain

$$E[\sin \Delta_k t' E[\sin \Delta_k t]_T]_{T'} = 1/2\, S(\Delta_k) \cos (\Delta_k N\tau/2) \qquad (21)$$

By substituting (21) and (22) into (13) and then into (11), we obtain

$$E[e^2(t)]_{T'} = \sum_k B_k^2(1 - S(\Delta_k) \cos \Delta_k N\tau/2) \qquad (22)$$

The factor $\cos \Delta_k N/2$ will disappear if the modulation factor is computed at the middle of each time interval T. For some constituents like K_1 and K_2, the satellites have a frequency difference of plus and minus Δ; since one of the satellites is usually larger than the other, the result expressed in (22) will not be affected. By expanding the expression for $E[e^2(t)]_{T'}$, we find that a pair of satellites will produce $-2E[\cos \Delta_k (t + t')]_{T'}$, and according to equations (20) and (21) this term is equal to zero.

The result of $E[e^2(t)]_{T'}$, expressed in equation (22) indicates that when $N = 0$, $h(t) = h'(t)$ and therefore $E[e^2(t)]_{T'}$ 0; and when N is large, $E[e^2(t)]_{T'} = \sum_k B_k^2$ which is twice the variance of each satellite constituent. Since the variance of a cosine function is 1/2, the break-even point for the application of the modulation factor is when

$$1 - S(\Delta_k) \cos \Delta_k N\tau/2 = 1/2 \qquad (23)$$

and the solution of (23) is

$$N\tau = .3\, F \qquad (24)$$

where

$$F = 2\pi/\Delta_k. \qquad (25)$$

Therefore, the modulation factor must be updated at an interval of less than .3 F to be effective. If the interval is longer than .3 F, the application of the modulation factor will increase the error of the prediction rather than improving the prediction.

4. PRACTICAL APPLICATION

In most practical applications, the interval T is less than a year and, therefore, equation (22) can be simplified as

$$E[e^2(t)]_{T'} = \sum_k B_k^2 (1 - \sin(\Delta_k N\tau)/\Delta_k N\tau) \quad (26)$$

If we consider only the satellites with a frequency difference of 1 cycle per 18.6 years, the equation (26) can be simplified further as

$$E[e^2(t)]_{T'} = (1 - \sin \Delta N\tau/\Delta N\tau) \sum_k B_k^2 \quad (27)$$

In many places, the tidal response within each tidal frequency band is nearly constant, therefore

$$\sum_k B_k^2 = (K_1)^2 \sum_i R_{1i}^2 + (M_2)^2 \sum_j R_{2j}^2 \quad (28)$$

where K_1 et M_2 are the amplitudes of the constituents, and R_{1i} is the amplitude ratio of the i^{th} satellite in the diurnal tidal period with respect to K_1, and R_{2j} is the same ratio with respect to M_2 for the satellites in the semi-diurnal tidal period. The values of R_{1i} and R_{2j} can be derived from the equilibrium tide, and $\sum_k B_k^2$ is obtained as

$$\sum_k B_k^2 = .03787\,(K_1)^2 + .00288\,(M_2)^2 \quad (29)$$

Therefore, the percentage error of the modulation factor will be much larger in an area where the diurnal tide is dominant. By substituting (28) and (29) into (27), we obtain a quick estimate of $E[e^2(t)]_{T'}$ as

$$E[e^2(t)]_{T'} = (1 - \sin \Delta N\tau/\Delta N\tau)\,[(.19K_1)^2 + (.054\,M_2)^2] \quad (30)$$

The expected error σ can be expressed as the square root of $E[e^2(t)]_{T'}$, which can be approximated as

$$\sigma = (1 - \sin\Delta N\tau/\Delta N\tau)^{1/2}\,.19K_1\,(1 + .14\,M_2/K_1) \quad \text{if } K_1 > 0.3\,M_2 \quad (31)$$

or

$$\sigma = (1 - \sin\Delta N\tau/\Delta N\tau)^{1/2}\,.054\,M_2\,(1 + 1.76\,K_1/M_2) \quad \text{if } M_2 > 4\,K_1 \quad (32)$$

The following table gives value of the factors in equations (30) and (31), and it shows that the error is larger in places where the diurnal tide is predominant.

N (month)	$U = (1 - \sin\Delta N\tau/\Delta N\tau)^{1/2}$	.19U U × 100	.054 U × 100
1	.011	.209	.059
2	.014	.266	.076
3	.034	.646	.184
6	.069	1.311	.373
9	.103	1.957	.556
12	.138	2.622	.745

5. CONCLUSION

This study concludes that the stepwise updating of the modulation factor in tidal prediction will not produce any significant bias in the prediction. The variance of the error due to this stepwise computation is approximately $2(1 - \sin\Delta T/\Delta T)v^2$, where T is the time interval during which the modulation factor is held constant, Δ is $2\pi/18.6$ years, and v^2 is the variance of the predicted tide. The standard prediction program used by the Canadian Hydrographic Service, Department of Fisheries and Oceans, updates the value of the modulation factor at the beginning of every two months. The largest amplitude of K_1 and M_2 found in Canadian waters is about 100 cm and 600 cm; therefore, the expected error should be within .5 cm.

Reference

DRONKERS. J.J. (1964) : Tidal Computations, North-Holland Publishing Company, Amsterdam.