

## THE USE OF NODAL CORRECTIONS IN THE CALCULATION OF HARMONIC CONSTANTS

by Gabriel GODIN (\*)

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### ABSTRACT

The nodal corrections are effective for  $K_1$ ,  $O_1$  and  $K_2$ . The components  $Q_1$ ,  $J_1$ ,  $OO_1$  and  $NO_1$  ( $M_1$ ) are stabilized at sites where the tide is predominantly linear and where third order effects are minimal. The effectiveness of the corrections for  $Nu_2$  and  $Mu_2$  is not apparent but this fact has little practical importance. It is not prudent to apply any a priori corrections to  $2N_2$ ,  $N_2$  and  $L_2$ .  $S_2$  may at times be modulated by its non linear interaction with  $M_2$ .  $P_1$  could theoretically be modulated by its interaction with  $K_1$ , but no example of this occurrence could be found. The component  $M_2$ , which usually is the major component, exhibits much variability; it may be demodulated by the nodal corrections when the local tide is linear. The amplitude corrections for  $M_2$  become excessive when non linear effects become preponderant although the phase corrections continue to help.

### INTRODUCTION

The components of the tide-generating forces consist of hundreds of distinct harmonics of very close frequency while they differ widely in amplitude. Only a handful dominate, while the remaining ones are barely detectable. The latter are created by long period modulations and perturbations in the orbits of the Moon and of the Earth: the excentricity of the Moon's orbit and the orientation of its major axis vary over an interval of 8.8 years, the plane of the Moon's orbit oscillates about the ecliptic over a period of 18.6 years, the Earth's axis of rotation precesses over an interval of 26,000 years and so on. From a computing point of view, this results in the fact that most of the major components of the tide are surrounded by minor satellites whose frequencies nearly coincide with their own.

(\*) Centro de Investigacion Cientifica y de Educacion Superior de Ensenada (CICESE), Ensenada B.C., Mexico. *Mailing address* : 2936 Arles Mews, Mississauga, Ontario, Canada L5N 2N2.

The presence of such satellites cannot be overlooked if the objective of a harmonic analysis of the water level observed at a given site is to reveal the "harmonic constants" of the local tide. Failing to do so, the annual samples of the revealed constants, when plotted on a time scale, will exhibit marked periodicities both in their amplitude and in their phase lag, indicating that they definitely do not have fixed values. This ever-present but slow variation in some of the components of the tide is known as the "nodal modulation" (attributing it wholly to the oscillations of the Moon's orbit). Attempts have been made since the days of Darwin (1883) to remedy this situation by applying "nodal corrections" to the components which manifest such behaviour. It is our intent to verify the capacity of such corrections to stabilize the revealed amplitude and phase lag of these components into effective constants.

### THE POTENTIAL OF THE TIDE-GENERATING FORCES AND ITS USE FOR THE CALCULATION OF NODAL CORRECTIONS

The potential of the tide-generating forces for the Moon or the Sun is :

$$V = -\frac{3kM}{2r}\left(\frac{a}{r}\right)^2 (\cos^2\xi - 1/3) - \frac{kM}{2r}\left(\frac{a}{r}\right)^3 \cos\xi (5\cos^2\xi - 3) = V_2 + V_3 \quad (1)$$

where:

$k$  = universal constant of gravitation

$M$  = mass of the perturbing celestial body

$r$  = distance between the centre of the Earth and that of the celestial body

$a$  = distance from the centre of the Earth to the point of observation

$\xi$  = angular distance on the celestial sphere between the position of the point of observation and that of the celestial body.

The ratio  $(a/r)$  being very small, it follows that  $V_3$ , known as the third order term, is much smaller than  $V_2$ . It is completely negligible in the case of the Sun, but it does contribute small but not totally insignificant harmonics in the case of the Moon.  $\xi$  in the usable form of (1) is expressed as a function of the colatitude  $\theta$  and of the longitude  $\phi$  of the point of observation and of the perturbing star. The expansion of  $V_2$  in terms of these variables yields a sum of terms whose frequencies fall into three groups (bands): low frequency, diurnal and semidiurnal. The same factor involving  $\theta$ , the colatitude, affects all the components within the same band. Correspondingly, if the response of the ocean to these forces is strictly linear, it should be nearly identical for all the components lying within the same band since they cover a very narrow range of frequency. A similar expansion of the third order term  $V_3$  yields small terms lying in the three bands just mentioned plus terdiurnal frequencies which are uniquely attributable to its effect. The most important of the latter frequencies are known as  $M_3$  and  $MO_3$ ; they are detectable in records which are little affected by non linear effects. If these effects matter, the non linear contribution  $NK_3(N_2 + K_1)$ , differing in frequency by  $1.29 \times 10^{-5}$  cycle/hour or 1 cycle/8.8 years, interferes with  $M_3$  while  $MO_3(M_2 + O_1)$  and  $2MK_3(2M_2 - K_1)$  affect

MO<sub>3</sub>, all three having exactly the same frequency. When it comes to nodal modulations, V<sub>3</sub> contributes small terms which fall into the groups dominated by:

Q<sub>1</sub>, NO<sub>1</sub> (M<sub>1</sub>) and J<sub>1</sub> in the diurnal band

and

2N<sub>2</sub>, N<sub>2</sub> and L<sub>2</sub> in the semidiurnal band.

The more important third order terms differ in frequency from those just mentioned by 1 cycle/8.8 years, a minuscule but distinct frequency difference which should become obvious if the observations cover a long enough interval. In spite of falling in the same bands as the second order terms, the latter contributions are affected by a different colatitude factor: therefore the forces they generate do not have the same geographical distribution as those originating from the second order term. The purely solar terms:

P<sub>1</sub>, S<sub>2</sub>, R<sub>2</sub> and T<sub>2</sub>

will not carry any third order tint. The lunar and lunisolar harmonics:

O<sub>1</sub>, K<sub>1</sub>, Nu<sub>2</sub>, Mu<sub>2</sub>, M<sub>2</sub>, K<sub>2</sub>

which are important in most tidal records, do contain a third order contribution but it is so small that it is effectively negligible. We are left with:

Q<sub>1</sub>, NO<sub>1</sub> (M<sub>1</sub>), J<sub>1</sub>, 2N<sub>2</sub>, N<sub>2</sub> and L<sub>2</sub>

which have satellites originating from both V<sub>2</sub> and V<sub>3</sub>.

The nodal corrections are calculated from the development of the tide-generating potential in the following way. Each major component is surrounded by its satellites: their amplitudes are written as ratios r<sub>j</sub> to that of the dominant one and their frequency difference is denoted by D<sub>j</sub>. It is assumed that the response in amplitude and phase of the local tide is the same for all the members of the group which is affected by a common colatitude factor:

$$\begin{aligned}
 Af_2(\theta) [\cos (U + \sigma t - g) + \sum_{j=1}^k r_j \cos (U + \sigma t + D_j t - g)] \\
 + Bf_3(\theta) [\sum_{j=k}^m r_j \cos (U + \sigma t + D_j t - g')]
 \end{aligned} \tag{2}$$

where :

- U = astronomical argument of the major component for the chosen time origin
- σ = its frequency
- t = time elapsed between the central time of the observation interval and the chosen time origin
- r<sub>j</sub> = ratio of the amplitude of the jth satellite to that of the major component
- D<sub>j</sub> = difference in frequency of the jth satellite
- A = amplitude response for the second order term
- g = Greenwich phase lag for the second order term
- B = amplitude response for the second order term
- g' = Greenwich phase lag for the third order term
- θ = colatitude of the point of observation
- f<sub>2</sub>(θ) = second order term of the potential (colatitude dependent)
- f<sub>3</sub>(θ) = third order term of the potential.

We can make no headway with (2) unless we neglect the third order contribution. This is the case for  $K_2$ , which we use as an example. From the development of the tide-generating potential (CARTWRIGHT and TAYLER, 1971), we can write out the ratios and frequency differences for  $K_2$  in Table 1.

TABLE 1  
Ratios and frequency differences (expressed as Doodson numbers) for the components  $K_2$

K <sub>2</sub> Group				
Colatitude factor	Doodson numbers τ s h p N' p'	Equilibrium amplitude V/G cm	Frequency difference	Ratio
f <sub>2</sub> (θ)	2 2 0 0 0 0	7.993		
	0 1 0	2.382	0 1 0	0.298
	0 2 0	0.259	0 2 0	0.032
f <sub>3</sub> (θ)	- 1 0 0	0.019	- 1 0 0	0.002

**Calculation of f and u**

$A_{K_2} [ \cos (U + \sigma t - g) + 0.298 \cos (U + \sigma t - g + N') + 0.032 \cos (U + \sigma t - g + 2N') ]$   
 Take year 1962. Time origin  $t_0 = 22.5$  May 1960  
 $N' = 188.82^\circ$ . Rate of change of  $N' = 0.0001470940$  cycle/day  
 Number of days elapsed : 2 years or 730 days  
 $N'(1962) = 188.82^\circ + 38.66^\circ (0.1074 \text{ cycle}) = 227.48^\circ$   
 $A_{K_2} [ \cos (U + \sigma t - g) + 0.298 \cos (U + \sigma t - g + 227.48^\circ) + 0.032 \cos (U + \sigma t - g + 94.95^\circ) ]$   
 Consequently :  $f(1962) = 0.82$      $u(1962) = -13^\circ$   
 $V =$  potential of the tide-generating forces  
 $G =$  acceleration due to gravity (usually denoted by  $g$ )

Regardless of the magnitude of the response to the third order term, we can safely neglect it in the case of  $K_2$ . We are left with an expression of the form:

$$H [ \cos (U + \sigma t - g) + \sum_j r_j \cos (U + \sigma t + D_j t - g) ] \quad (3)$$

where we have written  $H$  for  $Af_2(\theta)$ .  $H$  and  $g$  should be the amplitude and Greenwich phase lag (harmonic constants) for  $K_2$  in the area of interest. They are modulated by a factor  $f(N)$  and a variable phase  $u(N)$  where  $N$  denotes the year of observation, the variations in  $f$  and  $u$  being very slow: both  $f(N)$  and  $u(N)$  can be deduced from (3). They are calculated for the central time  $t_0$  of the observations. The  $D_j$ 's cause a phase shift  $e_j$ , for each of the satellites. (3) is written as:

$$f(N) H \cos [U - g + \sigma t + u(N)] \quad (4)$$

where  $f(N)$  and  $u(N)$  are calculated from:

$$f(N) = \sqrt{1 + \sum_j r_j \cos e_j)^2 + (\sum_j r_j \sin e_j)^2} \quad (5)$$

$$u(N) = \arctan \left( \frac{\sum_j r_j \sin e_j}{1 + \sum_j r_j \cos e_j} \right) \quad (6)$$

The annual sample of  $K_2$  gives  $f(N)H$ ,  $g-u(N)$ ; the nodal corrections consist in dividing the annual amplitude by  $f(N)$  and adding  $u(N)$  to the sample Greenwich phase lag.  $f$  and  $u$  are calculated from (5) and (6) during the course of a harmonic analysis using information on the  $r_j$ 's and  $D_j$ 's which is stored in memory. Tables of  $f$  and  $u$  have been calculated by the German Hydrographic Office in Hamburg till the year 2000 (DHI, 1967). We see that only second order nodal modulations can be taken into account in this manner. It means that the "constants" for  $Q_1$ ,  $NO_1$  ( $M_1$ ),  $J_1$ ,  $2N_2$ ,  $N_2$  and  $L_2$  might still show third order oscillations if these exist at the site under study.

### CHECK ON THE EFFECTIVENESS OF THE NODAL CORRECTIONS

Their effectiveness can be assessed by observing the trends in the harmonic constants calculated at a specific station before and after their application; this requires at least 10 years of good quality observations for the site. Such long series are becoming increasingly available in computer compatible format for the more important harbours of the world. Experience teaches us that stations facing the open ocean are relatively free of non linear effects, while those in rivers are most affected; it also teaches us that there exists an infinite variety in tidal records and that we must be very careful before generalizing. Yet some general features do emerge from their study in spite of the actual complexity of the situation and we shall strive to explain these in terms of physical reasoning in the last paragraph. It turns out that some components respond very well to the nodal corrections regardless of the presence or absence of non linear effects. Others are not at all improved by the same corrections and may even be worsened by them. The corrections become an academic exercise in the case of the minor components, other sources of variability importing much more. Finally, there are components which, theoretically, are free of modulations; yet these do exhibit some periodicity at times.

In order to do this verification, we have sought stations which have an exposure to the open ocean, others where non linear effects are slight and finally a pair of stations where these are preponderant. We show in Figure 1 the amplitude of  $O_1$ ,  $K_1$  and in Figure 2 both the amplitude and phase lag of  $K_2$  before and after the nodal corrections. The stations selected are Manzanillo (facing the Pacific on the west coast of Mexico), Tofino (facing the Pacific in the more northerly latitude of Canada), Quebec (inside the Saint Lawrence River) and Baltra (Galapagos). We have included  $K_1$  at Baltra ( $O_1$  is effectively 0 at that site) to underline the fact that even though some eight years of observations are available, these barely suffice to do the verification. However, the few points obtained do indicate that at Baltra, like everywhere else, the corrections do accomplish their task. At Manzanillo, the corrected  $O_1$  and  $K_1$  are de facto constants.  $K_2$  is also demodulated although more unstable; we note its small size. The corrections proved effective even at Quebec where the tide is fully distorted by non linear effects. What remains does show

AMPLITUDE

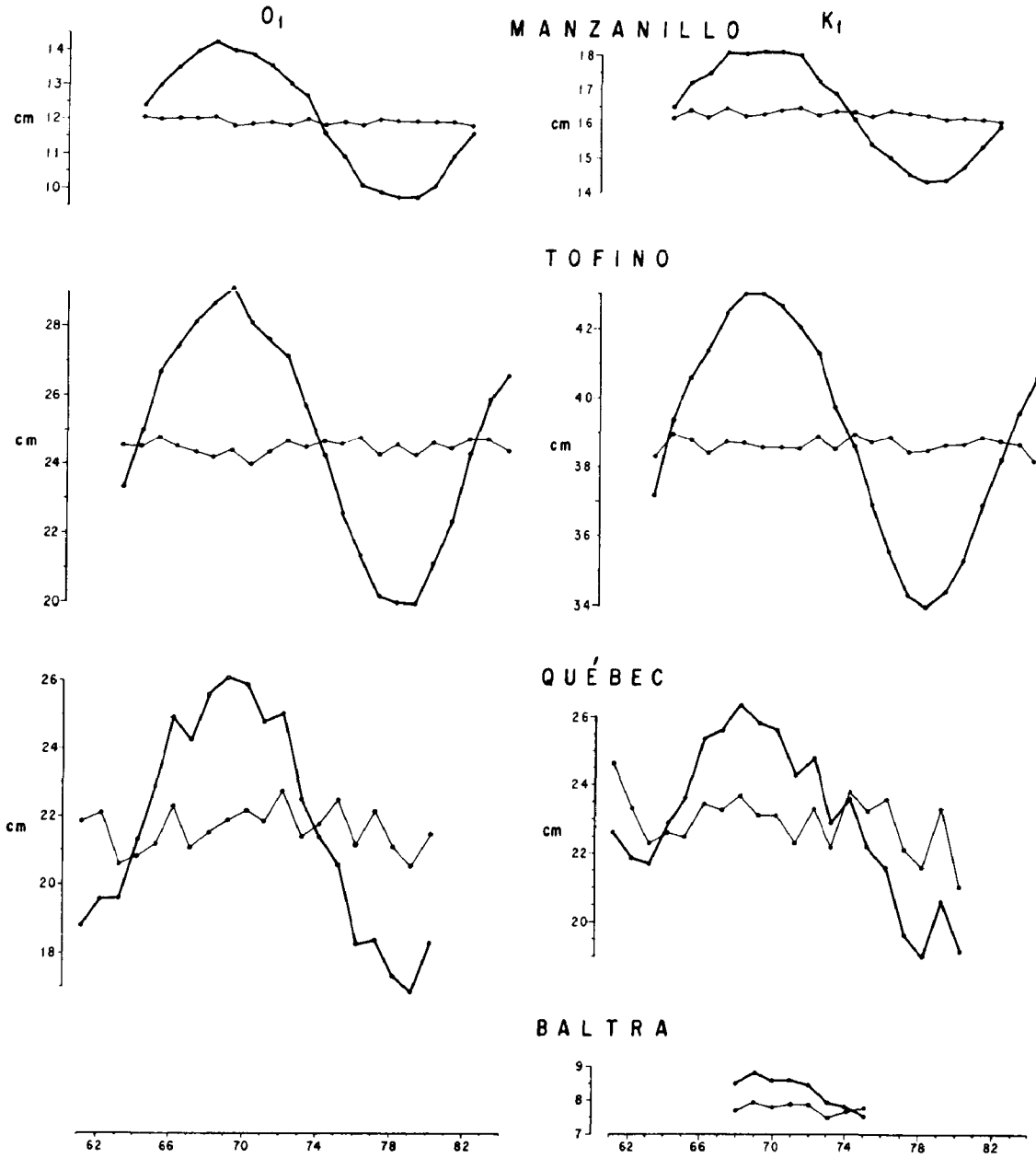


FIG. 1. — Amplitude of the diurnal components  $O_1$  and  $K_1$  before (open circles, thick lines) and after (closed circles, thin lines) the second order nodal corrections.  $O_1$  is not shown for Baltra because it has a value near zero. The abscissa gives the year number, 19 being omitted.

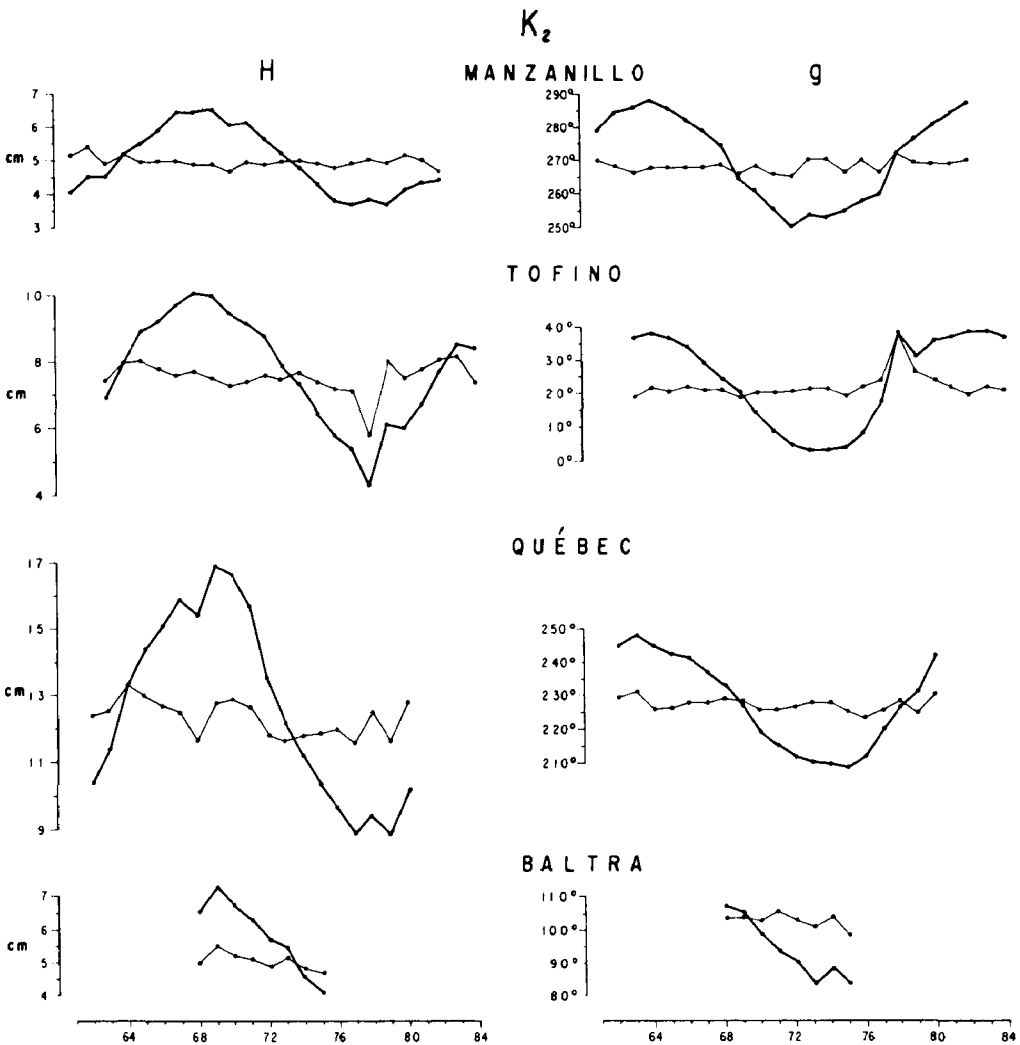


FIG. 2. — Amplitude and Greenwich phase lag for the semidiurnal component  $K_2$  before and after the second order nodal corrections.

much variability but no periodicity. There are some problems with the data of Tofino for the years 1978 and 1979, but otherwise the corrections are effective.

We give in Figure 3 plots for the lesser components  $Q_1$ ,  $J_1$ ,  $OO_1$ ,  $Nu_2$  and  $Mu_2$  before and after the corrections. Only the raw values for the amplitude of  $Mu_2$  and  $Nu_2$  are shown for Manzanillo, because it is impossible to distinguish between the raw and corrected values. We also show the phase lag for  $Mu_2$ . The corrections for the diurnal components are clearly effective at Manzanillo. At Quebec the intrinsic variability of the signal at the site makes it impossible to check on the corrections; in fact, their application is not even relevant. At Manzanillo itself, the question ceases to be relevant in the case of  $Nu_2$  and  $Mu_2$ .

Figure 4 gives plots of the amplitude and phase lag of  $N_2$  and  $L_2$ . We show the second order corrections only in the case of  $N_2$ : these corrections are unable

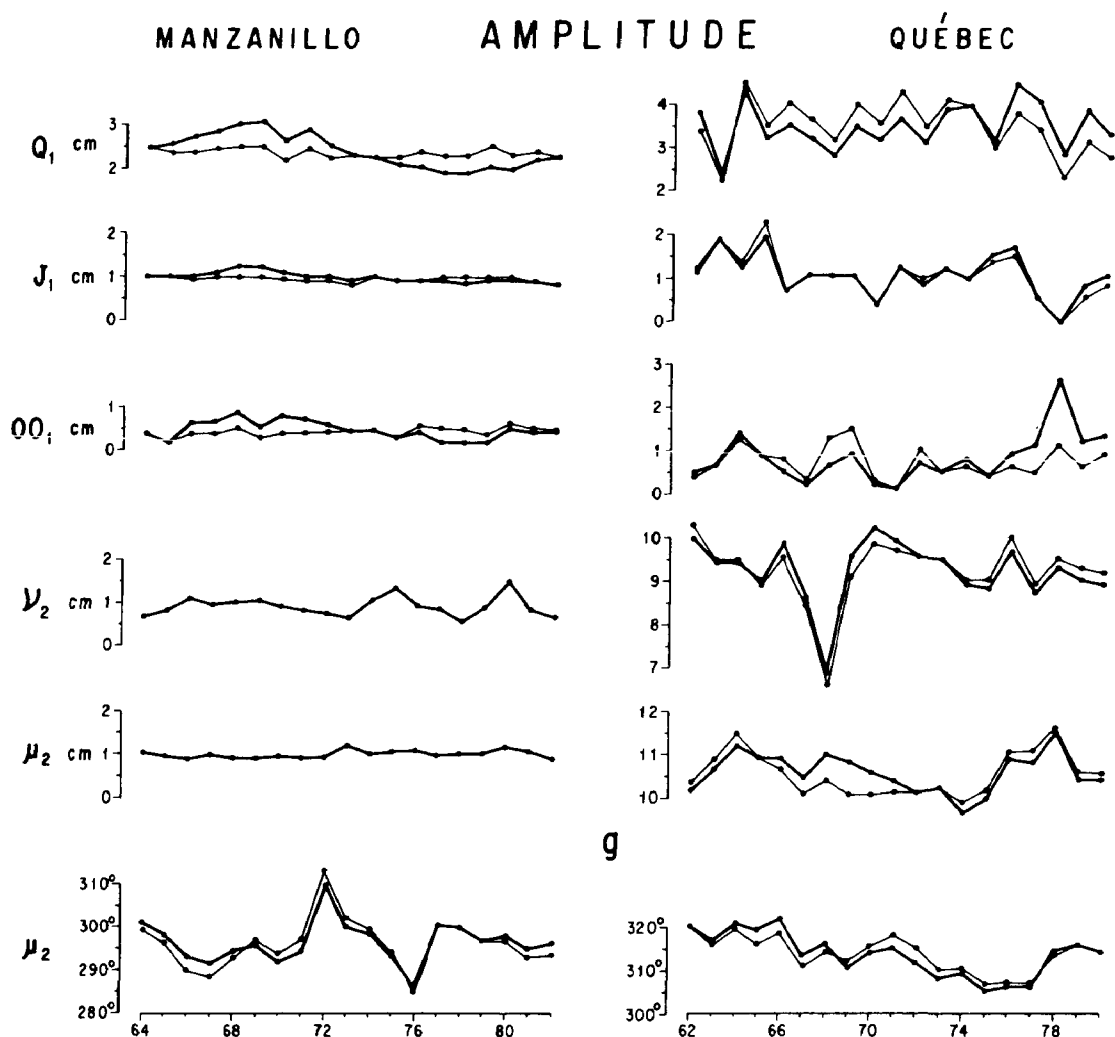


FIG. 3. — Amplitude and Greenwich phase lag (for  $Mu_2$  only) of the minor components  $Q_1$ ,  $J_1$ ,  $OO_1$ ,  $Mu_2$  and  $Nu_2$ , before and after the corrections. (Similar profiles are obtained for the component  $NO_1$ ).

to reduce the 9 year oscillations present in it. Non linear effects being minimal at Manzanillo, these oscillations must be due to the third order contribution. Friction creates the new component  $2ML_2$  at Quebec which coincides exactly in frequency with  $N_2$ : therefore, we have two extra contributions at this frequency, in addition to the linear response at  $N_2$ . We have plotted the amplitude of  $L_2$  on a magnified scale for Manzanillo to show its extraordinary regularity in spite of its microscopic size; the one at Quebec is given on a regular scale. The 9 year periodicity in  $L_2$  is present in both their amplitude and their phase lag. The phase lag of  $L_2$  at Manzanillo suggests the presence and interference of a component of a magnitude equal to that of the second order  $L_2$  (GODIN and GUTTIÉRREZ, 1985a).  $2MN_2$  dominates at the  $L_2$  frequency at Quebec as can be deduced from the trends in a response diagram (GODIN and GUTIÉRREZ, 1985b). In both cases the second order linear  $L_2$  fails to dominate at this frequency.



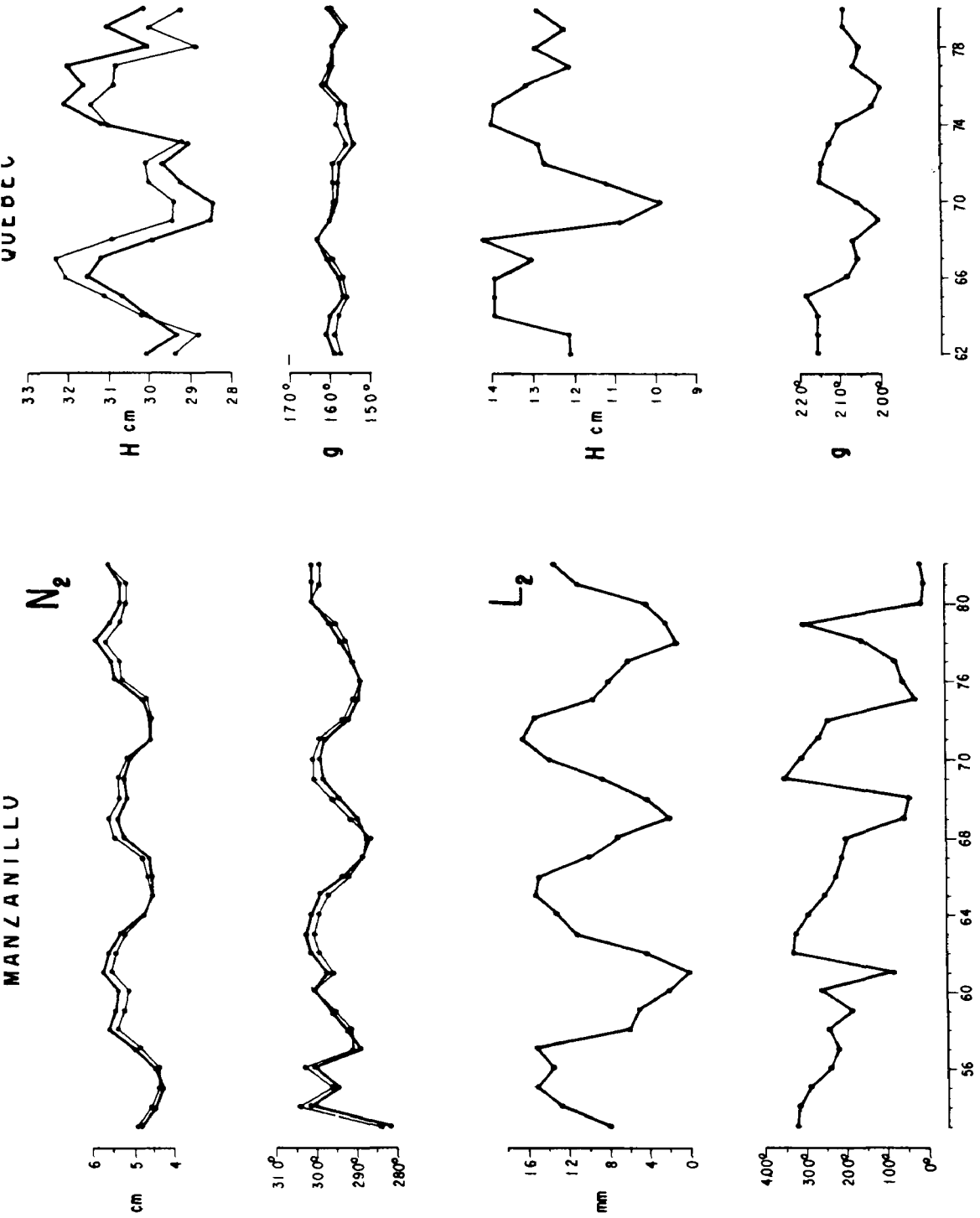


FIG. 4. — Amplitude and Greenwich phase lag of the components  $N_2$  and  $L_2$ ; the second order corrections are applied only to  $N_2$  because of their obvious lack of effect in the cases under consideration.

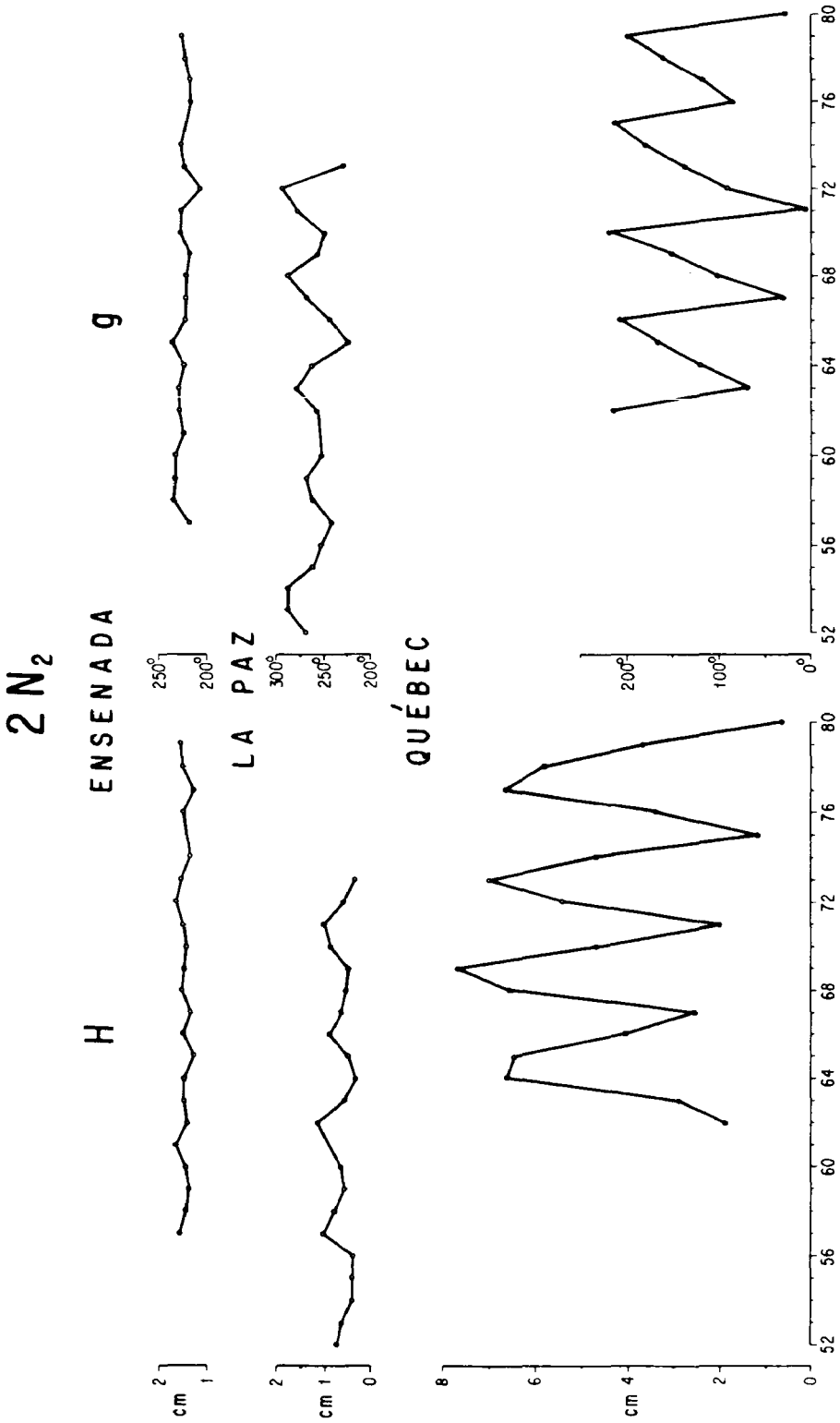


FIG. 5. — Raw amplitude and Greenwich phase lag of the component  $2N_2$ ; the four year interference pattern is created by the frictional harmonic  $2MK_2$ , which differs very slightly in frequency from  $2N_2$ .

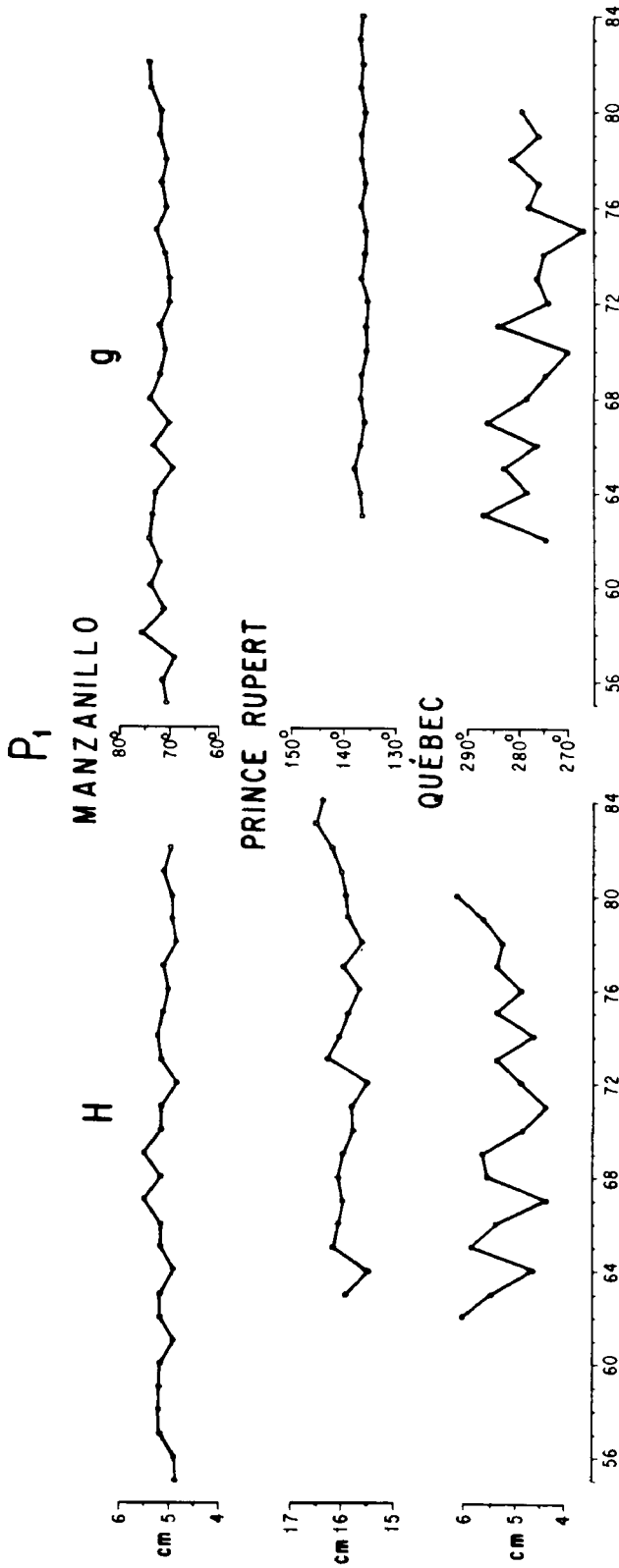


FIG. 6. — Raw amplitude and Greenwich phase lag of the component  $P_1$ , which is virtually unmodulated by background satellites.

It is current practice to apply nodal corrections based on the expected modulation of the linear  $L_2$ ; when this is done, the corrected values may be worse than the original ones. We give  $2N_2$  in Figure 5. As a linear contributor,  $2N_2$  represents a minor term in the tide. In the physical reality of records, it is prone to disturbances from the third order term and from the non linear harmonics  $2NM_2$  and  $2MK_2$ .  $2NM_2$  has exactly the same frequency as  $2N_2$  and adds its contribution to it vectorially.  $2MK_2$  differs ever so slightly in frequency and creates an interference pattern which has a period of 4 years. We show  $2N_2$  for Ensenada (facing the Pacific) which has a good ocean exposure, La Paz (inside a bay at the entrance of the Gulf of California) where slight non linear effects make themselves felt and Quebec where they are overwhelming. The interference pattern, even though of an amplitude of less than 3 mm, is apparent at La Paz while it dominates totally at Quebec. Consequently, the question of the effectiveness of the corrections for  $2N_2$ , when it is truly itself, does not matter since it is so small, while it becomes irrelevant at any site where there is even a whiff of non linearities.

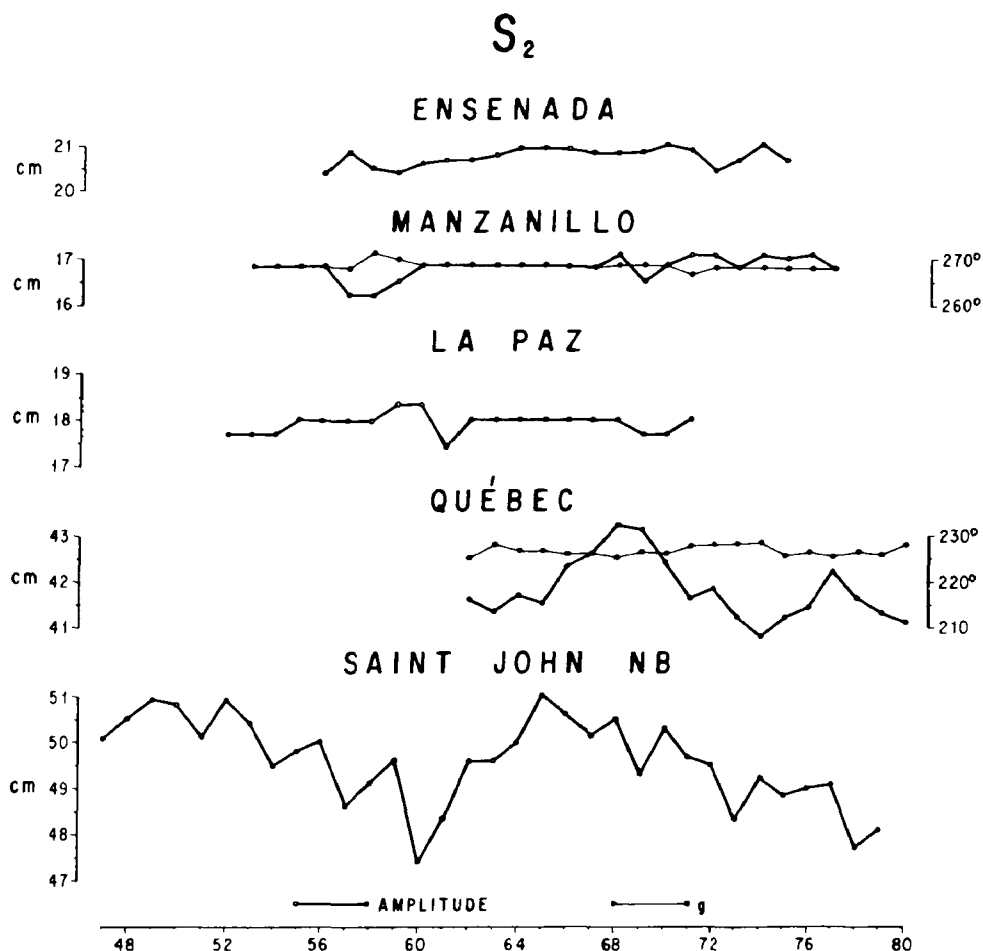


FIG. 7. — Raw amplitude and Greenwich phase lag (for Manzanillo and Quebec only) of the major solar component  $S_2$ . It is unmodulated by background satellites, but it may be modulated by fluctuations in the major lunar component  $M_2$  or by the fresh water discharge when friction is effective.

We now turn our attention to the components which are virtually free of modulations: these are  $P_1$  and  $S_2$ . The revealed  $P_1$  (Fig. 6) at Manzanillo, Prince Rupert (facing the Pacific at a more northerly latitude than Tofino) and Quebec, is unmodulated; it is more variable at Quebec, but it still exhibits no periodicity. The situation for  $S_2$  is not as clear cut. In Figure 7 we show the amplitude and the phase lag of  $S_2$  at Quebec and Manzanillo as well as the  $S_2$  amplitude for Ensenada, La Paz and Saint John NB (in the middle of the Bay of Fundy where the tide is strong and where non linearities turn out to be important). Its amplitude is definitely more variable at Manzanillo than that of  $P_1$ ; it shows some vague periodicity at La Paz and it definitely is periodic at Quebec and Saint John NB. On the other hand, the  $S_2$  phase lag is rather constant everywhere; this is why we have shown it only for Manzanillo and Quebec, the two extreme cases. We may justly suspect non linear effects for such behaviour and a study of these indicates that quadratic friction causes the current component due to  $M_2$  to control a good part of the damping of the other components when it dominates. At Quebec, the

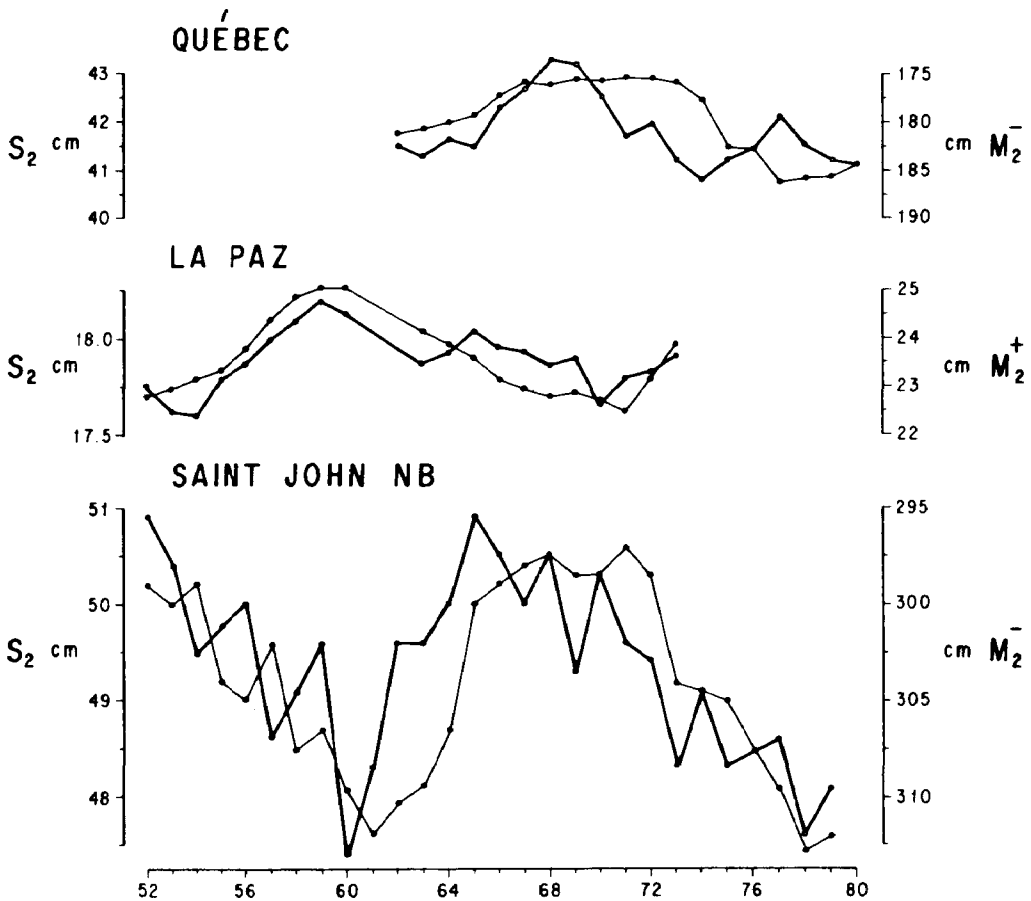


FIG. 8. — Plot of the raw amplitude of  $S_2$  (full circles, thin lines) on a background of the raw amplitude of  $M_2$  (open circles, thick lines). A minus sign on  $M_2$  indicates that the scale of amplitude of  $M_2$  is inverted.

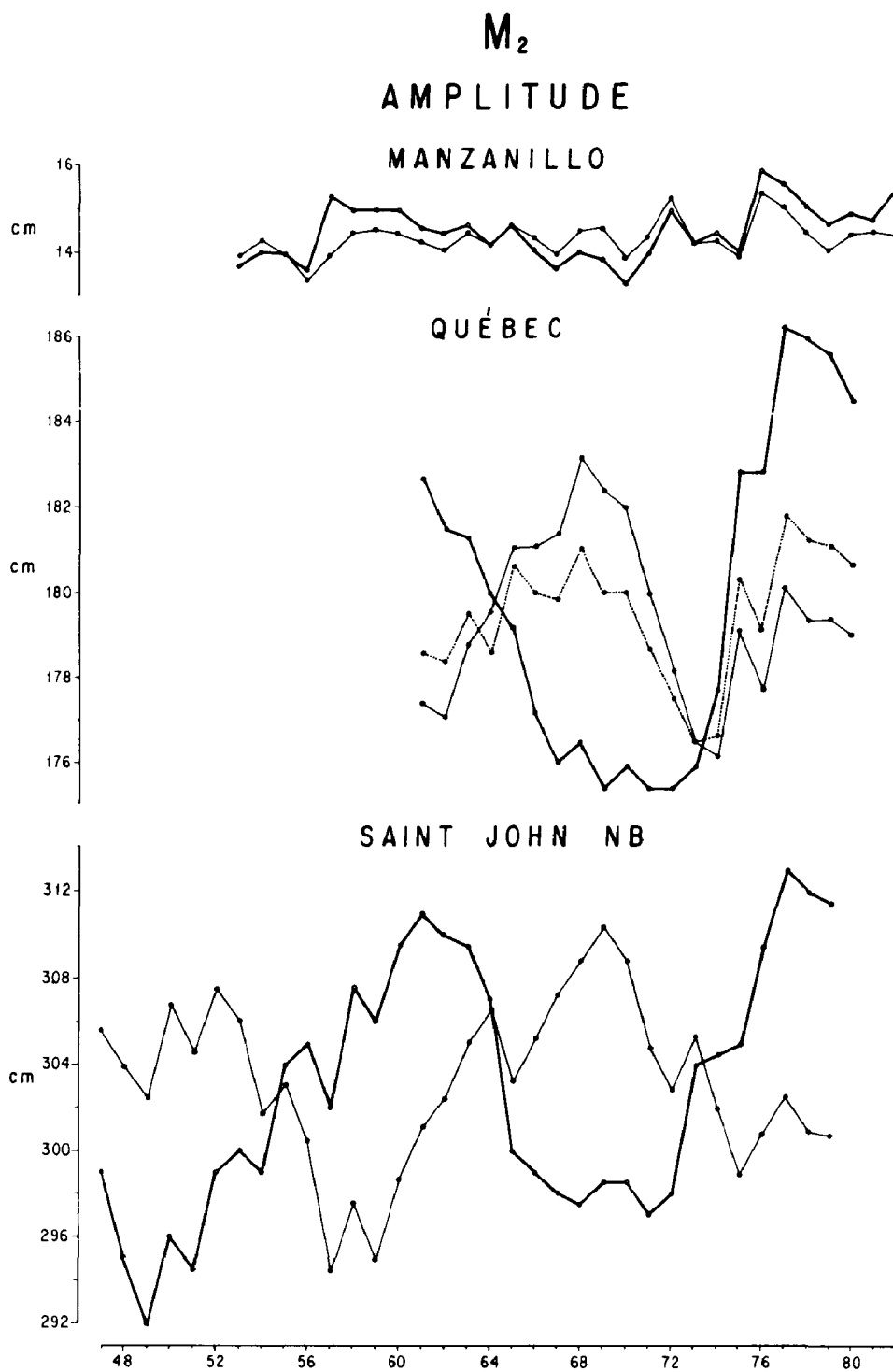


FIG. 9. — Raw and corrected amplitude of the component  $M_2$ . The dotted curve for Québec represents the corrected values, taking 70% of the second order nodal corrections suggested for the amplitude of  $M_2$ .

damping of  $S_2$  is due not only to  $M_2$  but also to the fresh water discharge: the latter acts as a universal leveller, damping equally all the components in the same band (GODIN, 1985). To check on this, we plot in Figure 8 the amplitude of  $S_2$  over the amplitude of  $M_2$  as a background for Quebec, La Paz and Saint John NB; the  $M_2$  scale is inverted for Quebec and Saint John. The vertical scale for La Paz has been considerably amplified to show the overall correlation between the  $M_2$  and  $S_2$  amplitudes; there is a short period oscillation in  $S_2$  for La Paz and Quebec, which is not reflected in  $M_2$ . The inverse correlation of  $M_2$  and  $S_2$  at Saint John is evident: the calculated coefficient of correlation has value  $-0.75$ . At Quebec we see some vague correlation between  $S_2$  and  $M_2$ : it is marginal at most.  $S_2$  correlates directly with  $M_2$  at La Paz, with a coefficient of correlation equal to  $+0.76$ . We shall seek the reasons for such behaviour in the amplitude of  $S_2$  in the next paragraph; all we retain for the moment is that the amplitude of  $S_2$  may be modulated under some circumstances.

We are now left with  $M_2$ . It is usually the most important component of the tide and it does have some second order modulations. Since it is of large magnitude, the accuracy of the corrections matters very much in practice. Figure 9 shows  $M_2$  for Manzanillo, Quebec and Saint John. The most striking feature of  $M_2$  is that it is highly variable and that this variability is unrelated to the modulations. The corrections do reduce the variability of the phase lag in all cases; the effectiveness of the corrections on the  $M_2$  amplitude is not as clear cut. They reduce the amplitude modulation somewhat at Manzanillo. The same corrections induce countermodulations at Quebec and Saint John; some countermodulation is noticeable at La Paz around 1955, but it is not discernible afterwards. This suggests that the amplitude corrections may be excessive in some cases. To underline this fact we reduce the amplitude corrections to 70 % of those suggested by (5) for Quebec and show the adjusted curve in Figure 9. The overshooting is reduced somewhat but the cure is not perfect. It suggests that the adjustments to the corrections may be a function of time.

*A few solid facts emerge from our inspection:*

- a) The components  $K_1$ ,  $O_1$  and  $K_2$  have their periodicities removed by the nodal corrections even at sites where the tide is most distorted.
- b) The components  $Q_1$ ,  $J_1$ ,  $OO_1$  and  $NO_1$  ( $M_1$ ) are also stabilized by the same corrections at sites where the tide is predominantly linear and where third order effects are minimal. CARTWRIGHT (1975), using a much finer method of analysis than the conventional harmonic method, noted that  $M_1$  (of third order) is noticeable in the northeast Atlantic.
- c) The effectiveness of the corrections for  $Nu_2$  and  $Mu_2$  is not apparent, but this fact had little practical importance.
- d) It is not prudent to apply any form of corrections to  $2N_2$ ,  $N_2$  and  $L_2$  till something is known about their characteristics at the site under study. All three are prone to third order disturbances; in addition, non linear terms affect the magnitude and phase lag of both  $2N_2$  and  $L_2$ .  $2N_2$  often shows a clear 4 year interference pattern whenever  $2MK_2$  is present.  $N_2$  may contain the non linear contribution  $2ML_2$  which effectively is  $2M(2MN_2)_2$  in shallow water.
- e) The component  $P_1$  behaved as a harmonic constant in all the cases examined. However, we shall see in the following paragraph that in cases where the

diurnal currents are strong, the presence of a fresh water discharge or non linear effects contribute to the diurnal band in a tangible way; the  $P_1$  should start exhibiting modulations analogous to those of  $K_1$ .

- f) The component  $S_2$  may exhibit periodicities in its amplitude at sites where non linear effects exist.
- g) The component  $M_2$  has its modulations reduced at sites where the tide is linear. At sites where non linear effects are sensible, the corrections to its phase lag are still useful while those to its amplitude may turn out to be excessive. In all cases,  $M_2$  always shows much variability.

### NON LINEAR EFFECTS IN TIDES

Friction affects the movement of tides everywhere, even in the deepest ocean. It becomes of major importance in the shallower basins. Friction combines with the fresh water discharge in rivers to distort the tidal signal even more. The equations of hydrodynamics for a canal of variable width and depth are:

$$u \frac{\partial u}{\partial t} + g \frac{\partial Z}{\partial x} = -u \frac{\partial u}{\partial x} - \frac{Ku|u|}{H+Z} \quad (7)$$

$$\frac{\partial}{\partial x} [B(H+Z)u] = -\frac{\partial}{\partial t} (BZ) \quad (8)$$

where:

$u$  = current

$Z$  = vertical displacement

$H$  = mean depth

$K = G/C^2$  where  $C$  is the Chézy coefficient of friction

$B$  = width of the canal

$x$  = position along the  $x$  axis

$t$  = time

$G$  = acceleration due to gravity (usually denoted by  $g$ ).

The term  $Ku|u|/(H+Z)$  represents the effect of friction: it is proportional to  $u^2$ , inversely proportional to the instantaneous depth  $H+Z$  and acts in a direction opposite to that of the current. It is non linear because it involves the product  $u|u|$  of the variable  $u$ . Other non linear terms in (7) and (8) are  $u\partial u/\partial x$ , the convective term, and  $(\partial/\partial x)(BZu)$ , a flux term involving the volume of water occupied by the vertical displacement. The substitution of typical values of  $u$ ,  $K$ ,  $H$ ,  $Z$ ,  $B$  in (7) and (8) suggests that friction is the more important non linear contribution in general. The other non linear terms become important in zones where there exist abrupt gradients in the field of currents or when the vertical displacement  $Z$  becomes comparable with the depth  $H$ .

The product  $u|u|$  can be well approximated by GODIN and GUTIÉRREZ (1985b):

$$u|u| \approx \frac{U^2}{2} \left[ \sum_j (m + \frac{3}{4m} a_j + \frac{3}{2m} \sum_{k \neq j} a_k^2) a_j \cos \sigma_j \right] \quad (9a)$$



$$+ \frac{3}{4m} \sum_j a_j^2 \sum_{k \neq j} a_k [\cos(2\sigma_j + \sigma_k) + \cos(2\sigma_j - \sigma_k)] \tag{9b}$$

$$+ \frac{6}{4m} \sum_{j \neq k \neq n} a_j a_k a_n [\cos(\sigma_j + \sigma_k + \sigma_n) + \cos(\sigma_j + \sigma_k - \sigma_n) + \cos(\sigma_j - \sigma_k + \sigma_n) + \cos(\sigma_j - \sigma_k - \sigma_n)] \tag{9c}$$

$$+ \frac{1}{4m} \sum_j a_j^3 \cos 3\sigma_j \tag{9d}$$

where:

U = maximum possible velocity of the current at point x, assuming that the current is due exclusively to the tide. Consequently, it can be represented by a superposition of harmonics and U is the sum of the amplitudes of the harmonics present in the signal.

a<sub>j</sub> = ratio of the amplitude of the jth component of the tidal current to the maximum possible amplitude U

σ<sub>j</sub> = abbreviation for the phase of the component σ<sub>j</sub>t - b<sub>j</sub> where b<sub>j</sub> is its phase lag

m = a constant for the approximation of u|u| by (9). The approximation is optimum for m = 0.7.

A numerical experiment involving 6 semidiurnal components observed in the current running in the body of the Bay of Fundy reproduces u|u|, normalized to the range (-1, +1) with a root mean square error of 0.025, or with an average accuracy of 97.5 %. Approximation (9) is therefore most useful to understand the effect of quadratic friction on the tide.

When fresh water discharge is present, it contributes a current -u<sub>o</sub> = -a<sub>o</sub>U directed downstream. In this instance, the approximation to u|u| becomes:

$$u|u| \approx \frac{U^2}{2} \left[ -ma_o - \frac{a_o^3}{m} + \sum_j \left\{ m + \frac{3a_o^2}{m} + \frac{3a_j^2}{4m} + \frac{3}{2m} \sum_{k \neq j} a_k^2 \right\} a_j \cos \sigma_j - \frac{3a_o}{2m} \left\{ \sum_j a_j^2 (1 + \cos 2\sigma_j) + 2 \sum_{k \neq j} a_j a_k [\cos(\sigma_j + \sigma_k) + \cos(\sigma_j - \sigma_k)] \right\} \right] \tag{10}$$

We dropped triple products of the a<sub>j</sub>'s in (10) because they are smaller than those involving products with the ratio of the fresh water discharge a<sub>o</sub>.

We have similar expressions for (∂/∂x) (u<sup>2</sup>/2) and (∂/∂x) (Zu) :

$$\left( \frac{\partial}{\partial x} \right) \left( \frac{u^2}{2} \right) \approx \frac{1}{4} \left[ \sum_j a_j^2 (1 + \cos 2\sigma_j) + 2 \sum_{j,k}^{j \neq k} a_j a_k [\cos(\sigma_j + \sigma_k) + \cos(\sigma_j - \sigma_k)] \right] \left( \frac{\partial U^2 / \partial x}{2a_o Z U / \partial x} \right) \tag{11}$$

where Z means here the maximum local elevation. We used the same ratios for u<sup>2</sup>/2 and Zu: this is permissible only away from zones of amphidromy.

$u\partial u/\partial x$  creates the difference which is observed between the speed of propagation of high and low water in rivers. (11) indicates that it, as well as  $(\partial/\partial x)$  (Zu), create components like  $MS_4$  and  $MS_f$ . The factor  $1/(H+Z)$  in the friction term creates different regimes of flow and of frictional interactions during high and low water in the upstream portions of rivers. The convective term  $u\partial u/\partial x$  also acts as a damping factor in that area of rivers; both  $u\partial u/\partial x$  and  $(\partial/\partial x)$  (Zu) draw energy away from the tide. The friction term in its form (9) and (10) damps the incoming tide: the factor of  $\cos \sigma_j$  in both expressions acts on the  $j$ th component. Because of the non linearity of friction, all the components are involved in this damping. It also creates new harmonics like  $2MS_6$ ,  $2MS_2$ ,  $MNS_6$ ,  $MNS_2$ ,  $M_6$ , etc., in embayments. When the fresh water discharge becomes important, friction raises the local mean level and creates new harmonics like  $MS_4$  and  $MS_f$  identical to those created by (11). In contrast to the other non linear effects, friction creates some harmonics which coincide exactly with some of the components of the incoming tide like  $Mu_2$ ,  $L_2$ ,  $2N_2$ , etc. We give in Table 2 the most important of these compound harmonics.

TABLE 2

Compound harmonics created by friction which coincide with some of the components of the incoming tide

Diurnal band		Semidiurnal band	
Original harmonic	Harmonic created by friction	Original harmonic	Harmonic(s) created by friction
$2Q_1$	$2OK_1$	$Eps_2$	$MNS_2$
$\phi_1$	$2KP_1$	$2N_2$	$2NM_2, 2MK_2^*$
$OO_1$	$2KO_1$	$Mu_2$	$2MS_2$
		$N_2$	$2ML_2$
		$Nu_2$	$MLS_2$
		$Lda_2$	$SNM_2, 2MNU_2$
		$L_2$	$2MN_2$
		$Zet_2$	$MSN_2^*$
		$Eta_2$	$MKN_2$

\* Differs slightly in frequency.

Only the most powerful components of the current will succeed in creating compound harmonics. Although the law of friction is "quadratic", mathematically it involves an odd function. We see in (9) that it creates linear and cubic terms involving at most the interaction of three components (not necessarily different); the latter terms contribute the compound harmonics. To see why the weaker components of the current are much less effective, we consider the following plausible values of the ratios  $a_j$ 's:

$$a_{M2} = 0.65 \quad a_{S2} = 0.11 \quad a_{N2} = 0.13.$$

We can calculate from (9) the relative magnitude of the coefficient giving rise to the following compound harmonics :

$$\begin{array}{lll} M_6 & : & 0.27 \\ 2MS_6 & : & 0.15 \\ 2MN_6 & : & 0.15 \end{array} \quad \begin{array}{lll} S_6 & : & 0.00 \\ 2SM_6 & : & 0.03 \\ MNS_6 & : & 0.06 \end{array} \quad \begin{array}{lll} N_6 & : & 0.00 \\ 2NM_6 & : & 0.03 \end{array}$$

We normally expect  $2MS_2 (= Mu_2)$ ,  $2MN_2 (= L_2)$ ,  $2MK_2 (= 2N_2)$  to be the dominant contributors in the semidiurnal band; as a rule, the diurnal currents are not as strong as the semidiurnal ones, so we should expect even  $2KO_1$  and  $2OK_1$

to be rarely of importance.  $N_2$  should theoretically be disturbed by  $2ML_2$ ; since under frictional circumstances,  $L_2$  is dominated by  $2MN_2$ ,  $N_2$  should interact with itself through  $2MN_2$  in a manner which cannot be apparent from its analyzed value. To complicate the situation even further,  $2N_2$ ,  $N_2$  and  $L_2$  are also perturbed by third order effects which it is not possible to estimate a priori. The minor components  $Eps_2$ ,  $Lda_2$ ,  $Zet_2$  and  $Eta_2$ , because of the non linear contributions they may contain, can scarcely be expected to behave as true tidal components.  $Mu_2$  itself is drowned eventually by  $2MS_2$  in rivers. For instance, using the slope of the response diagrams as a guide, we deduce that  $2MS_2 \approx 19$  cm at Quebec while  $Mu_2 \approx 8$  cm.

If we inspect the damping portion (9 a) of (9), we see that all the components act through the term involving the summation in  $k \neq j$  to damp the  $j$ th wave. Inserting orders of magnitude for the  $a_j$ 's, as we have just done in the above example, we would see that  $M_2$  will contribute significantly to the damping of its companions while they have little influence on  $M_2$  itself or on their other companions. This brings to the fore the damping of  $S_2$ : the larger the  $M_2$  current, the stronger the damping of  $S_2$  will be. We saw this to be the case for  $S_2$  at Saint John NB in Figure 8. At La Paz, we obtained a positive correlation! La Paz lies between two zones of semidiurnal amphidromies, one inside the Gulf of California, the other off Acapulco. The nodal modulation of  $M_2$  may result in stronger currents in the area of La Paz when the vertical  $M_2$  passes through a minimum: this can be checked only by current observations. In rivers (10) indicates that the discharge damps all the components in the same band equally. Quebec lies in a transition zone where the tidal currents are still important while the discharge also makes itself felt. What we have there is some control of the minor components by  $M_2$  and the reflection of different annual mean values for the discharge. The non linear character of the damping causes the range of the modulation of  $M_2$  to become reduced where friction starts distorting the signal; this explains why the full nodal corrections cause some counter modulations in some instances.

We retain the following from our considerations on the friction term :

- a) Only the stronger components of the current can give rise to noticeable compound harmonics of frictional origin,
- b) Frictional harmonics may frequently interfere with  $2N_2$ ,  $Mu_2$  and  $L_2$ . Friction also creates complex interactions at the frequencies of  $N_2$  and  $Nu_2$ . The minor components  $Eps_2$ ,  $Lda_2$ ,  $Zet_2$ ,  $Eta_2$  may be masked by frictional harmonics,
- c) In embayments, the stronger components of the current control much of the damping of the others: consequently  $S_2$  may be modulated by  $M_2$ .

It is not appropriate to introduce the concept of background noise when inspecting the spectra of tidal records, i.e. a signal of approximately constant amplitude but random phase right across the tidal bands. In zones of undistorted tides, like the west coast of Mexico, extremely weak signals like  $M_3$  are fully detectable and their variance is appreciably smaller than their revealed amplitude (GODIN and GUTIÉRREZ, 1985a). The signal variability increases at the frequencies where it is the strongest, most frequently at the frequency of  $M_2$ . We can understand the higher variability of the larger components by attributing it to the fact that they are the ones interfering most strongly with the others: their energy is drawn away from them in all directions and this process cannot be expected to be smooth and regular.

In the foregoing, we have assumed that the semidiurnal currents dominate around the station investigated. We now consider the possible, but rare, occurrence when the diurnal currents equal or exceed in magnitude the semidiurnal ones : in this case frictional effects should become noticeable inside the diurnal band. Friction would create  $2K_{O_1}$  and  $2O_{K_1}$  which coincide with  $OO_1$  and  $2Q_1$  : the equilibrium ratios of these two components should show abnormal values. If we consider other types of non linearities, we would have :

$$MK_1 = O_1 \quad MO_1 = K_1 \quad SK_1 = P_1 \quad SO_1 \quad NO_1 \quad NK_1 = Q_1 \quad KO_2 = M_2.$$

If these are present they should not be as obvious as those in the semidiurnal band.  $MK_1$  and  $MO_1$  have modulations similar to those of  $O_1$  and  $K_1$ .  $2KO_1$  should have large modulations, but these parallel those already present in  $OO_1$ . The only unmistakable clue to their presence should be the behaviour of  $P_1$  : it should start being modulated by  $K_1$ .  $2Q_1$  should have an equilibrium ratio which strays from the common trend, but since it lies at the low extremity of the band there may be other reasons why this could occur. We now understand that even if non linear effects are created by the diurnal currents, these should not give as clear cut an indication of their presence in the diurnal band as those in the semidiurnal one.

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## APPENDIX

### GEOGRAPHICAL LOCATION OF THE TIDAL STATIONS INVESTIGATED

Prince Rupert .....	54°19'N	93°54'W
Tofino .....	49°09'N	125°55'W
Quebec.....	46°50'N	71°10'W
Saint John, New Brunswick.....	45°16'N	66°04'W
Ensenada .....	31°51'N	116°38'W
La Paz .....	24°10'N	110°21'W
Manzanillo .....	19°03'N	114°20'W
Baltra, Santa Cruz, Galapagos ..	0°42'S	90°02'W