

A METHOD FOR APPROXIMATING THE NODAL MODULATIONS OF REAL TIDES

by M. AMIN (*)

SUMMARY

A simple method for approximating the nodal modulations of tidal constituents generated in shallow waters is described. The changes in the nodal modulations from their equilibrium values are found to be mostly due to bottom friction. In this method, functional approximations of interaction coefficients are obtained with the help of resolved non-linear tides of frictional origin. New nodal terms and contributions to the nodal terms of gravitational origin are estimated by using the interpolated values of interaction coefficients for respective frequencies. Four years of observations can provide a reasonably accurate estimate of nodal terms and nodal modulations as such, which normally requires at least 18.6 years of observations. Tidal predictions are shown to be quantitatively improved when newly derived modulations are used in place of the conventional equilibrium modulations.

INTRODUCTION

In the analysis and prediction of tides, the conventional method of accounting for unresolved nodal terms is based on using the modulations of the equilibrium tide due to the gravitational forces. This approach is justified by the assumption that the response of the oceanic system is expected to remain almost constant within a narrow band of frequencies covering 1 cycle per 18.6 years. Thus the relationship between the principal tide and its nodal terms in the observed tide (output) remains the same as in the input equilibrium tide. However, the input to the oceanic system is not limited to the equilibrium tide; there are contributions from radiational tides (CARTWRIGHT, 1968 and ZETLER, 1971), atmospheric tides (CARTWRIGHT, 1968 and AMIN, 1982) and non-linear tides. The spectral distribution of the tidal energy is changed by non-linearities

(*) Institute of Oceanographic Sciences, Bidston Observatory, Birkenhead, Merseyside L43 7RA, U.K.

introduced by shallow water effects and bottom friction (GALLAGHER & MUNK, 1971, and PINGREE, 1983). The spatial distribution of forcing functions of these tides is different from that of the gravitational forces. CARTWRIGHT (1975) showed that the oceanic response varies if the spatial distribution of input forces varies, and thus changes in relationships among various tides are not as normally expected. PINGREE & GRIFFITHS (1981) and PINGREE (1983) noted that the effect of friction on the solar tide (S_2) and the lunar elliptic tide (N_2) is more than that on the principal lunar tide (M_2). The nodal terms are observed to be more affected than their principal tides by the bottom friction, and the modulations of the equilibrium tide cannot account for these changes (AMIN, 1985). CARTWRIGHT & EDDEN (1977) have suggested that triple interaction due to the bottom friction can be represented as a smooth function of frequency (called response function or admittance of third order) over a band of frequencies covering a tidal species. New nodal terms and contributions to the existing terms of gravitational origin can be estimated by utilising the interaction coefficients at the desired frequencies interpolated from these functional approximations.

This paper is concerned with estimating the contribution of non-linear terms to the principal tides and their nodal terms where data of sufficient length to resolve them is not available, so that nodal modulations can be corrected. Our investigations will be limited to the tidal regimes which are predominantly semi-diurnal, such as around Great Britain, where the effects of friction on nodal modulations are quite noticeable.

MATHEMATICAL BACKGROUND

The tides of deep water are of the form

$$\zeta(t) = \sum_i H_i \cos(V_i + \sigma_i t - G_i) = \sum_i Z(\sigma_i) H_i^{(e)} \cos(V_i + \sigma_i t - G_i) \quad (1)$$

where ζ is the surface elevation above the mean sea level, $H^{(e)}$ is the amplitude of a harmonic term in the equilibrium tide and H is its amplitude in the observed tide, V is the phase of the equilibrium term at an arbitrary time origin $t = 0$, σ is the speed of the harmonic term and G is the phase lag of the observed term with respect to its phase in the equilibrium tide. The ratio ($Z = H/H^{(e)}$) is the response (linear admittance) of the oceanic system. The harmonic terms of equation (1) correspond to the terms obtained from an expansion of the equilibrium tide and summation is over all terms which have a significant contribution to sea surface elevations. In shallow waters the tidal wave is influenced by additional forces. Neglecting the high order terms, Coriolis effects and with various degrees of approximations, the equations of continuity and motion can be written as:

$$\partial \zeta / \partial t = - \frac{\partial}{\partial x} (D u) - \frac{\partial}{\partial y} (D v) - \frac{\partial}{\partial x} (\zeta u) - \frac{\partial}{\partial y} (\zeta v) \quad (2)$$

$$\partial u / \partial t = -g \frac{\partial \zeta}{\partial x} - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{k}{\rho D} u(u^2 + v^2)^{1/2} \quad (3)$$

$$\partial v / \partial t = -g \frac{\partial \zeta}{\partial y} - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \frac{k}{\rho D} v(u^2 + v^2)^{1/2} \quad (4)$$

where u and v are depth averaged current components along the x - and y -axes respectively, D is the depth of the sea, g is the acceleration due to gravity, ρ is the density of water and k is the coefficient of bottom friction. Coriolis effects are neglected here because they do not influence the generation of new non-linear terms. An approximate solution of equations (2-4) for ζ , which satisfies the input given by equation (1), can be written in the form (AMIN, 1985):

$$\zeta(t) = \zeta^{(L)}(t) + \zeta^{(S)}(t) - \zeta^{(F)}(t) \quad (5)$$

where

$\zeta^{(L)}(t)$ = the linear tide as in equation (1).

$$\begin{aligned} \zeta^{(S)}(t) = & \sum_i Z''(\sigma_i, \sigma_i) \cdot (\sigma_i + \sigma_i) \frac{H_i^2}{2} \sin [2V_i + 2\sigma_i t - 2G_i + \Psi''(\sigma_i, \sigma_i)] \\ & + \sum_i \sum_{j \neq i} Z''(\sigma_i, \pm \sigma_j) (\sigma_i, \pm \sigma_j) H_i H_j \sin [(V_i \pm V_j) + (\sigma_i \pm \sigma_j)t \cdot (G_i \pm G_j) + \Psi''(\sigma_i, \pm \sigma_j)] \\ & + \text{high order terms} \end{aligned} \quad (6)$$

$$\begin{aligned} \zeta^{(F)}(t) = & \sum_i Z'''(\sigma_i) \left[\frac{3}{2} \sum_j H_j^2 - \frac{3}{4} H_i^2 \right] H_i \cos [V_i + \sigma_i t - G_i + \Psi'''(\sigma_i)] \\ & + \sum_i \sum_{j \neq i} Z'''(2\sigma_i, \pm \sigma_j) \frac{3}{4} H_i^2 H_j \cos [(2V_i \pm V_j) + (2\sigma_i \pm \sigma_j)t - (2G_i \pm G_j) + \Psi'''(2\sigma_i, \pm \sigma_j)] \\ & + \sum_i \sum_{j > i} \sum_k Z'''(\sigma_i, \sigma_j, \sigma_k) \frac{3}{2} H_i H_j H_k \cos [(V_i + V_j + V_k) + (\sigma_i + \sigma_j + \sigma_k)t \cdot (G_i + G_j + G_k) + \Psi'''(\sigma_i, \sigma_j, \sigma_k)] \\ & + \sum_i \sum_{j > i} \sum_k Z'''(\sigma_i, \sigma_j, -\sigma_k) \frac{3}{2} H_i H_j H_k \cos [(V_i + V_j + V_k) + (\sigma_i + \sigma_j - \sigma_k)t \cdot (G_i + G_j + G_k) + \Psi'''(\sigma_i, \sigma_j, -\sigma_k)] \\ & + \sum_i Z'''(\sigma_i, \sigma_i, \sigma_i) \frac{1}{4} H_i^3 \cos [3V_i + 3\sigma_i t - 3G_i + \Psi'''(\sigma_i, \sigma_i, \sigma_i)] + \text{high order terms} \end{aligned} \quad (7)$$

The shallow water tide $\zeta^{(S)}$ is due to the terms $\frac{\partial}{\partial x} (\zeta u)$ and $\frac{\partial}{\partial y} (\zeta v)$, $u \frac{\partial u}{\partial x}$ and $v \frac{\partial v}{\partial y}$,and frictional tide $\zeta^{(F)}$ is due to the terms $u (u^2 + v^2)^{1/2}$ and $v (u^2 + v^2)^{1/2}$ in equations (2-4).

The interaction coefficients Z'' and Z''' and phase response Ψ'' and Ψ''' are functions of g , D , k , the distance through which the tidal wave has travelled in time t , and the response of the sea system. The response of the sea changes both in the frequency and space domain. Interaction coefficient Z''' and phase response Ψ''' of a non-linear term of speed σ_n , generated by interaction of terms of speeds σ_i , σ_j and σ_k are discrete values of response functions $Z'''(\sigma)$ and $\Psi'''(\sigma)$ at $\sigma = \sigma_n$. Admittance functions Z' , Z'' and Z''' characterise linear, second order and third order interactions. We are only concerned with third order interaction, represented by equation (7), because the non-linear tides of the semi-diurnal species which are of significant size are generated by this mechanism. The first term of equation (7) shows how bottom friction modifies the input gravitational tides. In view of the many variables involved it is not easy to compute interaction coefficients analytically. It was suggested (AMIN, 1985) that a more simple and realistic approach is to equate the observed and analytical values of non-linear terms as

$$H_n = Z'''_n c H_i H_j H_k \quad (8)$$

$$G_n = (G_i + G_j - G_k) - \Psi'''_n \quad (9)$$

giving

$$Z'''_n = H_n / c H_i H_j H_k \quad (10)$$

$$\Psi'''_n = (G_i + G_j - G_k) - G_n \quad (11)$$

The constant c originates from the expansion of terms $u(u^2+v^2)^{1/2}$, $v(u^2+v^2)^{1/2}$. In case of non-linear tides of triple interaction $c = 3/4$ if $i=j$ (e.g. $2SM_2$) and $c = 3/2$ if $i \neq j$ (e.g. MSN_2).

The interaction coefficients and phase responses can thus be computed with the help of resolved non-linear tides and interacting tides which are responsible for their generation. Data sets from eight ports on the west coast of Great Britain (Table 1) were analysed to resolve non-linear tides and obtain empirical estimates for the interaction coefficients. A smooth functional relationship for their coefficients is important and is shown for the semi-diurnal band in Figure 1.

TABLE 1

Observations utilised to resolve the tidal constituents which are used to estimate the nodal terms

Port	Observations
Stornoway	4 years (1978-81)
Millport	8 years (1968-75)
Liverpool	18.6 years (1963-Aug. 1981)
Holyhead	8 years (1964-71)
Fishguard	9 years (1963-71)
Milford Haven	8.5 years (June 1961-72)
Ilfracombe	3.3 years (Feb. 1968-April 1971)
Avonmouth	2 years (1974-75)

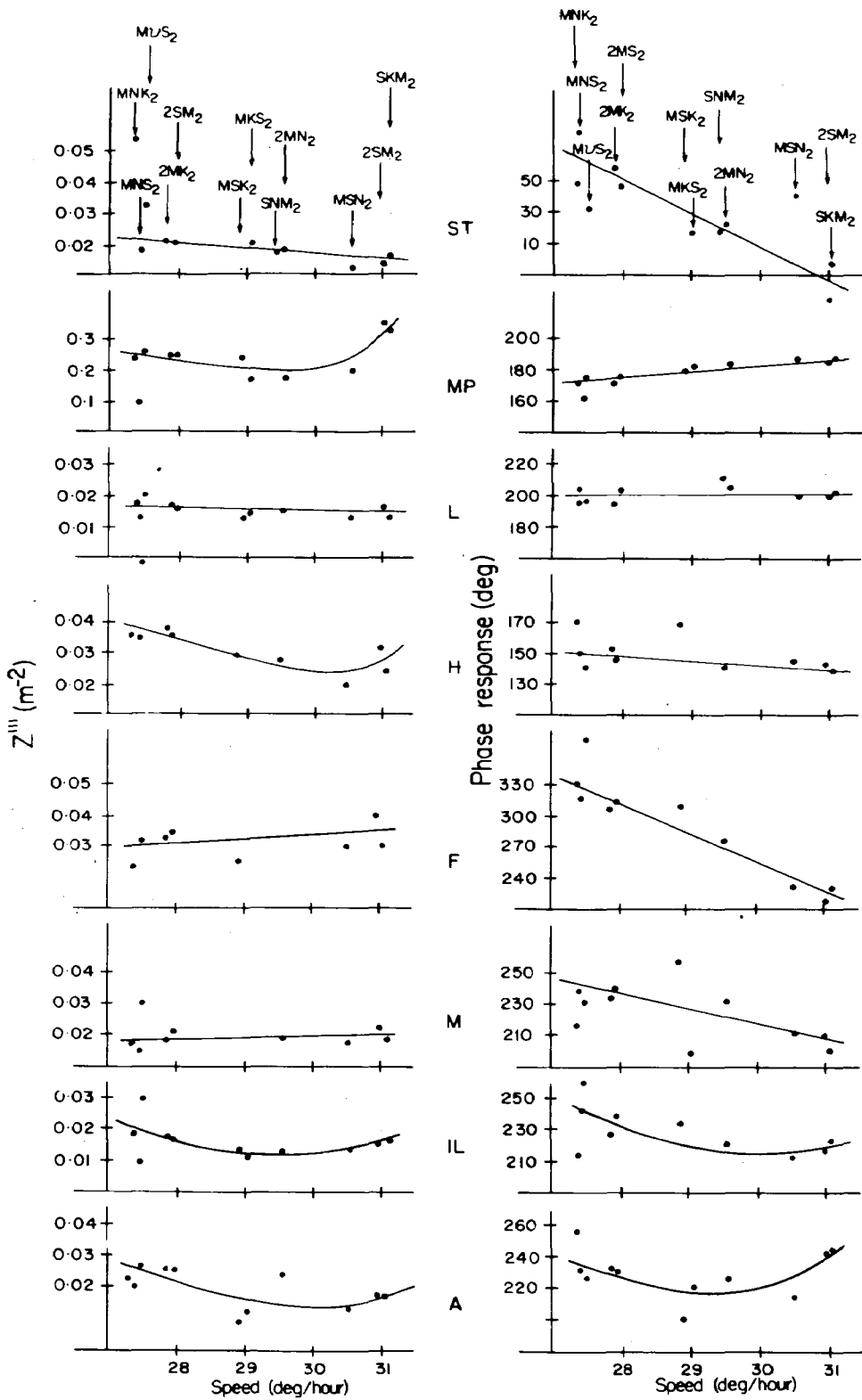


FIG. 1.— Functional approximations of interaction coefficients and phase responses of non-linear tides of the semi-diurnal species. ST, Stornoway; MP, Millport; L, Liverpool; H, Holyhead; F, Fishguard; M, Milford Haven; IL, Ilfracombe; A, Avonmouth.

THE NODAL TIDES OF M_2 AND S_2

In a study of perturbations of tidal harmonics of the Thames estuary, it was observed that nodal modulations were over-estimated when using equilibrium theory (AMIN, 1983). As a result the M_2 tide, deduced from observations, showed periodic variations which were opposite in phase to the equilibrium nodal modulations. Similar variations were also noted in the S_2 tide. Analyses of 19 years' observations of Liverpool and Newlyn sea levels confirmed the presence of previously unknown nodal terms near the M_2 and S_2 (AMIN, 1985).

On the British coast, contributions from the second order interaction can be neglected because O_1 and K_1 , and P_1 , are very small in comparison with M_2 and S_2 . It is reasonable to assume that the triple interaction, $(M_2 + M_2 - M_{2,-N})$, is responsible for the generation of $M_{2,N}$ and interactions $(S_2 \pm M_2 \mp M_{2,-N})$ are responsible for the development of $S_{2,\pm N}$. Here the second subscript is used to indicate a nodal term and specifies the difference between its argument number and the argument number of its principal tide which is defined by the first subscript.

Interaction coefficients can be interpolated from the function Z'' for the frequencies of constituents M_2 and S_2 and nodal terms $M_{2,\mp N}$ and $S_{2,\pm N}$. The new terms can be estimated by substituting the values of H and G of M_2 , S_2 and $M_{2,-N}$ in equations (8, 9). However, the problem of estimating the term $M_{2,-N}$, when observations are only for one or two years (less than 18 years), is a little more complex. There are two procedures which can be adopted to estimate this term. First, it can be approximated from M_2 by simply using the equilibrium relationship. This procedure is not quite accurate because experience has shown that the nodal term is over-estimated. Second, the observed M_2 tide can initially be adjusted for frictional effects obtained from the first term in equation (7) and eliminated from it. Then the nodal term can be estimated by using equilibrium theory, and the effect of friction on it is estimated and added. The nodal terms calculated in this way give a more accurate representation of the observed tide than that given by the first procedure because account is taken of the non-linear effects. The nodal terms estimated by this procedure are listed in Table 2, and nodal modulations computed by using these terms for years 1960-80 are shown in Figure 2. The differences between the modulation computed here and that given by the equilibrium tide, conventionally used in analysis and predictions, are also included in Figure 2. Modulations in the real tides are smaller than that in the equilibrium tide. Moreover, differences increase where tidal ranges increase, for example at the ports of Avonmouth and Liverpool. Another interesting feature is that the modulation of Stornoway tides is lightly increased in spite of this being a fairly linear port. Although this behaviour appears to be anomalous, it is also observed by PINGREE (personal communication) and he considers it is due to the nearby amphidrome.

TABLE 2

The estimated amplitudes (mm) and phase lags (degrees) of the nodal terms of M_2 and S_2 tides

(a) The M_2 tide						
Port	M_{2-N}		M_2		$M_{2,N}$	
	H	G	H	G	H	G
Stornoway	54	195.9	1396	197.35	2	328.0
Millport	33	341.0	1121	342.73	6	350.5
Liverpool	99	329.6	3121	323.53	10	247.6*
Holyhead	63	289.6	1810	291.88	4	328.2
Fishguard	52	209.2	1359	207.52	2	100.9
Milford Haven	76	176.5	2240	172.55	6	120.6
Ilfracombe	109	165.4	3070	162.24	10	115.2
Avonmouth	131	210.0	4249	201.53	35	154.0

(b) The S_2 tide						
Port	S_{2-N}		S_2		$S_{2,N}$	
	H	G	H	G	H	G
Stornoway	1	190.0	549	230.40	1	186.0
Millport	4	204.3	296	34.92	4	207.7
Liverpool	13	158.2	1008	7.87	5	111.1*
Holyhead	3	178.7	596	327.81	3	183.6
Fishguard	2	144.4	533	247.83	2	341.4
Milford Haven	5	357.7	820	216.63	5	349.6
Ilfracombe	8	344.8	1122	208.65	8	340.5
Avonmouth	23	52.0	1500	261.25	22	32.0

* These terms are resolved from observations of 18.6 years.

A COMPARISON OF SYNTHESISED TIDES WITH THE OBSERVED TIDES

The actual performance of the procedure for estimating the nodal terms and nodal modulations can only be tested by the accuracy of predictions. In this connection, the observed tides of Avonmouth were compared with: (a) tides synthesised using conventional modulations based on the equilibrium tide, and (b) tides synthesised using modulations as computed here. The differences (observed — synthesised) of Mean High Water Springs for years 1941-50 are shown in Figure 3. Only spring tides are examined because the M_2 tide is almost in phase with the S_2 tide during these occasions and their errors combine to give differences of significant magnitudes which are easy to observe. Effects of surges and noise are reduced by the process of averaging over a year. Improvements in predictions are clearly demonstrated, both near the peak of the nodal cycle around 1941 and when modulations are near a minimum in 1949. Moreover, these accuracies are achieved by using observational data from only two years (1974-75).

In absolute terms these improvements are very small, but their significance increases when they are associated with extreme tidal conditions. They gain additional significance in tidal power studies where energy output is calculated from the square of the tidal amplitudes.

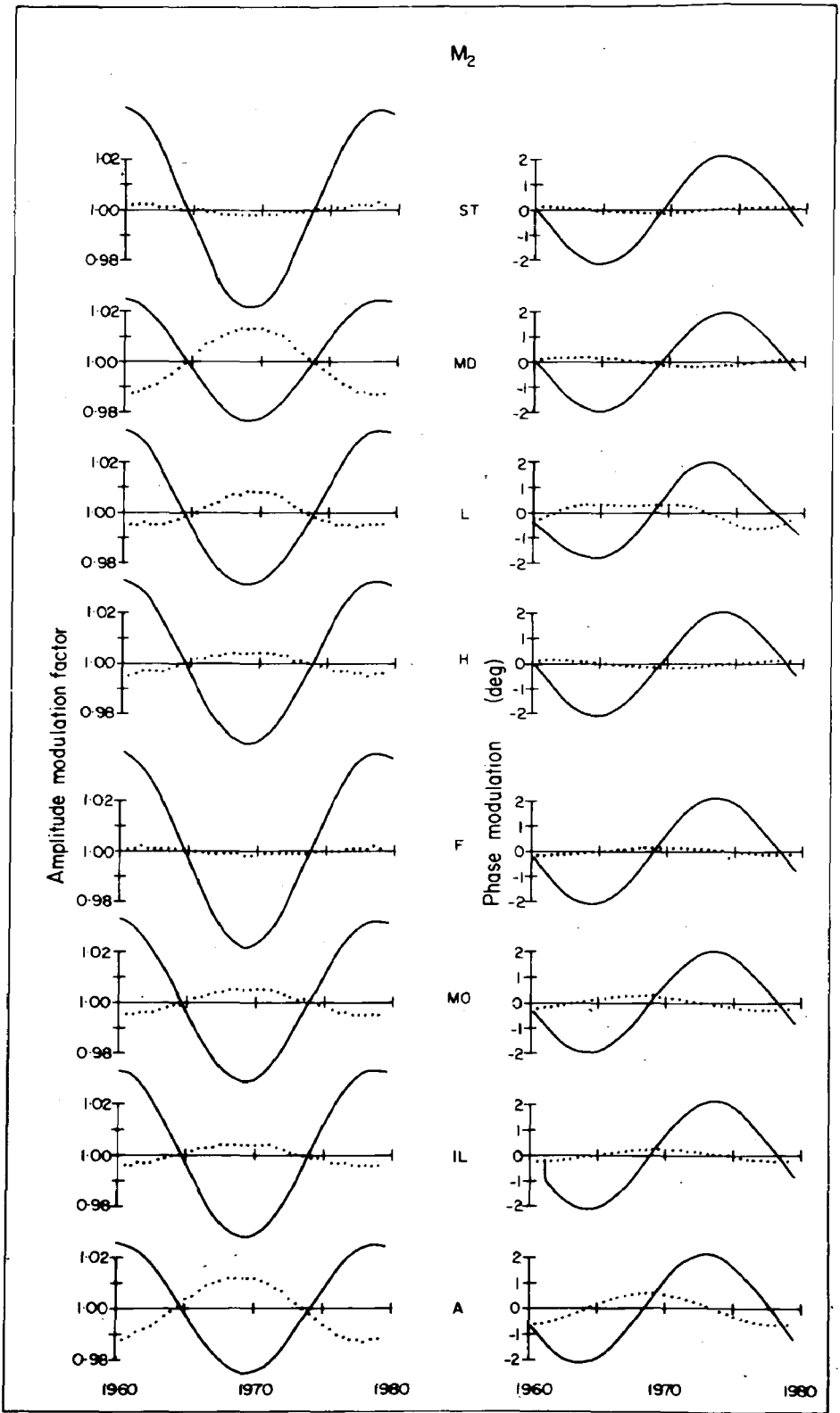


FIG. 2a.— Estimated nodal modulations of the observed M_2 tide (—), and difference between the observed and the equilibrium modulations of the M_2 (observed - equilibrium) tide (.....).

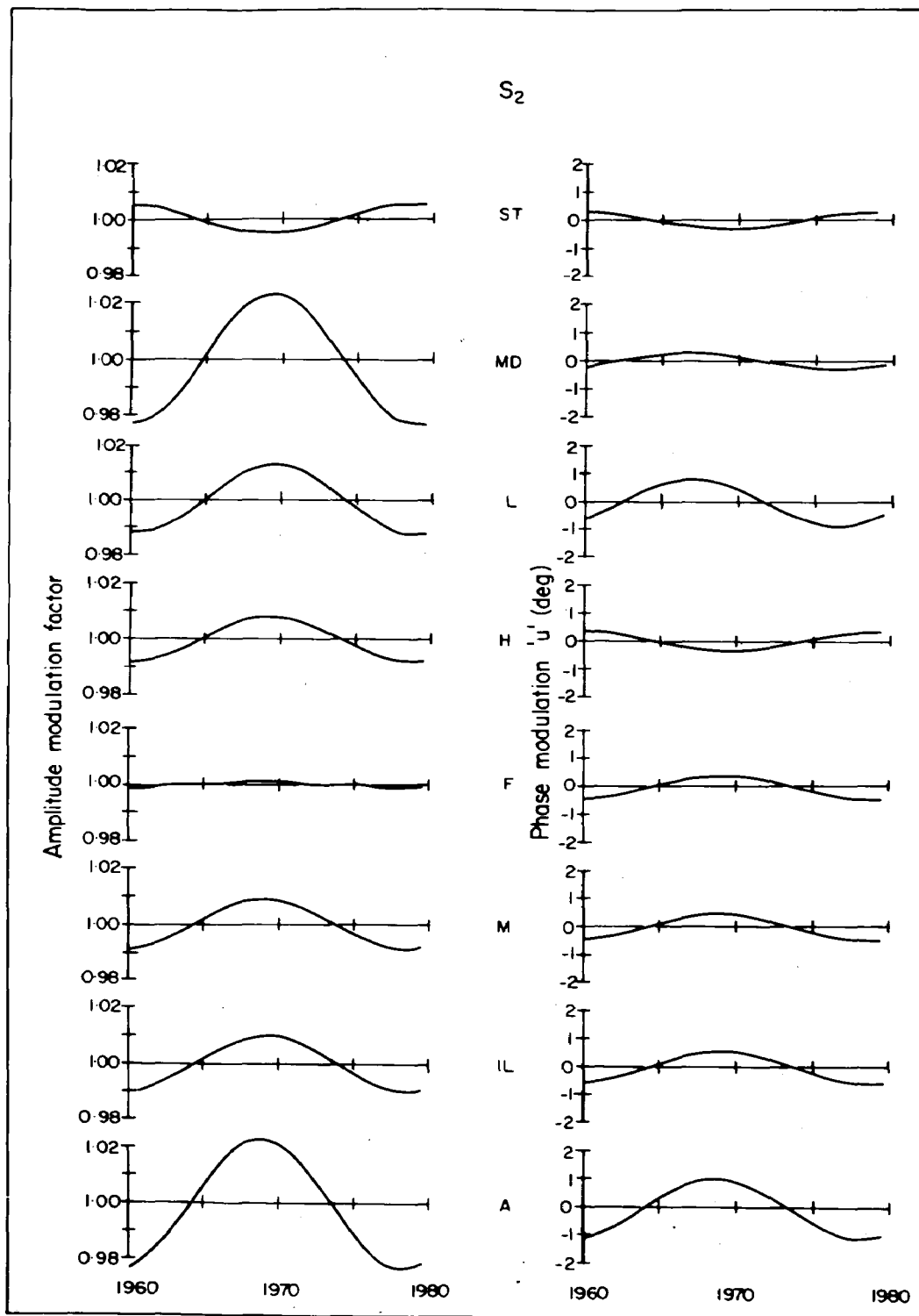


FIG. 2b.— Estimated nodal modulations of the observed S_2 tide. The equilibrium S_2 tide has no nodal modulations.

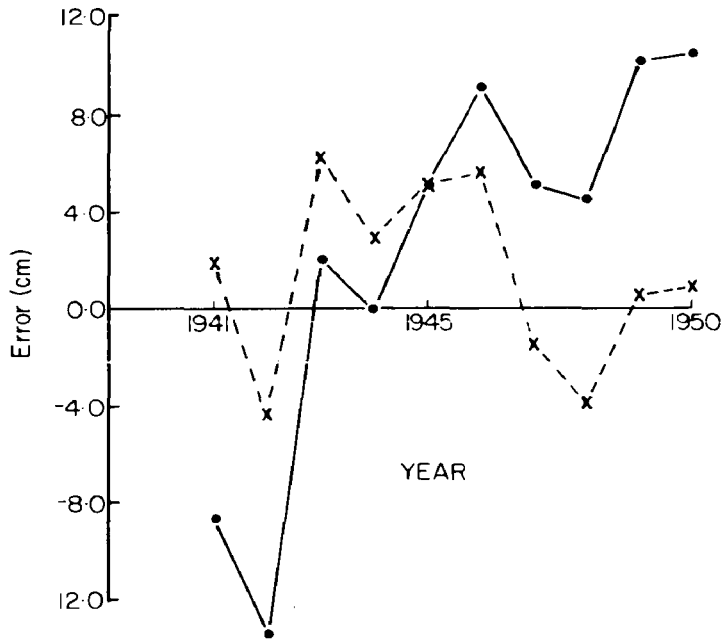


FIG. 3.— Differences (observed – synthesised) of Mean High Water Springs: ———, when conventional modulations based on the equilibrium tide are used; - - - - - , when the modulations as computed here are used.

CONCLUDING REMARKS

Functional approximations of interaction coefficients can be used to estimate nodal terms in the observed tide. Nodal modulations of the real tides are shown to differ from those of the equilibrium tides which are conventionally used in analysis and prediction of tides. The observed modulations are generally smaller than the modulations of the equilibrium tides. Stornoway is an exception where the modulation of the observed tide is greater than the modulation of the equilibrium tide. This anomalous behaviour is probably related to the influence of a nearby amphidromic system.

It is encouraging to find that this minor feature of a tidal regime which can only be studied by sophisticated numerical models can be detected by a relatively simple method. An experiment on Avonmouth tides has shown that the accuracy of predictions increases when the new modulations are used in computation. Therefore, it is suggested that assumptions made in estimating interaction coefficients are reasonable and the method of deriving nodal terms is valid.

However, the efficiency of the method may be affected by some of the necessary assumptions. Firstly, non-linear constituents used in estimating interaction coefficients are assumed to be of single origin, i.e. third order. This is

not strictly true for all constituents because many of these constituents have input either from double interaction (e.g. OO_2 contributes to MNK_2) or from the equilibrium tide (e.g. at the frequency of MNS_2). In some cases, linear and non-linear components (e.g. μ_2 and $2MS_2$) can be separated (see AMIN, 1983), but some error is inevitable in such a process. Secondly, the effect of friction on nodal term $M_{2,N}$ has to be estimated, and eliminated from it for its subsequent use in the derivation of the other terms.

A successful implementation of this procedure can improve the harmonic method of analysis and prediction significantly. It will also bring the Harmonic Method much closer to the Response Method (MUNK and CARTWRIGHT, 1966).

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